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Random Binomial Tree Models and Options

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Abstract: In this study, we introduce the simplest random binomial tree model. Usual binomial tree model is prescribed by pair of numbers (u, d) , where u denotes the increase rate of the stock over the fixed period of time and d denotes the decrease rate, with $0 < d < 1 < u$. We call the pair (u, d) an environment of the binomial tree model. A pair (U_n, D_n) , where $\{U_n\}$ and $\{D_n\}$ be the sequences of independent, identically distributed random variables with $U_n > 1$ and $0 < D_n < 1 < U_n$ for all n , is called a random environment and binomial tree model with random environment is called random binomial tree model. In this paper, we define and study European call option for such models.

Key words: Binomial tree model, random walk in random environment, European call option

INTRODUCTION

A binomial tree model is extremely important, popular and useful technique for pricing an option (Cox *et al.*, 1979; Rendleman and Bartter, 1979). In the binomial model, the market is composed of a non-risky asset B (bond), corresponding to the investment into a savings account in a bank and of a risky asset S (stock), corresponding for example, to a quoted stock in the exchange. For the sake of simplicity, we suppose that the time intervals have the same length $\Delta t = t_n - t_{n-1}$, i.e., $N \times \Delta t = T$ and the interest rate is constant over the period $(0, T)$, that is $r_n = r/N$ for every interval $\Delta = t_n - t_{n-1}$. Then the dynamics of the bond is given by:

$$B_n = B_{n-1}(1+r/N), \quad N = 1, 2, \dots, N \quad (1)$$

So that:

$$B_n = B_0(1+r/N)^n$$

It is evident that B_n converges to $B_0 \times e^r$. The details of compound interest are available in study of Hull (2011) and Pascucci (2011).

For the risky asset, we assume that the dynamics is stochastic. In particular, we assume that when passing from time t_{n-1} to time t_n the stock can only increase or decrease its value with constant increase and decrease rates. Let u indicates the increase rate of the stock over the period $[t_{n-1}, t_n]$ and d indicates the decrease rate, where $0 < d < 1 < u$.

We point out that we have:

$$S_n = \begin{cases} uS_{n-1}, & \text{With probability } p, \\ dS_{n-1}, & \text{With probability } 1-p \end{cases} \quad (2)$$

Hence, a “trajectory” of the stock is vector such as (for example, in the case $N = 6$):

$$(S_0, uS_0, udS_0, u^2dS_0, u^3dS_0, u^4dS_0, u^4d^2S_0)$$

It is well-known (for example (3, 4) that the binomial tree model is arbitrage –free and complete if and only if condition:

$$d < 1 + r < u \quad (3)$$

The binomial tree model is a natural bridge, overture to continuous models for which it is possible to derive the Black-Scholes option pricing formula (Black and Scholes, 1973; Georgiadis, 2011; Merton, 1973).

There are two types of options: calls and puts. A call option gives the holder the right to buy the underlying asset for a certain price by a certain date. A put option gives the holder the right to sell the underlying asset by a certain date for a certain price. There are four possible positions in option markets: A long position in a call, a short position in a call, a long position in a put and a short position in a put. Taking a short position in an option is known as writing it. Options are currently traded on stocks, stock indices, foreign currencies, futures contracts and other assets. Options can be either American or European, a distinction that has nothing to do with geographical location. American options can be

exercised at any time up to the expiration date, whereas European options can be exercised only on the expiration date itself. European options are generally easier to analyze.

Random binomial tree model

Random walks in a random environment: Random walks in a random environment on the integers Z was introduced and studied by Solomon (1975). Let (α_n) be a sequence of independent, identically distributed random variables with $0 \leq \alpha_n \leq 1$ for all n . The random walk in a random environment on the integers Z is the sequence $\{X_n\}$ where $X_0 = 0$ and inductively $X_{n+1} = X_n + 1$ ($X_n - 1$), with probability αX_n , $(1 - \alpha X_n)$. Solomon have proved that randomizing the environment in some sense slows down the random walk. Later, Menshikov and Peteritis (2001) studied random walks in a random environment on a regular, rooted, colored tree and the asymptotic behavior of the walks was classified for ergodicity or transience in terms of the geometric properties of the matrix describing the random environment.

