Best Proximity Pairs in Fuzzy Normed Spaces

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Abstract: This study has considered the problem of finding a best approximation pair, i.e., two points which achieve the minimum distance between two nonempty sets in fuzzy normed spaces. We defined F-proximinal set and F-approximately compact relative to any set and study the existence and uniqueness of best proximity points.

Key words: Fuzzy normed spaces, F-best approximation pair, F-proximinal pair, F-quasi-Chebyshev, F-approximately compact, F-boundedly compact

INTRODUCTION

The best proximity pair evolves as a generalization of the concept of best approximation. Recently, Bauschke et al. (2004), Kim (2006), Cuenya and Bonifacio (2008) and Mohsenalhosseini et al. (2011) obtained some results on characterization and finding the best proximity points in linear normed spaces. Shams et al. (2009) studied the best approximation pairs in probabilistic normed spaces. This paper attempted to investigate the concept of best approximation pairs in fuzzy normed spaces and get some results on existence and compactness of the best proximity sets.

PRELIMINARIES

Definition 2.1: A binary operation \([0, 1] \times [0, 1] \rightarrow [0, 1]\) is said to be a continuous \(t\)-norm if \((0, 1, *)\) is a total monotone with unit 1 such that \(a \ast b \geq c \ast d\) whenever \(a \leq c, b \leq d\) and \(a, b, c, d \in [0, 1]\).

If \(a \in \mathbb{R}^+\), then we define:

\[
t_a(x) =
\begin{cases}
0 & \text{if } x \leq a \\
1 & \text{if } x > a
\end{cases}
\]

Definition 2.2: 3-tuple \((X, N, *)\) is said to be a fuzzy normed space if \(X\) is a vector space, \(*\) is a continuous \(t\)-norm and \(N\) is a fuzzy set on \(X \times (0, \infty)\) satisfying the following conditions for every \(x, y \in X\) and \(t, s > 0\):

- \(N(x, t) > 0\)
- \(N(x, 0) = 1 = N(0, t)\)
- \(N(\alpha x, t) = N(x, t/|\alpha|)\), for all \(\alpha \neq 0\)
- \(N(x, t+s) \geq N(x+y, t+s)\)
- \(N(x, t)\) is a nondecreasing function on \(\mathbb{R}\) and \(\lim_{t \to 0} N(x, t) = 1\)

In addition, if for \(t > 0\), \(x \sim N(x, t)\) is a continuous map on \(X\); then \((X, N, \sim)\) is called a strong fuzzy normed space.

Lemma 2.1: Let \(N\) be a fuzzy norm. Then:
- \(N(x, t)\) is nondecreasing with respect to \(t\) for each \(x \in X\)
- \(N(x+y, t) = N(y-x, t)\)

Definition 2.3: Let \((X, N, *)\) be a fuzzy normed space. The open ball \(B(x, r, t)\) and the closed ball \(B[0, r, t]\) with the center \(x \in X\) and radius \(0 < r < 1\), \(t > 0\) are defined as follows:

- \(B(x, r, t) = \{y \in X : N(x-y, t) > 1-r\}\)
- \(B[0, r, t] = \{y \in X : N(x-y, t) \geq 1-r\}\)

Lemma 2.2: If \((X, N, *)\) is a fuzzy normed space. Then:
- The function \((x, y) \rightarrow x+y\) is continuous
- The function \((x, y) \rightarrow x\) is continuous

Example 2.1: Let \((X, \|\|)\) be a normed space. If we define \(a \ast b = ab\) or \(a \ast b = \min\{a, b\}\) and:

\[
N(x, t) = \frac{kt}{kt + m|x|}, k, m, n \in \mathbb{R}^+
\]

Then \((X, N, \ast)\) is a fuzzy normed space. In particular if \(k = m = n = 1\) we have:

\[
N(x, t) = \frac{t}{t + |x|}
\]

which is called the standard fuzzy norm induced by the norm \(\|\|\).
Definition 2.4: A sequence \( \{x_n\} \) in a fuzzy normed space \((X, N, \ast)\) is called a convergence sequence to \(x \in X\), if for each \(t > 0\) and each \(0 < \varepsilon < 1\) there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\), we have:

\[
N(x_n - x, t) > 1 - \varepsilon
\]

Definition 2.5: Let \((X, N, \ast)\) be a fuzzy normed space. For each \(t > 0\), a subset \(A \subseteq X\) is called \(F\)-bounded if there exists \(0 < r < 1\) such that \(N(x, t) > 1 - r\) for all \(x \in A\).

