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Some Common Fixed Point Theorems for Weakly $C_{f,g}$-contractive Mappings in Complete Metric Spaces

[1]V. Parvaneh and [2]H. Hosseinzadeh
[1]Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
[2]Department of Mathematics, Ardebil Branch, Islamic Azad University, Ardebil, Iran

Abstract: The aim of this paper is to present some common fixed point theorems for $C_{f,g}$-contractions in a complete metric space. Finally, some results for contractions of integral type are given.

Key words: Common fixed point, complete metric space, weak C-contraction

INTRODUCTION

The concept of C-contraction was defined by Chatterjea (1972) as follows.

Definition 1: A mapping $T:X\to X$ where (X, d) is a metric space is said to be a C-contraction if there exists $\alpha\in(0,1/2)$ such that for all $x, y\in X$ the following inequality holds:

$$d(Tx,Ty)\le\alpha(d(x,Ty)+d(y,Tx)).$$

Chatterjea (1972) has proved that, if (X, d) is a complete metric space, then every C-contraction on X has a unique fixed point. Choudhury (2009) introduced a generalization of C-contraction by the following definition.

Definition 2: A mapping $T:X\to X$, where (X, d) is a metric space is said to be a weakly C-contractive mapping if for all $x, y\in X$:

$$d(Tx,Ty)\le\frac{1}{2}(d(x,Ty)+d(y,Tx))-\phi((d(x,Ty),d(y,Tx)))$$

where, $\varphi:[0,\infty)^2\to[0,\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Choudhury (2009) has proved that, if (X, d) is a complete metric space, then every weak C-contraction on X has a unique fixed point.

For a survey of fixed point theory and related results we refer to Mujahid and Dragan (2010), Zhang and Song (2009), Moradi *et al.* (2011), Doric (2009), Nashine and Samet (2011), Mohamadi *et al.* (2009), Okoroafor and Osu (2006), Olaleru (2006) and Tiwari *et al.* (2012).

Let us note that the beautiful theory of fixed point is used frequently in other branches of mathematics and engineering sciences (Shakeri *et al.*, 2009).

The purpose of this study is to obtain a common fixed point theorem for four maps satisfying a certain contractive condition. Our result generalized the results of Chatterjea (1972) and Choudhury (2009).

Throughout this paper, let:
$\Omega=\{\varphi|\varphi:[0,\infty)^2\to\{0,\infty)$ is a continuous function such that $\varphi(x,y)=0$ iff $x=y=0\}$.

Definition 3: (a) Let (X, d) be a metric space and $T,S:X\to X$. If $w = Tx = Sx$, for some $x\in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S, (b) Let T and S be two self-mappings of a metric space (X, d). T and S are said to be weakly compatible if for all $x\in X$ the equality $Tx = Sx$ implies $TSx = STx$ (Beg and Abbas, 2006).

MAIN RESULTS

Definition 1: Two mappings $T,S:X\to X$, where (X,d) is a metric space are called weakly $C_{f,g}$-contractive (or weak $C_{f,g}$-contraction) if for all $x, y\in X$,

$$d(Tx,Sy)\le\frac{1}{2}(d(fx,Sy)+d(gy,Tx))-\phi(d(fx,Sy),d(gy,Tx)) \tag{1}$$

where, $\phi\in\Omega$.

Following is the main result of this study.

Theorem 1: Let (X, d) be a complete metric space and let E be a nonempty closed subset of X. Let $T,S:X\to X$ be two weakly $C_{f,g}$-contractive mappings (condition 1):

I $TE\subseteq gE$ and $SE\subseteq fE$.
II The pairs (S, f) and (T, g) be weakly compatible.

Corresponding Author: V. Parvaneh, Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran

Assume that f and g also are continuous functions on X. In addition, for all $x \in X$:

$$d(fTx, Tfx) \le d(fx, Tx) \text{ and } d(gSx, Sgx) \le d(gx, Sx) \quad (2)$$

and for all x, $y \in X$

$$d(fgx, gfy) \le d(gx, fy) \quad (3)$$

then, T, f, S and g have a unique common fixed point.

Proof: Let $x_0 \in E$ be arbitrary. Using (I), there exist tow sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = Tx_0 = gx_1$, $y_1 = Sx_1 = fx_2$, $y_2 = Tx_2 = gx_3, \ldots, y_{2n} = Tx_{2n} = gx_{1n+1}$, $y_{2n+1} = Sx_{2n+1} = fx_{2n+2}, \ldots$.

