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Estimation Parameters of Weighted Exponential Distribution with Presence of Outliers Generated from Exponential Distribution

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Abstract: This study deals with the estimation of parameters of the weighted exponential distribution with presence of outliers. The maximum likelihood and moment of the estimators are derived. These estimators are compared empirically using monte carlo simulation when all the parameters are unknown. There Bias and MSE are investigated with help of numerical technique.

Key words: Weighted exponential distribution, maximum likelihood estimator, moment estimator, outlier, newton-raphson method, monte-carlo simulation

INTRODUCTION

Recently a new class of Weighted Exponential (WE) distribution has been proposed by Gupta and Kundu (2009). Different methods may be used to introduce a shape parameter to an exponential model and they may result in a variety of Weighted Exponential (WE) distribution. For example, the gamma distribution and the generalized exponential distribution are different weighted versions of the exponential distribution.

The random variable X is said to have a Weighted Exponential (WE) distribution with the shape and scale parameters as $\alpha>0$ and $\lambda>0$, respectively, if it has the following probability density (PDF):

$$f_X(x;\alpha,\lambda)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right), x>0 \tag{1}$$

The corresponding Cumulative Distribution Function (CDF) for x>0, becomes:

$$F_X(x;\alpha,\lambda)=\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{\alpha+1}\left(1-e^{-(\alpha+1)\lambda x}\right)\right] \tag{2}$$

From now on a WE distribution with the shape and scale parameters as α and λ, respectively will be denoted WE (α, λ).

Note that the WE (α, λ) can be obtained exactly the same way Azzalini (1985) obtained the skew-normal distribution from two i.i.d. normal distribution. Suppose X_1 and X_2 are i.i.d. exp (λ), i.e., an exponential random variable with mean $1/\lambda$, then for $\alpha>0$ consider a new random variable $X = X_1$, if $\alpha X_1 \geq X_2$. Then X follows WE (α, λ), Dixit *et al.* (1996), assume that a set of random variables (X_1, X_2,..., X_n) represent the distance of an infected sampled plant from a plant from a plot of plants inoculated with a virus. Some of the observations are derived from the airborne dispersal of the spores and are distributed according to the exponential distribution. The other observations out of n random variables (say k) are present because aphids which are know to be carriers of Barley Yellow Mosaic Dwarf Virus (BYMDV) have passed the virus into the plants when the aphids feed on the sap. Dixit and Nasiri (2001) considered estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution. Also new Makhdoom (2011) obtained the estimation of the parameters of minimax distribution in the presence of one outlier. In this paper, we obtain the maximum likelihood and moment estimators of the parameters of the weighted exponential distribution in the presence of one outlier generated from exponential distribution.

Let the random variable X_1, X_2,..., X_{n-k} are independent, each having the probability density function f_1 (x):

$$f_1(x;\alpha,\lambda)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right),\ x>0$$

k random variables (as outlier) are also independent, have the probability density function f_2 (x):

$$f_1\ (x;\ \alpha,\ \lambda_2) = \lambda e^{-\lambda x}\ x>0$$

The joint density of X_1, X_2,..., X_n is given as:

$$f(x_1,x_2,\dots,x_n)=C\prod_{i=1}^{n}f_1(x_i).\sum_{\underset{\sim}{A}}\prod_{r=1}^{k}\frac{f_2(x_{A_r})}{f_1(x_{A_r})} \tag{3}$$

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Where:

$$C=\frac{k!(n-k)!}{n!}$$

and:

$$\sum_{\underline{A}}=\sum_{A_1=1}^{n-k+1}\sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}$$

(Dixit, 1989; Dixit *et al*., 1996; Dixit and Nasiri, 2001). The main of this paper is to focus on obtaining maximum likelihood estimators and moment estimators of WE (α, λ) with presence of outliers.

