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Finite Element Method to Generalized Thermoelastic Problems with Temperature-dependent Properties

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Abstract: In the context of Lord-Shulman theory, the generalized thermoelastic problem with temperature-dependent properties is investigated. Due to the nonlinearity of the governing equations, finite element method is adopted to solve such problem. The corresponding nonlinear finite element equation is derived by means of virtual displacement principle. As a concrete example, a thin slim strip subjected to a thermal shock is investigated in detail. The nonlinear finite element equation for this problem is solved directly in time domain. The variations of the considered variables are illustrated graphically. The results show that solving the derived nonlinear finite element equation directly in time domain is an efficient and accurate method for such problem, the temperature-dependent properties act to reduce the magnitudes of the considered variables, and taking the temperature-dependence of material properties into account in the investigation of generalized thermoelastic problem is very necessary and practical for accurately predicting the thermoelastic behaviors.

Key words: Generalized thermoelasticity, thermal shock, finite element method, thermal relaxation, virtual displacement principle

INTRODUCTION

In the classical coupled thermoelastic theory proposed by Biot (1956), due to the diffusive heat conduction equation, it predicts an infinite speed for heat propagating in elastic medium, which is physically unrealistic. To eliminate such inherent paradox, the generalized thermoelastic theories have been introduced by Lord and Shulman (1967) and Green and Lindsay (1972) since 1960's. In L-S theory, a wave-type new heat conduction law was postulated to replace the classical Fourier's law. This new law is the same as that presented by Catteneo (1958) and Vernotte (1961). It contains the heat flux vector as well as its time derivative and also contains a new constant that acts as a relaxation time. The G-L theory modified both the energy equation and the Duhamel-Neumann relation by introducing two relaxation times, and also modified the heat conduction equation by introducing the temperature-rate term, which doesn't violate the Fourier's law of heat conduction when the body under consideration has a center of symmetry.

Based on these generalized theories, a large number of efforts have been devoted to investigating generalized dynamic problems. Among them, Sherief and Dhaliwal (1981) concerned a one-dimensional thermal shock problem by the Laplace transform technique and its inverse transform. Sherief and Anwar (1994) dealt with the thermoelastic problem of a homogeneous isotropic thick plate of infinite extent with heating on a part of the surface by means of state space approach together with Laplace and Fourier integral transforms and their inverse counterparts. Dhaliwal and Rokne (1989) solved a thermal shock problem of a half-space with its plane boundary either held rigidly fixed or stress-free and an approximate small-time solution was obtained by using the Laplace transform method. Chen and Weng 1988 proposed a hybrid Laplace transform-finite element method model for the coupled transient behavior of generalized thermoelastic problem. They have used this method to study the generalized thermoelastic response of a square cylinder with elliptical hole in (1988) and an axisymmetric circular cylinder in (1989).

To explore the effect of temperature-dependent properties on predicting the dynamic behaviors of the generalized thermoelatic problems, Ezzat *et al.* (2004) investigated problems in generalized thermoelasticity with the modulus of elasticity dependent with temperature. Aouadi (2006) studied the effect of temperature dependence of the modulus of elasticity on the behavior of solutions in generalized thermo-piezoelectricity. Unfortunately, in the process of calculation in their works, the properties were taken as linear functions of reference temperature instead of the current temperature.

In the present study, the generalized thermoelastic problem with temperature-dependent material properties is formulated in the context of L-S theory. The problem is solved by means of finite element method and the derived finite element equations are solved directly in time domain.