We complete the proof in two steps:

Step 1: $\{y_n\}$ is Cauchy.

Consider two cases as follows:

- If for some n, $y_n = y_{n+1}$, then $y_{n+1} = y_{n+2}$. If not, then $y_{n+1} \ne y_{n+2}$. Let n = 2k Therefore, using condition (1), we have:

$$\begin{aligned}
&d(y_{2k+2}, y_{2k+1}) = d(Tx_{2k+2}, Sx_{2k+1}) \\
&\le \frac{1}{2}(d(fx_{2k+2}, Sx_{2k+1}) + d(gx_{2k+1}, Tx_{2k+2})) - \phi(d(fx_{2k+2}, Sx_{2k+1}), d(gx_{2k+1}, Tx_{2k+2})) \\
&= \frac{1}{2}(d(y_{2k+1}, y_{2k+1}) + d(y_{2k}, y_{2k+2})) - \phi(d(y_{2k+1}, y_{2k+1}), d(y_{2k}, y_{2k+2})) \\
&\le \frac{1}{2}d(y_{2k}, y_{2k+2}) - \phi(0, d(y_{2k}, y_{2k+2})) = \frac{1}{2}d(y_{2k+1}, y_{2k+2}) - \phi(0, d(y_{2k+1}, y_{2k+2})) \\
&< \frac{1}{2}d(y_{2k+1}, y_{2k+2})
\end{aligned}$$

which is a contradiction. Hence, we must have $y_{n+1} = y_{n+2}$, when, n is even. In a same way we can show that this equality holds, when n is odd. Therefore, in any case, if for an n, $y_n = y_{n+1}$, we always obtain $y_n = y_{n+2}$. Repeating the above process inductively, we obtain that $y_n = y_{n+k}$ for all $k \ge 1$ Therefore, in this case $\{y_n\}$ is a constant sequence and hence is a Cauchy one.

- If $y_n \ne y_{n+1}$, for every positive integer n, then for n = 2k, using condition (1), we obtain that:

$$\begin{aligned}
&d(y_{2k+1}, y_{2k}) = d(Tx_{2k}, Sx_{2k+1}) \\
&\le \frac{1}{2}(d(fx_{2k}, Sx_{2k+1}) + d(gx_{2k+1}, Tx_{2k})) - \phi(d(fx_{2k}, Sx_{2k+1}), d(gx_{2k+1}, Tx_{2k})) \\
&= \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) - \phi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})) \\
&\le \frac{1}{2}d(y_{2k-1}, y_{2k+1}) \\
&\le \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1}))
\end{aligned}$$

Hence,

$$d(y_{2k+1}, y_{2k}) \le d(y_{2k}, y_{2k-1})$$

If, n = 2k+1, similarly we can prove that:

$$d(y_{2k+2}, y_{2k+1}) \le \frac{1}{2}d(y_{2k}, y_{2k+2}) \le \frac{1}{2}(d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2})).$$

That is:

$$d(y_{2k+2}, y_{2k+1}) \le d(y_{2k+1}, y_{2k}).$$

Therefor, in general, $d(y_{n+1}, y_n)$ is a decreasing sequence of nonnegative real numbers and bounded from below and hence it is convergent.

Assume that:

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = r.$$

From the above argument,

$$d(y_{2k+1}, y_{2k}) \le \frac{1}{2}d(y_{2k-1}, y_{2k+1}) \le \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1})),$$

and if $k \to \infty$, we have:

$$r \le \lim_{k\to\infty} \frac{1}{2}d(y_{2k-1}, y_{2k+1}) \le r.$$

Therefore:

$$\lim_{k\to\infty} d(y_{2k-1}, y_{2k+1}) = 2r.$$

We have proved that:

$$\begin{aligned}
d(y_{2k+1}, y_{2k}) &\le \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) \\
&- \phi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k}))
\end{aligned}$$

Now, if $k \to \infty$ and using the continuity of φ we obtain

$$r \le \frac{1}{2}2r - \phi(2r, 0)$$

and consequently, $\varphi(2r, 0) = 0$. This gives us that,

$$r = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \quad (4)$$

by our assumption about φ.