JOINT DISTRIBUTION OF (X_1,..., X_n) WITH k OUTLIERS

The joint distribution of X_1,..., X_n in the presence of k outliers after some computations in Eq. 3 is given by:

$$f(x_1,x_2,\ldots,x_n)=C\prod_{i=1}^{n}\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\sum_{\underline{A}}\prod_{r=1}^{k}\frac{\lambda e^{-\lambda x_{A_r}}}{\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x_{A_r}}\left(1-e^{-\alpha\lambda x_{A_r}}\right)}$$

$$=C\left(\frac{\alpha+1}{\alpha}\right)^{n-k}\lambda^n e^{-\lambda\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-\alpha\lambda x_i}\right)\sum_{\underline{A}}\frac{1}{\prod_{i=1}^{n}\left(1-e^{-\alpha\lambda x_{A_r}}\right)} \qquad (4)$$

METHOD OF MOMENT

From Eq. 4, the marginal distribution of X is:

$$h(x;\alpha,\lambda)=\frac{k}{n}\lambda e^{-\lambda x}+\frac{n-k}{n}\frac{(\alpha+1)}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right) \qquad (5)$$

where, x>0, λ>0. Consider,

$$b=\frac{k}{n},\quad \bar{b}=\frac{n-k}{n}\ \text{and}\ b+\bar{b}=1$$

From Eq. 5 we get:

$$E(X)=\frac{k}{n}\frac{1}{\lambda}+\frac{n-k}{n}\frac{\alpha+1}{\alpha}\frac{1}{\lambda}-\frac{n-k}{n}\frac{1}{\alpha}\frac{1}{(\alpha+1)\lambda}$$

$$=\frac{1}{\lambda}\left(b+\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-\bar{b}\frac{1}{\alpha}\frac{1}{\alpha+1}\right)$$

$$E(X^2)=\frac{k}{n}\frac{2}{\lambda^2}+\frac{n-k}{n}\frac{\alpha+1}{\alpha}\frac{2}{\lambda^2}-\frac{n-k}{n}\frac{1}{\alpha}\frac{2}{(\alpha+1)^2\lambda^2}$$

$$=\frac{2}{\lambda^2}\left(b+\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-\bar{b}\frac{1}{\alpha}\frac{1}{(\alpha+1)^2}\right)$$

Let:

$$m_i^{'}=\frac{1}{n}\sum_{j=1}^{n}X_j^i,\quad D=\frac{m_2^{'}}{m_1^{'}}$$

and:

$$H=\frac{m_2^{'}}{m_1^{'2}}$$

$$D=\frac{b+\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-\bar{b}\frac{1}{\alpha}\frac{1}{(\alpha+1)^2}}{b+\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-\bar{b}\frac{1}{\alpha}\frac{1}{\alpha+1}}\frac{2}{\lambda} \qquad (6)$$

$$H=\frac{2b+2\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-2\bar{b}\frac{1}{\alpha}\frac{1}{(\alpha+1)^2}}{\left(b+\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-\bar{b}\frac{1}{\alpha}\frac{1}{\alpha+1}\right)^2} \qquad (7)$$

From Eq. 7, we have:

$$Hb^2+2Hb\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-2H\bar{b}^2\frac{1}{\alpha^2}-2Hb\bar{b}\frac{1}{\alpha(\alpha+1)}+H\bar{b}^2\left(\frac{\alpha+1}{\alpha}\right)^2$$

$$+H\bar{b}^2\frac{1}{\alpha^2(\alpha+1)^2}+2bH+2\bar{b}H\left(\frac{\alpha+1}{\alpha}\right)-2H\bar{b}\frac{1}{\alpha}\frac{1}{(\alpha+1)^2} \qquad (8)$$

$$=2b+2\bar{b}\left(\frac{\alpha+1}{\alpha}\right)-2\bar{b}\frac{1}{\alpha}\frac{1}{(\alpha+1)^2}$$

MAXIMUM LIKELIHOOD ESTIMATORS OF (α, λ_1, λ_2)

The joint distribution of X_1,..., X_n in the presence of k outliers after some computations in Eq. 3 is given by:

$$f(x_1,\ldots,x_n)=C\left(\frac{\alpha+1}{\alpha}\right)^{n-k}\lambda_1^{\,n-k}\lambda_2^{\,k}\prod_{i=1}^{n}e^{-\lambda_1 x_i}(1-e^{-\alpha\lambda_1 x_i})$$

$$\sum_{\underline{A}}\prod_{r=1}^{k}\frac{e^{x_{A_r}(\lambda_1-\lambda_2)}}{1-e^{-\alpha\lambda_1 x_{A_r}}} \qquad (9)$$