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BASIC EQUATIONS

In the absence of body force and inner heat source, the L-S type generalized thermoelastic governing equations are:

$$\begin{aligned} &\sigma_{ij,j} = \rho \ddot{u}_i, \quad \sigma_{ij} = c_{ijkl}\varepsilon_{kl} - a_{ij}\theta, \\ &\eta = a_{kl}\varepsilon_{kl} + \nu\theta, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \\ &q_{i,i} = -T_0\dot{\eta}, \quad q_i + \tau\dot{q}_i = -\kappa_{ij}\theta_{,j} \end{aligned} \tag{1}$$

In the above equations, a superimposed dot denotes the derivative with respect to time, a comma followed by a suffix denotes material derivative and the summation convention is used. σ_{ij} are the components of the stress tensor, ε_{ij} the components of the strain tensor, u_i the components of displacement vector, C_{ijkl} the elastic constants, a_{ij} the thermal moduli, k_{ij} the coefficients of thermal conductivity, η the entropy density, q_i the components of heat flux vector, τ the thermal relaxation time, θ = T-T_0, T the absolute temperature, T_0 the initial reference temperature, v = $\rho C_E T_0^{-1}$, ρ the mass density, CE the specific heat at constant strain, λ, μ are Lame's constants, e = ε_{ii} is the dilatation, γ = (3λ+2μ) and α_t is the coefficient of linear thermal expansion. It should be noted that when τ = 0 the L-S theory reduces to the classical coupled thermoelasticity.

Our goal is devoted to investigating the effect of temperature-dependent material properties on thermoelastic behaviors, therefore we assume:

$$\begin{aligned} &\lambda = \lambda_0 f_1(\theta), \mu = \mu_0 f_2(\theta), \kappa = \kappa_0 f_3(\theta), \\ &\gamma = \gamma_0 f_4(\theta) \end{aligned} \tag{2}$$

where, λ_0, μ_0, k_0 and γ are constants, $f_i(\theta)$ (I = 1, 2, 3, 4) are given non-dimensional functions of temperature, in case of temperature-independent properties, $f_i(\theta)\equiv 1$ and $\lambda = \lambda_0$, $\mu = \mu_0$, $k = k_0$, $\gamma = \gamma_0$. Except the properties in (2), the other material properties are assumed to be independent of temperature.

Rishin *et al*. (1973) investigated the relationship between modulus of elasticity of several sprayed coatings and temperature, and they found the modulus of elasticity decreases monotonically with the increasing of temperature. For simplicity and without loss of generality, we assume:

$$f_i(\theta) = f(\theta) = 1 - \alpha\theta (i = 1, 2, 3, 4) \tag{3}$$

where, α is an empirical material constant.

For generalized thermoelastic problems, the integral transform techniques are often used to get the solutions of such problems. However, this method encounters loss of precision which is believed to be caused by discretization error and truncation error introduced inevitably in the process of numerical inverse Laplace and Fourier transforms. To overcome the defect of the above method, we are encouraged to formulate our problem by finite element method and solve the derived nonlinear finite element equations directly in time domain as reported by Tian *et al*. (2006).

FINITE ELEMENT FORMULATIONS

Rewrite the constitutive equations in matrix form as follows:

$$\{\sigma\} = [c]\{\varepsilon\} - \{a\}\theta, \eta = \{a\}^T\{\varepsilon\} + \nu\theta \tag{4}$$

The generalized heat conduction law can be written in matrix form as:

$$\{q\} + \tau\{\dot{q}\} = -[\kappa]\{\theta'\} \tag{5}$$

where $\theta' = \theta_{,i}$.

We introduce two sets of shape functions $[N^e_1]$ and $[N^e_2]$ and the displacement {u} and the temperature θ on the element level can be respectively expressed as:

$$\{u\} = [N_1^e]\{u^e\} \quad \theta = \{N_2^e\}^T\{\theta^e\} \tag{6}$$

In terms of the geometrical equation ε_{ij} = [B]{u^e} as well as $\theta' = \theta_{,i}$, it yields:

$$\{\varepsilon\} = [B_1]\{u^e\} \quad \{\theta'\} = [B_2]\{\theta^e\} \tag{7}$$

where, $[B_1]$ and $[B_2]$ relate respectively to the first order derivative of components in $[N^e_1]$ and $[N^e_2]$ with respect to material coordinates, and they can be explicitly specified once the components of $[N^e_1]$ and $[N^e_2]$ are given.