Now, it is sufficient to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose opposite, that is $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $y_{2m(k)}$ and $y_{2n(k)}$ of y_{2n} such that n(k) is smallest index for which n(k)>m(k) and:

$$d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon. \tag{5}$$

This means that:

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon. \tag{6}$$

From (5) and triangle inequality:

$$\begin{aligned}\varepsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-2}) \\ &+ d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &< \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).\end{aligned}$$

Letting k→∞ and using Eq. 4 we can conclude that:

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon. \tag{7}$$

Moreover, we have:

$$d(y_{2n(k)-1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)}) | \leq d(y_{2n(k)-1}, y_{2n(k)}) \tag{8}$$

and

$$| d(y_{2m(k)-1}, y_{2n(k)-1}) - d(y_{2m(k)}, y_{2n(k)-1}) | \leq d(y_{2m(k)-1}, y_{2m(k)}) \tag{9}$$

and

$$| d(y_{2m(k)-2}, y_{2n(k)}) - d(y_{2m(k)-1}, y_{2n(k)}) | \leq d(y_{2m(k)-2}, y_{2m(k)-1}). \tag{10}$$

Using Eq. 6, 7, 8, 9 and 10, we get:

$$\begin{aligned}\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) &= \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) \\ &= \lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) = \varepsilon.\end{aligned}$$

Using Eq. 1 we have:

$$\begin{aligned}d(y_{2m(k)-1}, y_{2n(k)}) &= d(Tx_{2n(k)}, Sx_{2m(k)-1}) \\ &\leq \frac{1}{2}(d(fx_{2n(k)}, Sx_{2m(k)-1}) + d(gx_{2m(k)-1}, Tx_{2n(k)})) \\ &- \phi(d(fx_{2n(k)}, Sx_{2m(k)-1}), d(gx_{2m(k)-1}, Tx_{2n(k)})) \\ &= \frac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)})) \\ &- \phi(d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2m(k)-2}, y_{2n(k)})).\end{aligned}$$

Making k→∞ in the above inequality and taking into account (10) and by the continuity of φ, we have:

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \phi(\varepsilon, \varepsilon)$$

and hence, φ(ε, ε) = 0. By our assumption about φ, we have ε = 0 which is a contradiction.

Step 2: Existence of coincidence point and common fixed point.

Since, (X, d) is complete and $\{y_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Since, E is closed and $\{y_n\} \subseteq E$, we have $z \in E$.

Also, we know that

$$\begin{aligned}z &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = \lim_{n\to\infty} Sx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n}.\end{aligned}$$

Since, f and g are continuous,

$$fy_n \to fz, gy_n \to gz \tag{11}$$

On the other hand, from 2 and 3 we conclude that:

$$\begin{aligned}d(Ty_{2n+1}, gz) &\leq d(Ty_{2n+1}, fy_{2n+2}) + d(gy_{2n+1}, fy_{2n+2}) + d(gy_{2n+1}, gz) \\ &= d(Tfx_{2n+2}, fTx_{2n+2}) + d(fgx_{2n+3}, gfx_{2n+2}) + d(gy_{2n+1}, gz) \\ &\leq d(Tx_{2n+2}, fx_{2n+2}) + d(gx_{2n+3}, fx_{2n+2}) + d(gy_{2n+1}, gz) \\ &= d(y_{2n+2}, y_{2n+1}) + d(y_{2n+2}, y_{2n+1}) + d(gy_{2n+1}, gz).\end{aligned}$$

Therefore, from 4 and 11:

$$\lim_{n\to\infty} d(Ty_{2n+1}, gz) = 0. \tag{12}$$

Also, using 2 we have,

$$\begin{aligned}d(Ty_{2n+1}, fz) &\leq d(Ty_{2n+1}, fy_{2n+2}) + d(fy_{2n+2}, fz) \\ &= d(Tfx_{2n+2}, fTx_{2n+2}) + d(fy_{2n+2}, fz) \\ &\leq d(Tx_{2n+2}, fx_{2n+2}) + d(fy_{2n+2}, fz) \\ &= d(y_{2n+2}, y_{2n+1}) + d(fy_{2n+2}, fz).\end{aligned}$$