In Eq. 4, if let k = 1(for one outlier), the joint distribution of X_1,..., X_n is given by:

$$f(x_1,\ldots,x_n)=\frac{1}{n}\left(\frac{\alpha+1}{\alpha}\right)^{n-1}\lambda_1^{\,n-1}\lambda_2 e^{-\lambda_1\sum_{i=1}^{n}x_i}$$

$$\prod_{i=1}^{n}\left(1-e^{-\alpha\lambda_1 x_i}\right)\sum_{A=1}^{n}\frac{e^{(\lambda_1-\lambda_2)x_{A_1}}}{1-e^{-\alpha\lambda_1 x_{A_1}}} \qquad (10)$$

The Log-likelihood function of the observed sample is:

$$\ln L(\alpha,\lambda_1,\lambda_2) = -\ln(n) + (n-1)\ln\left(\frac{\alpha+1}{\alpha}\right) + (n-1)$$
$$\ln(\lambda_1) + \ln(\lambda_2) - \lambda_1\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln\left(1-e^{-\alpha\lambda_1 x_i}\right) \quad (11)$$
$$+\ln\left(\sum_{A_1=1}^{n}\frac{e^{(\lambda_1-\lambda_2)x_{A_1}}}{1-e^{-\alpha\lambda_1 x_{A_1}}}\right),$$

The MLE's of α, λ_1 and λ_2 say $\hat{\alpha}, \hat{\lambda}_1$ and $\hat{\lambda}_2$, respectively which is obtained as the solutions of:

$$g_1 = \frac{\partial \ln L(\alpha,\lambda_1,\lambda_2)}{\partial\alpha} = \frac{1-n}{\alpha(\alpha+1)} + \lambda_1\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda_1 x_i}}{1-e^{-\alpha\lambda_1 x_i}} + \frac{\sum_{A_1=1}^{n}\frac{-\lambda_1 x_{A_1} e^{-\alpha\lambda_1 x_{A_1}+\lambda_1 x_{A_1}-\lambda_2 x_{A_1}}}{\left(1-e^{-\alpha\lambda_1 x_{A_1}}\right)^2}}{\sum_{A_1=1}^{n}\frac{e^{(\lambda_1-\lambda_2)x_{A_1}}}{1-e^{-\alpha\lambda_1 x_{A_1}}}} = 0$$

$$g_2 = \frac{\partial \ln L(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_1} = \frac{n-1}{\lambda_1} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{\alpha x_i e^{-\alpha\lambda_1 x_i}}{1-e^{-\alpha\lambda_1 x_i}} + \frac{\sum_{A=1_1}^{n}\frac{x_{A_1} e^{(\lambda_1-\lambda_2)x_{A_1}}\left(1-e^{-\alpha\lambda_1 x_{A_1}}\right) - \alpha x_{A_1} e^{-\alpha\lambda_1 x_{A_1}} e^{(\lambda_1-\lambda_2)x_{A_1}}}{\left(1-e^{-\alpha\lambda_1 x_{A_1}}\right)^2}}{\sum_{A_1=1}^{n}\frac{e^{(\lambda_1-\lambda_2)x_{A_1}}}{1-e^{-\alpha\lambda_1 x_{A_1}}}} = 0$$

$$g_3 = \frac{\partial \ln L(\alpha,\lambda_1,\lambda_2)}{\partial\lambda_2} = \frac{1}{\lambda_2} + \frac{\sum_{A=1_1}^{n}\frac{-x_{A_1} e^{(\lambda_1-\lambda_2)x_{A_1}}\left(1-e^{-\alpha\lambda_1 x_{A_1}}\right)}{\left(1-e^{-\alpha\lambda_1 x_{A_1}}\right)^2}}{\sum_{A_1=1}^{n}\frac{e^{(\lambda_1-\lambda_2)x_{A_1}}}{1-e^{-\alpha\lambda_1 x_{A_1}}}} = 0$$