Random binomial tree model: As mentioned above for binomial tree model during each time step the value of stock either moves up with a certain probability by u times or moves down by d times with a certain probability. Let us call the pair (u, d) environment of binomial tree model. Note that in this model pair (u, d) is the same for any moment of time.

Now we define random environment and random binomial branch model as follows. Let $\{U_n\}$ and $\{D_n\}$ be the sequences of independent, identically distributed random variables with $U_n > 1$ and $0 < D_n < U_n$ for all n . This pair (U_n, D_n) is called a random environment.

The random binomial tree model in a random environment (U_n, D_n) is defined as model such that during n th moment of time the value of stock either moves up with a certain probability by U_n times or moves down by D_n times with a certain probability for some realization of these random variables.

Simplest random binomial tree model: Let $\{U_n\}$ and $\{D_n\}$ be the sequences of independent, identically distributed random variables such that the random variable U_n takes only two values u_1 and u_2 respectively the random variable D_n takes two values d_1 and d_2 . Thus the pair of random variables (U_n, D_n) describes two possible environments (u_1, d_1) and (u_2, d_2) .

To avoid complicated mathematical formulations let introduce and explore simplest example of random environment in binomial tree model using tossing a coin.

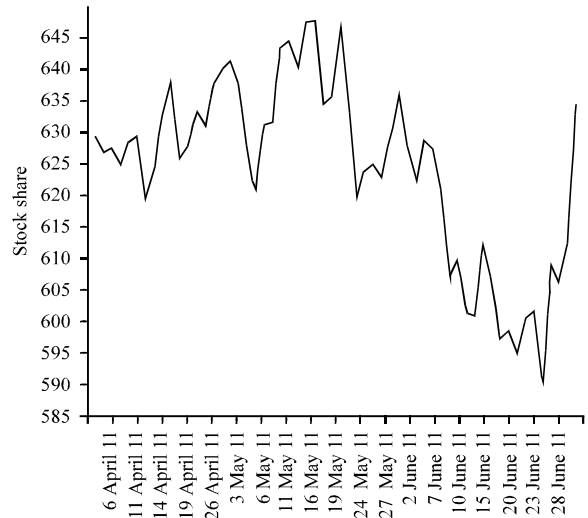


Fig. 1: Stock shares of Rolls Royce for period from 6th April till 28th June 2011

Let us consider two environments (u_1, d_1) and (u_2, d_2) . We will toss repeatedly a coin and if the result is “Head” we will apply first environment, in opposite case the second. That is we will choose environment randomly and we will call such model simplest random binomial tree model.

To justify necessity consideration of such models consider Fig. 1 which represents the movements of stock shares.

By examining the graph on the Fig. 1, one can see that there are two environments instead of one, namely $(u_1, d_1) = (1.04; 0.98)$ and $(u_2, d_2) = (1.02; 0.96)$. For this case we have two environments and they chosen randomly:

H, T, H, T, H, H, H, T, H, T, T, T, H, H, T, T, T, ...

One can see that this model more exactly reflects the trajectory of stock.

Theorem 1: The simplest random binomial tree model with two possible environments (u_1, d_1) and (u_2, d_2) is arbitrage-free and complete if and only if the following conditions holds:

$$\max \{d_1, d_2\} < 1 + r < \min \{u_1, u_2\} \tag{4}$$

where, r is the interest rate.

Proof: The proof immediately follows from similar proposition for binomial tree models with single environment. For example proposition available in study of Hull (2011) and Pascucci (2011).

Options on simplest random binomial tree models: Below we will consider simplest random binomial tree model where the randomness of environment is defined by tossing a coin (non necessarily fair) N times. Let Ω_N be a sample space with outcome $\omega = (\omega_1, \omega_2, \dots, \omega_N)$, where $\omega_i \in \{H, T\}$, $i = 1, 2, \dots, N$, observing “Heads” or “Tails”. Note that below T is a period of time and T is Tail. Let $\Pr \{H\} = \alpha$ and $\Pr \{T\} = 1-\alpha$, where $0 \leq \alpha \leq 1$. For any outcome $\omega \in \Omega_N$ we compute price of an option, i.e., construct a random variable and option price for random binomial tree model we define as expectation of corresponding random variable. In this study we consider European call option for simplest random binomial tree model. Other options are considered similarly.