Therefore, from 4 and 11:

$$\lim_{n\to\infty} d(Ty_{2n+1}, fz) = 0. \tag{13}$$

From Eq. 1:

$$\begin{aligned}d(Ty_{2n+1}, Sz) &\leq \frac{1}{2}(d(fy_{2n+1}, Sz) + d(gz, Ty_{2n+1})) \\ &- \phi(d(fy_{2n+1}, Sz), d(gz, Ty_{2n+1}))\end{aligned}$$

If in the above inequality, n→∞, from 11 and 13 we have:

$$d(fz, Sz) \leq \frac{1}{2}(d(fz, Sz) + 0) - \phi(d(fz, Sz), 0).$$

So:

$$\frac{1}{2}(d(fz, Sz)) \leq -\phi((d(fz, Sz), 0))$$

and hence, Sz = fz. We can analogously prove that Tz = gz.

Also:

$$\lim_{n\to\infty} d(Ty_{2n+1}, gz) = \lim_{n\to\infty} d(Ty_{2n+1}, fz) = 0$$

consequently: fz = gz, therefore Tz = gz = fz = sz = t

Now we show that z is a common fixed point.

Using weak compatibility of the pair (T, f) and (S,g) we have Tt = ft and gt = St. So,

$$d(Tt,t) = d(Tt,Sz) \le \frac{1}{2}(d(ft,Sz) + d(gz,Tt)) - \phi(d(ft,Sz),d(gz,Tt))$$
$$= \frac{1}{2}(d(Tt,t) + d(t,Tt)) - \phi(d(Tt,t),d(t,Tt)).$$

That is, φ(d(T, St), d(t, Tt)) =0 and this implies that Tt = t. Therefore ft = Tt = t.

Analogously,

$$d(t,St) = d(Tz,St) \le \frac{1}{2}(d(fz,St) + d(gt,Tz)) - \phi(d(fz,St),d(gt,Tz))$$
$$= \frac{1}{2}(d(t,St) + d(St,t)) - \phi(d(t,St) + d(St,t))$$

That is, φ(d(t, St), d(St, t)) =0 and this implies that St = t. Therefore, gt = St = t.

Hence, gt = St = t = ft = Tt.

It is easy to show that t is unique.

Example 1: Let X = R(The set of all real numbers) be endowed with the Euclidean metric. Suppose that T:X→X is defined by:

$$T(x) = \begin{cases} -x/16, & -\infty \le x \le 0, \\ 0, & 0 \le x \le \infty, \end{cases}$$

and Sx = 0 for all x∈R.

We define functions f, g: X→X by:

$$f(x) = \begin{cases} x/2, & -\infty \le x \le 0, \\ 0, & 0 \le x \le \infty, \end{cases}$$

and:

$$g(x) = \begin{cases} 0, & -\infty \le x \le 0, \\ x/2, & 0 \le x \le \infty, \end{cases}$$

and function φ:[0,∞)^2→[0,∞)by φ(t,s) = t+s/8.

One can easily obtains that for all x∈X,

$$d(fTx,Tfx) \le d(fx,Tx) \text{ and } d(gSx,Sgx) \le d(gx,Sx),$$

and for all x,y∈X,

$$d(fgx,gfy) \le d(gx,fy).$$

Now, we have the following four cases:

- x,y∈X(-∞, 0). Then we have

$$d(Tx,Sy) = \frac{1}{16}|x|$$
$$\le \frac{\frac{1}{2}|x-0| + |0 + x/16|}{8}$$
$$= \frac{|\frac{1}{2}x - 0| + |0 + x/16|}{4} - \frac{|\frac{1}{2}x - 0| + |0 + x/16|}{8}$$
$$\le \frac{|\frac{1}{2}x - 0| + |0 + x/16|}{2} - \frac{|\frac{1}{2}x - 0| + |0 + x/16|}{8}$$
$$= \frac{1}{2}(d(fx,Sy) + d(gy,Tx)) - \phi(d(fx,Sy),d(gy,Tx)).$$

- x∈(-∞, 0) and y∈[0, ∞). Then we have

$$d(Tx,Sy) = \frac{1}{16}|x|$$
$$\le \frac{\frac{1}{2}|x-0| + |y/2 + x/16|}{8}$$
$$= \frac{|\frac{1}{2}x - 0| + |y/2 + x/16|}{4} - \frac{|\frac{1}{2}x - 0| + |y/2 + x/16|}{8}$$
$$\le \frac{|\frac{1}{2}x - 0| + |y/2 + x/16|}{2} - \frac{|\frac{1}{2}x - 0| + |y/2 + x/16|}{8}$$
$$= \frac{1}{2}(d(fx,Sy) + d(gy,Tx)) - \phi(d(fx,Sy),d(gy,Tx)).$$