There is no closed-form solution to this system of equations, so we will solve for $\hat{\alpha}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ iteratively, using the Newton-Raphson method for root finding. In our case we will estimate $\hat{\theta} = (\hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2)$ iteratively:

$$\hat{\theta}_{i+1} = \hat{\theta}_i - G^{-1} g$$

where, g and G are the vector of normal equations and matrix of second derivatives, respectively,

$$g = [g_1\ g_2\ g_3]$$

$$G = \begin{bmatrix} \frac{\partial g_1}{\partial\alpha} & \frac{\partial g_1}{\partial\lambda_1} & \frac{\partial g_1}{\partial\lambda_2} \\ \frac{\partial g_2}{\partial\alpha} & \frac{\partial g_2}{\partial\lambda_1} & \frac{\partial g_2}{\partial\lambda_2} \\ \frac{\partial g_3}{\partial\alpha} & \frac{\partial g_3}{\partial\lambda_1} & \frac{\partial g_3}{\partial\lambda_2} \end{bmatrix}$$

The Newton-Raphson algorithm converges, as our estimates of θ, λ_1 and λ_2 change by less than a tolerated amount with each successive iteration, to $\hat{\theta}, \hat{\lambda}_1$ and $\hat{\lambda}_2$.

MAXIMUM LIKELIHOOD ESTIMATORS OF (α, λ)

Here, without loss of generality we assume $\lambda_1 = \lambda_2 = \lambda$. Then the Newton-Raphson method be reduced to estimation of $\theta = (\alpha, \lambda)$ in WE (α, λ) with presence one outlier (k = 1) generate from exponential distribution. In this case the log-likelihood function based on the observed sample $\{x_1,..., x_n\}$ is :

$$l(x_1,\ldots,x_n;\alpha,\lambda) = -\ln(n) + (n-1)\ln\left(\frac{\alpha+1}{\alpha}\right) + n\ln(\lambda) - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln\left(1-e^{-\alpha\lambda x_i}\right) + \ln\left(\sum_{A_1=1}^{n}\frac{1}{\left(1-e^{-\alpha\lambda x_{A_1}}\right)}\right) \quad (12)$$

The elements of vector of normal equations (g) and matrix of second derivatives (G) are as following:

$$g_1 = \frac{\partial l}{\partial\alpha} = \frac{1-n}{\alpha(\alpha+1)} + \lambda\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{1-e^{-\alpha\lambda x_i}} - \lambda\frac{\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{\left(1-e^{-\alpha\lambda x_i}\right)^2}}{\sum_{i=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_i}}} \quad (13)$$

$$g_2 = \frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \alpha\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{\left(1-e^{-\alpha\lambda x_i}\right)} - \alpha\frac{\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{\left(1-e^{-\alpha\lambda x_i}\right)^2}}{\sum_{i=1}^{n}\frac{1}{\left(1-e^{-\alpha\lambda x_i}\right)}} \quad (14)$$

$$\frac{\partial g_1}{\partial\alpha} = \frac{(n-1)(2\alpha+1)}{\left(\alpha^2+\alpha\right)^2} - \lambda^2\sum_{i=1}^{n}\frac{x_i^2 e^{-\alpha\lambda x_i}}{\left(1-e^{-\alpha\lambda x_i}\right)^2} + \lambda\frac{\left(\sum_{i=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_i}}\right)\left(\sum_{i=1}^{n}\frac{x_i^2\lambda e^{-\alpha\lambda x_i}\left(1-e^{-\alpha\lambda x_i}\right)^2 - 2\left(1-e^{-\alpha\lambda x_i}\right)\lambda x_i^2 e^{-2\alpha\lambda x_i}}{\left(1-e^{-\alpha\lambda x_i}\right)^4}\right)}{\left(\sum_{i=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_i}}\right)^2}$$

$$\frac{\partial g_2}{\partial\lambda} = \frac{-n}{\lambda^2} + \alpha^2\sum_{i=1}^{n}\frac{x_i^2 e^{-\alpha\lambda x_i} - 2x_i^2 e^{-2\alpha\lambda x_i}}{(1-e^{-\alpha\lambda x_i})^2} - \alpha^2\left[\frac{\left(\frac{x_{A_1}^2 e^{-\alpha\lambda x_{A_1}}\left(1+e^{-\alpha\lambda x_{A_1}}\right)}{\left(1-e^{-\alpha\lambda x_{A_1}}\right)^3}\right)\left(\sum_{A_1=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_{A_1}}}\right) - \left(\sum_{A_1=1}^{n}\frac{x_{A_1} e^{-\alpha\lambda x_{A_1}}}{\left(1-e^{-\alpha\lambda x_{A_1}}\right)^2}\right)^2}{\left(\sum_{A_1=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_{A_1}}}\right)^2}\right]$$