The variational form of Eq. 7 is:

$$\delta\{\varepsilon\} = [B_1]\delta\{u^e\}, \delta\{\theta'\} = [B_2]\delta\{\theta^e\} \tag{8}$$

In the absence of body force, the virtual displacement principle of the generalized thermoelastic problems in the context of L-S theory can be formulated as:

$$\begin{aligned} &\left(\int_V \begin{bmatrix} \sigma_{ij}\delta\varepsilon_{ij} + (q_i + \tau\dot{q}_i)\delta\theta_{,i} \\ -T_0(\dot{\eta} + \tau\ddot{\eta})\delta\theta \end{bmatrix} dV\right) \\ &= -\int_V \rho\ddot{u}_i\delta u_i dV + \int_{A_\sigma} \overline{T}_i\delta u_i dA \\ &\quad + \int_{A_q} \overline{q}\delta\theta dA. \end{aligned} \tag{9}$$

where, $\bar{T}_i$ represents the components of traction vector, and $\bar{q}$ the heat flux vector. Substituting Eq. 4-8 into Eq. 9, we arrive at:

$$\begin{bmatrix} M^e_{mm} & 0 \\ M^e_{\theta m} & M^e_{\theta\theta} \end{bmatrix} \begin{Bmatrix} \ddot{u}^e \\ \ddot{\theta}^e \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ C^e_{\theta m} & C^e_{\theta\theta} \end{bmatrix} \begin{Bmatrix} \dot{u}^e \\ \dot{\theta}^e \end{Bmatrix} + \begin{bmatrix} K^e_{mm} & -K^e_{m\theta} \\ 0 & K^e_{\theta\theta} \end{bmatrix} \begin{Bmatrix} u^e \\ \theta^e \end{Bmatrix} = \begin{Bmatrix} T^e_m \\ -T^e_\theta \end{Bmatrix}. \tag{10}$$

Where:

$$\begin{aligned} \left[K^e_{mm}\right] &= \int_V [B_1]^T [c][B_1]\, dV, \\ \left[K^e_{mm}\right] &= \int_V [B_1]^T [c][B_1]\, dV, \\ \left[K^e_{\theta\theta}\right] &= \int_V [B_2]^T [k][B_2]\, dV, \\ \left[C^e_{\theta m}\right] &= \int_V T_0 \{N^e_2\}\{a\}^T [B_1]\, dV, \\ \left[C^e_{\theta\theta}\right] &= \int_V T_0 \{N^e_2\} \nu \{N^e_2\}^T\, dV, \\ \left[M^e_{\theta m}\right] &= \int_V T_0 \{N^e_2\} \tau \{a\}^T [B_1]\, dV, \\ \left[M^e_{\theta\theta}\right] &= \int_V T_0 \{N^e_2\} \tau \nu \{N^e_2\}^T\, dV, \\ \left[M^e_{mm}\right] &= \int_V \left[N^e_1\right]^T \rho \left[N^e_1\right] dV, \\ \{T^e_m\} &= \int_{A_\sigma} \left[N^e_1\right]^T \{\bar{T}\}\, dA, \\ \{T^e_\theta\} &= \int_{A_q} \{N^e_2\} \bar{q}\, dA \end{aligned} \tag{11}$$

RESULTS AND DISCUSSIONS

To apply the foregoing method to a concrete example, we consider a generalized thermoelastic problem of a thin slim strip subject to a thermal shock. The schematic of the strip is shown in Fig. 1 and the strip is composed of copper material. All the edges of the strip are traction free and the strip is subject to a thermal shock $\theta = \theta_0 H(t)$ at left edge EH, where H(t) is Heaviside step function and θ_0 is the magnitude of thermal shock. Assume the other three edges EF, FG and HG are heat insulation, i.e., along EF, FG and HG, q = 0. As shown in Fig. 1, the Cartesian coordinates system xoy is established, and the coordinate origin is placed at the mid-point of EH.