- x,y∈[0, ∞) . Then we have

$$d(Tx,Sy) = 0$$
$$\le \frac{|0-0| + |y/2 - 0|}{8}$$
$$= \frac{|0-0| + |y/2 - 0|}{4} - \frac{|0-0| + |y/2 - 0|}{8}$$
$$\le \frac{|0-0| + |y/2 - 0|}{2} - \frac{|0-0| + |y/2 - 0|}{8}$$
$$= \frac{1}{2}(d(fx,Sy) + d(gy,Tx)) - \phi(d(fx,Sy),d(gy,Tx)).$$

- x∈[0, ∞) and y∈(-∞, 0). Then we have:

$$d(Tx,Sy) = 0$$
$$\le \frac{|0-0| + |y/2 - 0|}{8}$$
$$= \frac{|0-0| + |y/2 - 0|}{4} - \frac{|0-0| + |y/2 - 0|}{8}$$
$$\le \frac{|0-0| + |y/2 - 0|}{2} - \frac{|0-0| + |y/2 - 0|}{8}$$
$$= \frac{1}{2}(d(fx,Sy) + d(gy,Tx)) - \phi(d(fx,Sy),d(gy,Tx)).$$

So mappings T and S satisfy relation (1) and all conditions of Theorem 1 are hold and T, S, f and g have a unique common fixed point (x = 0).

Taking f = g in Theorem 1, we obtain the following.

Corollary 1: Let (X, d) be a complete metric space and let E be a nonempty closed subset of X. Let T, S are such that for all x,y∈X:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(fy,Tx)) - \phi(d(fx,Sy),d(fy,Tx)) \quad (14)$$

where, T, S and f be such that:

- TE⊆fE and SE⊆fE.
- The pairs (T, f) and (T, g) be weakly compatible.

Assume that f is a continuous function on X. In addition, for all x∈X:

$$d(fTx,Tfx) \le d(fx,Tx) \text{ and } d(fSx,Sfx) \le d(fx,Sx) \quad (15)$$

and for all x,y∈X:

$$d(f^2x,f^2y) \le d(fx,fy) \quad (16)$$

then, T, f and S have a unique common fixed point.

Taking T = S in Theorem 1, we have the following result.

Corollary 2: Let (X, d) be a complete metric space and let E be a nonempty closed subset of X. Let T:X→X be such that for all x,y∈X:

$$d(Tx,Ty) \le \frac{1}{2}(d(fx,Ty)+d(gy,Tx)) - \phi(d(fx,Ty),d(gy,Tx)) \quad (17)$$

where, T, f and g be such that:

- TE⊆fE and SE⊆gE.
- The pairs (T,f) and (T,g) be weakly compatible.

Assume that f and g also are continuous functions on X. In addition, for all x∈X:

$$d(fTx,Tfx) \le d(fx,Tx) \text{ and } d(gTx,Tgx) \le d(gx,Tx), \quad (18)$$

and for all x,y∈X:

$$d(fgx,gfy) \le d(gx,fy) \quad (19)$$

then, T, f and g have a unique common fixed point.

Taking T = S and f = g in Theorem 2.4, the following result is obtained.

Corollary 3: Let (X,d) be a complete metric space and let E be a nonempty closed subset of X. Let T:X→X be such that for all x,y∈X:

$$d(Tx,Ty) \le \frac{1}{2}(d(fx,Ty)+d(fy,Tx)) - \phi(d(fx,Ty),d(fy,Tx)) \quad (20)$$

where, T and f be such that:

- TE⊆ fE.
- The pair (T, f) be weakly compatible

Assume that T also is continuous on X. In addition, for all x∈X:

$$d(fTx,Tfx) \le d(fx,Tx) \quad (21)$$

and for all x,y∈X,

$$d(f^2x,f^2y) \le d(fx,fy). \quad (22)$$

Then, T and f have a unique common fixed point.

Remark 1: Taking T = S and f = g = I_x (the identity mapping on X) and X = E in Theorem 1, we obtain the result of Choudhury (2009) which has been mentioned above.