Recall that the objective of the analysis is to calculate the option price at the initial node of the tree.

Options on single-period binomial tree: Firstly, we consider European call option for simplest random single-period binomial tree model. In this case $N = 1$, i.e., $\Delta t = T$ and $\Omega_1 = \{H, T\}$ and random environments (u_1, d_1) and (u_2, d_2) correspond to outcomes “Head” and “Tail” respectively. Then we can compute option prices $f(H)$ and $f(T)$ assuming that condition (4) holds using well known formula (Cox *et al.*, 1979; Pascucci, 2011):

$$f = e^{-r\Delta t} [p f_u + (1-p) f_d] \tag{5}$$

Where:

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{6}$$

Then option price f for simplest random single-period binomial tree model we define as expectation of random variable $f: \Omega_1 \rightarrow R$, i.e.,

$$f = \alpha f(H) + (1-\alpha) f(T) \tag{7}$$

Example 1: Let us consider simplest random binomial tree model with two environment $u_1 = 1.1$, $d_1 = 0.9$ and $u_2 = 1.4$, $d_2 = 0.7$. If a stock price is currently \$20 and the risk-free interest rate is 12% per annum with continuous compounding, find the value of 3-months of European call option with a strike price of \$21 for simplest random single-period binomial tree model.

We consider the probability of turning out of the first environment as $\alpha = 0.9$ and of the second $1-\alpha = 0.1$. Such as the second environment has high u and low d , this kind of situation indicates to a crisis or too good economical state which happens seldom. So the price of an option calculated by using two environments, considering casual and extraordinary situations of market, is a more realistic one.

Using formulas from Eq. 5-7 one can compute that:

$$f(H) = 0.633; f(T) = 3.206$$

and

$$f = \alpha f(H) + (1-\alpha) f(T) = 0.890$$

Options on two-period binomial tree: In this case $N = 2$, i.e., $2 \times \Delta t = T$ and $\Omega_2 = \{(H, H); (H, T); (T, H); (T, T)\}$ and random environments (u_1, d_1) and (u_2, d_2) correspond to outcomes “Head” and “Tail” respectively.

For outcomes (H, H) and (T, T) the value if corresponding options one can compute directly using well known formula for the case of two time steps of usual binomial tree model:

$$f = e^{-2r\Delta t} [p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd}] \tag{8}$$

where, f_{uu} the value of the stock after two up movements, f_{ud} after one up and one down and f_{dd} after two down movements. For outcome (H, T) repeated application of equation (5) gives:

$$f_{u1} = e^{-r\Delta t} [p_2 f_{u1u2} + (1-p_2) f_{u1d2}] \tag{9}$$

$$f_{d1} = e^{-r\Delta t} [p_2 f_{d1u2} + (1-p_2) f_{d1d2}] \tag{10}$$

Substituting from Eq. 9 and 10 into 5, we get:

$$f(H, T) = e^{-2r\Delta t} [p_1 p_2 f_{u1u2} + p_1 (1-p_2) f_{u1d2}] + (1-p_1) [p_2 f_{d1u2} + (1-p_1) (1-p_2) f_{d1d2}] \tag{11}$$

$$f = \alpha^2 f(H, H) + 2\alpha(1-\alpha) f(H, T) + (1-\alpha)^2 f(T, T)$$

and it is evident that $f(H, T) = f(T, H)$.

Then the option price f for simplest random two-step binomial tree model is defined as expectation of random variable $f: \Omega_2 \rightarrow R$, i.e.,

$$f = \alpha^2 f(H, H) + 2\alpha(1-\alpha) f(H, T) + (1-\alpha)^2 f(T, T)$$

Example 2: Now let us look at the example 1 in a two steps model. Using formulas (8) and (11) one can compute that $f(H, H) = 1.282$; $f(H, T) = f(T, H) = 3.490$; $f(T, T) = 3.819$ and the value of our option will be $f = \alpha^2 f(H, H) + 2\alpha(1-\alpha) f(H, T) + (1-\alpha)^2 f(T, T) = 1.705$.