$$\frac{\partial g_1}{\partial \lambda}=\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{1-e^{-\alpha\lambda x_i}}-\lambda\alpha\sum_{i=1}^{n}\frac{x_i^2 e^{-\alpha\lambda x_i}}{\left(1-e^{-\alpha\lambda x_i}\right)^2}-\frac{\sum_{A_1=1}^{n}\frac{x_{A_1}e^{-\alpha\lambda x_{A_1}}}{\left(1-e^{-\alpha\lambda x_{A_1}}\right)^2}}{\sum_{A_1=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_{A_1}}}}$$

$$+\lambda\alpha\left[\frac{\left(\frac{x_{A_1}^2 e^{-\alpha\lambda x_{A_1}}\left(1+e^{-\alpha\lambda x_{A_1}}\right)}{\left(1-e^{-\alpha\lambda x_{A_1}}\right)^3}\right)\left(\sum_{A_1=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_{A_1}}}\right)-\left(\sum_{A_1=1}^{n}\frac{x_{A_1}e^{-\alpha\lambda x_{A_1}}}{\left(1-e^{-\alpha\lambda x_{A_1}}\right)^2}\right)^2}{\left(\sum_{A_1=1}^{n}\frac{1}{1-e^{-\alpha\lambda x_{A_1}}}\right)^2}\right];$$

$$\frac{\partial g_2}{\partial \alpha}=\frac{\partial g_1}{\partial \lambda};$$

The Newton-Raphson algorithm converges, as our estimates of α and λ change by less than a tolerated amount with each successive iteration, to $\hat{\theta}$ and $\hat{\lambda}$.

Note in case of no outlier presence, Gupta and Kundu (2009) obtained the estimation of parameters weighted exponential by Maximum likelihood and Moment method.

NUMERICAL EXPERIMENTS AND DISCUSSIONS

In this study, we have addressed the problem of estimating parameters of Weighted Exponential distribution in presence of one outlier. In order to have some idea about Bias and Mean Square Error (MSE) of methods of moment and MLE, we perform sampling experiments using a SAS. The results are given in Table 1 and 2, when $\alpha = 3$, $\lambda = 0.5$, $\alpha = 2$ and $\lambda = 1$. We report the average estimates and the MSEs based on 1000 replications. It is observed that the maximum likelihood estimators are better than the moment estimators; these are true for all values of n. Therefore we suggest using the maximum method for estimating parameters of the weighted exponential distribution in the presence of one outlier.

Table 1: $\alpha = 3$, $\lambda = 0.5$

n	Bias $\hat{\lambda}_{MOM}$	MSE $\hat{\lambda}_{MOM}$	Bias $\hat{\lambda}_{MLE}$	MSE $\hat{\lambda}_{MLE}$
10	-0.7854	1.8761	0.8191	1.2531
15	-0.7915	1.6413	0.9171	1.0611
20	-0.6219	1.3211	0.7432	0.9328
25	-0.5618	1.1918	0.6161	0.4319
30	-0.2191	0.9715	0.3371	0.3191
35	-0.4321	0.8171	0.4181	0.2781
40	-0.1919	0.5321	0.2191	0.1345
45	-0.2849	0.3191	0.2991	0.0597
50	-0.1965	0.1817	0.1431	0.0397

Table 2: $\alpha = 2$, $\lambda = 1$

n	Bias $\hat{\lambda}_{MOM}$	MSE $\hat{\lambda}_{MOM}$	Bias $\hat{\lambda}_{MLE}$	MSE $\hat{\lambda}_{MLE}$
10	-0.1371	1.5918	0.9414	1.2191
15	-0.3391	1.4718	0.7519	1.1918
20	-0.0191	1.0191	0.5191	0.9981
25	-0.2191	0.9887	0.4181	0.7761
30	-0.7519	0.8852	0.4011	0.5217
35	-0.7817	0.7519	0.2191	0.4319
40	-0.4751	0.5191	0.1911	0.3191
45	-0.0319	0.4139	0.1719	0.2971
50	-0.1219	0.3118	0.0823	0.1918

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