APPLICATIONS

In this part, from previous obtained results, we will deduce some common fixed point results for mappings satisfying a contraction condition of integral type in a complete metric space.

Branciari (2002) obtained a fixed point result for a single mapping satisfying an integral type inequality. Afterwards, Altun *et al*. (2007) established a fixed point theorem for weakly compatible mappings satisfying a general contractive inequality of integral type.

Similar to Nashine and Samet (2011), we denote by ϒ the set of all functions φ:[0,+∞)→[0,+∞) satisfying the following conditions:

- φ is a Lebesgue integrable mapping on each compact subset of [0,+∞)
- For all ε>0, we have:

$$\int_0^{\varepsilon} \varphi(t)dt > 0$$

Corollary 4: Let T and S satisfy the conditions of Theorem 1, except that condition (1) be replaced by the following:

There exists a $\varnothing\in\Upsilon$ such that:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \int_0^{\phi(d(fx,Sy),d(gy,Tx))} \varphi(t)dt \quad (23)$$

Then, T, S, f and g have a unique common fixed point.

Proof: Consider the function $\Phi(x)=\int_0^x \varphi(t)dt$. Then Eq. 23 changes to the following:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \Phi(\phi(d(fx,Sy),d(gy,Tx)))$$

and putting $\Psi = \Phi\phi$ and applying Theorem 1, we obtain the proof (it is easy to verify that $\Psi\in\Omega$).

Corollary 5: If in the above corollary, Eq. 23 be replaced by the following:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \phi(\int_0^{d(fx,Sy)} \varphi(t)dt, \int_0^{d(gy,Tx)} \varphi(t)dt)$$

then the result of corollary 4 is also hold.

Proof: Assume that:

$$\Phi(x)=\int_0^x \varphi(t)dt$$

Then the above condition will be the following:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \phi(\Phi(d(fx,Sy)),\Phi(d(gy,Tx)))$$

Taking,

$$\Psi(x,y)=\phi(\Phi(x),\Phi(y))$$

and applying Theorem 1, we obtain the proof (it is obvious that $\Psi\in\Omega$).

As in Nashine and Samet (2011), let $N\in N$ be fixed. Let $\{\varphi_i\}_{1\le i\le N}$ be a family of N functions which belong to Υ. For all $t\ge0$, we define:

$$\begin{aligned}
I_1(t) &= \int_0^t \varphi_1(s)ds,\\
I_2(t) &= \int_0^{I_1(t)} \varphi_2(s)ds = \int_0^{\int_0^t \varphi_1(s)ds} \varphi_2(s)ds,\\
I_3(t) &= \int_0^{I_2(t)} \varphi_3(s)ds = \int_0^{\int_0^{\int_0^t \varphi_1(s)ds} \varphi_2(s)ds} \varphi_3(s)ds,\\
&\ldots,\\
I_N(t) &= \int_0^{I_{(N-1)}(t)} \varphi_N(s)ds
\end{aligned}$$

We have the following result.

Corollary 6: Let T and S satisfy the conditions of Theorem 1 and condition (1) be substituted by the following:

There exists a $\phi\in\Upsilon$ such that:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - I_N(\phi(d(fx,Sy),d(gy,Tx))) \quad (24)$$

Then, T, S, f and g have a unique common fixed point.

Proof: Consider the function $\Psi(x,y) = I_N\phi(x,y)$. Then the inequality 24 will be:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \Psi(d(fx,Sy),d(gy,Tx))$$

applying Theorem 1, we obtain the desired result (it is easy to verify that $\Psi\in\Omega$).

Corollary 7: Let T and S satisfy the conditions of Theorem 1, except that condition (1) be replaced by the following:

There exists a $\phi\in\Upsilon$ such that:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \phi(I_N(d(fx,Sy)),I_N(d(gy,Tx)))$$

Then, T, S, f and g have a unique common fixed point.

Proof: Let $\Psi(x,y) = \phi(I_N(x),I_N(y))$. Then the above inequality will be changed to:

$$d(Tx,Sy) \le \frac{1}{2}(d(fx,Sy)+d(gy,Tx)) - \Psi(d(fx,Sy),d(gy,Tx))$$

Using Theorem 1, we obtain the proof (it is easy to show that $\Psi\in\Omega$).

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