MAIN RESULTS

Definition 3.1: Let \(G\) and \(H\) be two nonempty subsets of a fuzzy normed space \((X, N, \ast)\). For \(t > 0\), let:

\[
N(G - H, t) = \sup \{N(g - h, t) : (g, h) \in G \times H\}
\]

and:

\[
N(g - H, t) = \sup \{N(g - h, t) : h \in H\}
\]

An element \((g_0, h_0) \in G \times H\) is said to be a \(F\)-best approximation pair relative to \((G, H)\) if:

\[
N(g_0 - h_0, t) = N(G - H, t)
\]

We shall denote by \(P_{G \times H}\) the set of all elements of \(F\)-best approximation pair relative to \((G, H)\) i.e.:

\[
P_{G \times H} = \{(g, h) \in G \times H : N(g - h, t) = N(G - H, t))\}
\]

Also an element \(g_0 \in A\) is said to be a \(F\)-best approximation to \(G\) from \(H\) if:

\[
N(g_0 - H, t) = N(G - H, t)
\]

We denote by \(P_{G}^{H}\) the set of all elements of \(F\)-best approximation to \(G\) from \(H\) i.e.:

\[
P_{G}^{H} = \{g \in G : N(g - H, t) = N(G - H, t))\}
\]

Definition 3.2: For a fuzzy normed space \(X\) and nonempty subsets \(G\) and \(H\), a sequence \(\{x_n\}\) is said to converge in distance to \(H\) if \(\lim_{n \to \infty} N(x_n, H, t) = N(G - H, t)\).

Definition 3.3: Let \((X, N, \ast)\) be a fuzzy normed space. For each \(t > 0\) and for nonempty subsets \(G\) and \(H\) of \(X\), \(G \times H\) is called \(F\)-proximinal relative to \((G, H)\) if \(P_{G \times H}^{H}\) is non-void. \(G \times H\) is called \(F\)-quasi Chebyshev pair if \(P_{G \times H}^{H}\) is a compact set. Also \(G\) is called \(t\)-proximinal relative to \(H\) if \(P_{G}^{H}(t)\) is non-void for some \(H \subseteq X\) and \(G\) is called \(F\)-quasi Chebyshev relative to \(H\), if \(P_{G}^{H}(t)\) is a compact set for some \(H \subseteq X\).

Example 3.1: Let \(X = \mathbb{R}\). For \(a, b \in [0, 1]\), let \(a \ast b = ab\). Define \(N : \mathbb{R} \times (0, +\infty) \to [0, 1]\), by \(N(x, t) = t + |x|\). Then \((\mathbb{R}, N, \ast)\) is the standard fuzzy normed space. Let \(G = [0, 2]\) and \(H = [3, 4]\), then for each \(g \in G\) and \(h \in H\), it is easy to see that \(N(3 - 2, t) = N(g - h, t)\). So \(N(3 - 2, t) = N(G - H, t)\) and \(N(3 - H, t) = N(G - H, t)\). Hence, for each \(t > 0\), \((3, 2)\) is a \(F\)-best approximation pair relative to \((G, H)\) and \(3\) is a \(F\)-best approximation to \(G\) from \(H\).

Lemma 3.1: Let \(G\) and \(H\) be nonempty subsets of a strong fuzzy normed space \((X, N, \ast)\) and \(G\) is compact. Then \(G \cap H\) is nonempty if and only if \(N(G - H, t) = 1\) for all \(t > 0\).