Options on n-period binomial tree: Suppose that a tree with N -time steps is used to value a European call option with strike price K and life T . Each step is of length T/N , i.e., $N \times \Delta t = T$. For usual binomial tree model, if there have been j upward movements and $N-j$ downward movements

on the tree, the final stock price is $S_0 u^j d^{N-j}$, where u is the proportional up movement, d is the proportional down movement and S_0 is the initial stock price. Then option price f is computed as follows:

$$e^{-rT} \sum_{j=0}^N \binom{N}{j} p^j (1-p)^{N-j} \max(S_0 u^j d^{N-j} - K, 0) \quad (12)$$

Now consider simplest random binomial tree model where the randomness of environment is defined by tossing a coin (non necessarily fair) N times. Let Ω_N be a sample space with outcome $\omega = (\omega_1, \omega_2, \dots, \omega_N)$, where $\omega_i \in \{H, T\}$, $i = 1, 2, \dots, N$, observing "Heads" or "Tails". Let $\Pr\{H\} = \alpha$ and $\Pr\{T\} = 1-\alpha$, where $0 \leq \alpha \leq 1$. For any outcome $\omega \in \Omega_N$ let $M(\omega)$ be the number of tails in tossing of coin N times. If $M(\omega) = 0$ or $M(\omega) = N$, then one can compute option price for European call using previous formula. Now we show how to compute option price for $0 < M(\omega) < N$. Let $M(\omega) = m$. Then using trivial identity $(a+b)^N = (a+b)^{N-1} (a+b)$, we can find option price $f_m(\omega)$ as following:

$$e^{-rT} \left(\sum_{j=0}^{N-1} \sum_{k=0}^j \frac{(N-1)!}{(N-1-j)! j!} p_1^j (1-p_1)^{N-1-j} p_2^k (1-p_2)^{j-k} \right) \max(S_0 u_1^j d_1^{N-1-j} u_2^k d_2^{j-k} - K, 0)$$

Similarly for $M(\omega) = m$, where $1 \leq m \leq N-1$, we can find option price $f_m(\omega)$ as following:

$$e^{-rT} \left(\sum_{j=0}^{N-1} \sum_{k=0}^j \frac{(N-m)!}{(N-m-j)! j!} p_1^j (1-p_1)^{N-m-j} p_2^k (1-p_2)^{m-k} \right) \max(S_0 u_1^j d_1^{N-m-j} u_2^k d_2^{m-k} - K, 0) \quad (13)$$

where, p_1 and p_2 one can compute using formula (6) for corresponding environments (u_1, d_1) and (u_2, d_2) , respectively.

Let $A_m = \{\omega \in \Omega_N; M(\omega) = m\}$. It is well known that:

$$\Pr(A_m) = \binom{N}{m} \alpha^{N-m} (1-\alpha)^m$$

Now option price f for simplest random N -period binomial tree model we define as expectation of random variable $f: \Omega_N \rightarrow \mathbb{R}$, i.e.,:

$$f = \binom{N}{m} \alpha^{N-m} (1-\alpha)^m f_m \quad (14)$$

where, $m = 0, 1, \dots, N$.

CONCLUSIONS

One way of deriving the famous Black-Scholes-Merton result for valuing a European option on a non-dividend-paying stock is by allowing the number of time steps in a binomial tree to approach infinity. This study has introduced random binomial tree model and derived formula for computing European call options for such models. Similarly one can define and study other options. To derive formula for the valuation of a European call option for random binomial tree models similar to the Black-Scholes-Merton formula for usual binomial tree models, we have to investigate:

$$f = \binom{N}{m} \alpha^{N-m} (1-\alpha)^m f_m$$

by allowing the number of time steps to approach infinity. Since in the case of random binomial tree models we have "doubled" randomness. Also it is an interesting problem to derive the Black-Scholes-Merton differential equation with random coefficients. All these problems will be subjected in coming papers.

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