Proof: Let for all \(t > 0\), \(N(G - H, t) = 1\). As \(X\) is first countable, there exists a sequence \(\{g_n\}\) in \(G\) such that \(N(g_n - H, t) = N(G - H, t)\). Hence, \(G\) is compact set, there exists a subsequence \(\{g_{n_k}\}\) and \(g_t \in G\) such that \(g_{n_k} \to g_t\) and then \(N(g_{n_k} - H, t) = N(g_t - H, t)\). Therefore:

\[
N(g_t - H, t) = N(G - H, t)
\]

for all \(t > 0\). Hence, \(N(g_t - H, t) = 1\) and so \(g_t \in G \cap H\). Conversely, suppose there exists a \(g_t \in G \cap H\), then for all \(t > 0\), \(N(g_t - H, t) = 1\) and so:

\[
N(G - H, t) = 1
\]

for all \(t > 0\).

Definition 3.4: Let \((X, N, \ast)\) be a fuzzy normed space and \(G\) and \(H\) be nonempty subsets of \(X\). We say that the subset \(G\) is \(F\)-approximately compact relative to \(H\) if every sequence \(g_n \in G\) with the property that, for all \(t > 0\), \(N(g_n - H, t) = N(G - H, t)\), has a subsequence convergent to an element of \(G\).

Theorem 3.1: Let \(G\) and \(H\) be nonempty subsets of a strong fuzzy normed space \((X, N, \ast)\) and \(G\) is \(F\)-approximately compact relative to \(H\), then \(G\) is a \(F\)-proximinal set relative to \(H\).

Proof: By definition, there exists \(\{g_n\} \subseteq G\) such that \(N(g_n - H, t) = N(G - H, t)\). Since, \(G\) is \(F\)-approximately compact relative to \(H\), so there exists a subsequence \(g_{n_k}\) and \(g_t \in G\) such that \(g_{n_k} \to g_t\). Since, \((X, N, \ast)\) is a strong fuzzy normed space, we have, \(N(g_{n_k} - H, t) = N(g_t - H, t)\). Hence, for all \(t > 0\), \(N(g_{n_k} - H, t) = N(G - H, t)\).
**Theorem 3.2:** Let $G$ be a $F$-approximately compact relative to $H$ then, $G$ is $F$-quasi Chebyshev relative to $H$.

**Proof:** Let $\{g_i\}$ be a sequence in $P'_G(H)$. It is obvious that there exists a subsequence $\{g_{i_k}\}$ and $g_i \in G$ such that $g_{i_k} \rightarrow g_i$ and this complete the proof.

**Theorem 3.3:** Let $G$ and $H$ be nonempty subsets of a Fuzzy normed space $(X, N, *)$. If $G$ is $F$-approximately compact and $H$ is compact, then $G \times H$ is $F$-approximately compact relative to $H$.

**Proof:** Let $g_i \in A$ be any sequence converging in distance to $G$ and let the sequence $h_i \in H$ for all $t > 0$ satisfies, $\lim N(g_i - h_i, t) = N(G-H, t)$. Since, $H$ is compact, $h_i \rightarrow h_i \in H$. Hence:

$$N(G-H, t) \geq N(g_i - h_i, t) \geq N(g_i - h_i - h_i, t) \geq N(G-H, t) \ast e_0$$

Then $g_i$ converges in distance to $g_e$ and, since $N(G-H, t) = N(G-h_i, t)$ and $G$ is approximately compact, $g_i \rightarrow g_e \in G$; that is, $g_i$ converges subsequentially to an element of $G$.

**Theorem 3.4:** Let $G$ and $H$ be nonempty subsets of a Fuzzy normed space $(X, N, *)$. If $G$ is $F$-approximately compact and $H$ is compact, then $G \times H$ is $F$-quasi Chebyshev set relative to $(G, H)$.

**Proof:** Let $(g_i, h_i) \in G \times H$ be any sequence in $P'_G(H)$. Then for every $t > 0$, $N(g_i - h_i, t) = N(G-H, t)$. Since, $H$ is compact, $h_i \rightarrow h_i \in H$. Hence:

$$N(G-H, t) \geq N(g_i - h_i, t) \geq N(g_i - h_i - h_i, t) \geq N(G-H, t) \ast e_0$$

Therefore, $\lim N(g_i - h_i, t) = N(G-h_i, t)$. Since, $G$ is $F$-approximately compact, $g_i \rightarrow g_e \in G$. Hence:

$$N(g_i - h_i, t) \rightarrow N(G-h_i, t)$$

**Lemma 3.2:** Let $G$ and $H$ be nonempty subsets of a fuzzy normed space $(X, N, *)$. If $G$ is $F$-approximately compact and $F$-bounded and $H$ is $F$-boundedly compact, then $G$ is $F$-approximately compact relative to $H$.

**Proof:** Let $\{g_i\}$ be a sequence converges in distance to $H$ and let $h_i \in H$ satisfies:

$$N(g_i - h_i, t) \rightarrow N(G-H, t)$$

As $\{g_i\}$ is $F$-bounded, so is $\{h_i\}$. Since, $H$ is $F$-boundedly compact, $h_i \rightarrow h_i \in H$. Hence:

$$N(g_i - h_i, t) \rightarrow N(G-H, t)$$

**Lemma 3.3:** Let $G$ and $H$ be nonempty subsets of a fuzzy normed space $(X, N, *)$. If $G$ is closed and $F$-boundedly compact and $H$ is $F$-bounded, then $G$ is $F$-approximately compact relative to $H$.

**Proof:** Suppose $\{g_i\}$ be a sequence such that $N(g_i - h_i, t) = N(G-H, t)$ and choose $h_i \in H$ such that:

$$N(g_i - h_i, t) \rightarrow N(G-H, t)$$

As $\{h_i\}$ is $F$-bounded, so is $\{h_i\}$; hence, $g_i \rightarrow g_e \in G$, which complete the proof.

**Theorem 3.5:** Let $G$ and $H$ be nonempty subsets of a fuzzy normed space $(X, N, *)$. If $G$ is $F$-proximinal and $H$ is compact, then $G \times H$ is $F$-proximinal pair relative to $(G, H)$.

**Proof:** Suppose $h_i \in H$ satisfies $\lim N(1_{G-h_i}, t) = (G-H, t)$. By compactness of $H$, $h_i \rightarrow h_i \in H$ so $N(G-h_i, t) = N(G-H, t)$. Since, $G$ is $F$-proximinal, there exists $g_e \in G$ such for all $t > 0$, $N(g_e - h_i, t) = N(G-h_i, t)$, so $N(g_e - h_i, t) = N(G-h_i, t)$.

**Theorem 3.4:** Let $G$ and $H$ be nonempty subsets of a fuzzy normed space $(X, N, *)$. If $G$ is $F$-proximinal and $F$-bounded and $H$ is closed and $F$-boundedly compact, then $G \times H$ is $F$-proximinal pair relative to $(G, H)$.

**Proof:** Suppose $h_i \in H$ satisfies $\lim N(1_{G-h_i}, t) = (G-H, t)$. Since, $G$ is $F$-bounded, $\{h_i\}$ must also be $F$-bounded so $h_i \rightarrow h_i \in H$.

**Lemma 3.5:** Let $G$ and $H$ be nonempty subsets of a fuzzy normed space $(X, N, *)$. If $G$ is approximately compact and $H$ is compact, then $K = \{g \in G : \exists h \in H, N(g-h, t) = N(G-h, t)\}$ is compact.

**Proof:** Let $g_i$ be a sequence in $K$ and for every $n \in N$ choose $\{h_i\}$ in $H$ such that an minimizes the distance from $G$ to $\{h_i\}$. Since $H$ is compact, $h_i \rightarrow h_i \in H$ Hence:

$$N(h_i - G, t) \geq N(g_i - h_i, t) \geq N(g_i - h_i - h_i, t) \geq N(G-h_i, t) \ast e_0$$

Then $\lim N(g_i - h_i, t) = N(h_i - G, t)$. Therefore, $\{g_i\}$ converges in distance to $h_i$. Since, $G$ is approximately compact, so it is converges subsequentially.
REFERENCES