To carry out the simulation, we set $\tau_0 = 0.02$, $\tau = 0.2$, T = 293K, $\theta_0 = 1$, $P_0 = 1$ and the material parameters of copper material are:

$$\lambda_0 = 7.76\times10^{10}\,\text{Nm}^{-2} \qquad \mu_0 = 3.86\times10^{10}\,\text{Nm}^{-2}$$
$$\alpha_t = 1.78\times10^{-5}\,\text{K}^{-1} \qquad \eta_1 = 8886.73\,\text{sm}^{-2}$$
$$C_E = 383.1\,\text{Jkg}^{-1}\text{K}^{-1} \qquad \rho = 8954\,\text{kgm}^{-3}$$

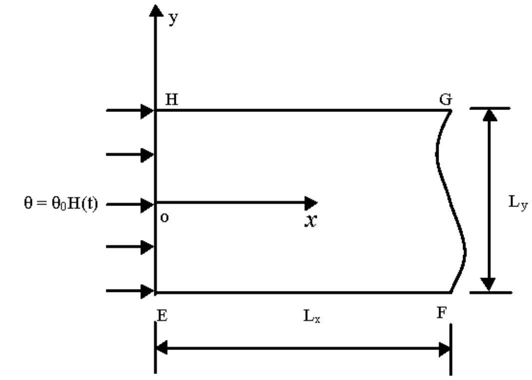

Fig. 1: Schematic of the thin slim strip subjected to a thermal shock

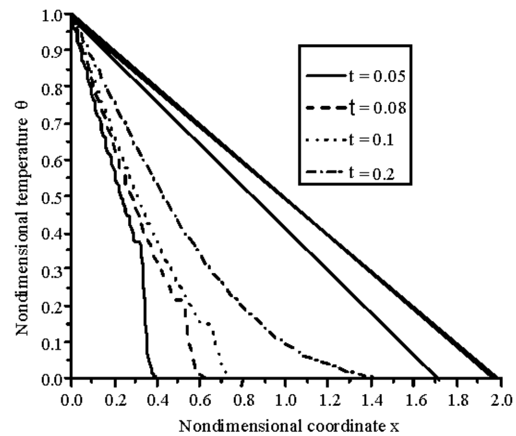

Fig. 2: Distributions of non-dimensional temperature along axis- at different times when $\alpha = 0$

The following non-dimensional variables: $x_i^* = c_1\eta_1 x_i$, $u_i^* = c_1\eta_1 u_i$, $t^* = c_1^2\eta_1 t$, $\tau^* = c_1^2\eta_1\tau$, $\theta^* = \theta/T_0$, $\sigma_{ij}^* = \sigma_{ij}/\mu_0$, $\eta_1 = \rho C_E/\kappa_0$ and $c_1^2 = (\lambda_0 + 2\mu_0)/\rho$ are introduced in the calculation for convenience.

Assume the dimension in the x-direction is taken to be much greater than that in the y-direction ($L_X >> L_y$) and set $L_x = 2$ while $L_y = 0.02$.

Calculated results by finite element method in time domain for the non-dimensional temperature, displacement and stress are illustrated graphically in Fig. 2-7 respectively, dropping the asterisk at the upper right corner of the non-dimensional variables for convenience.

Among Fig. 2-7, the first three ones are given to display the general variations of considered variables in case of temperature independent properties (i.e., $\alpha = 0$)

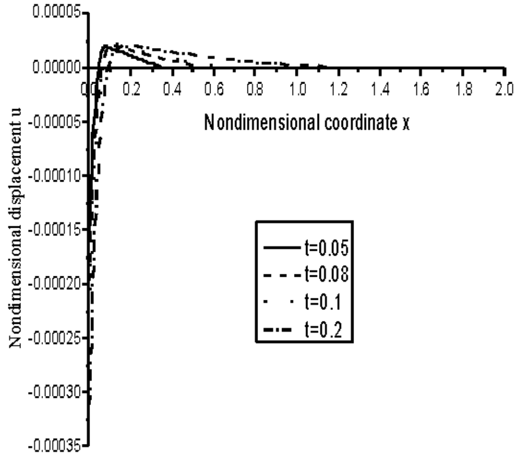

Fig. 3: Distributions of non-dimensional displacement along axis- at different times when $\alpha = 0$

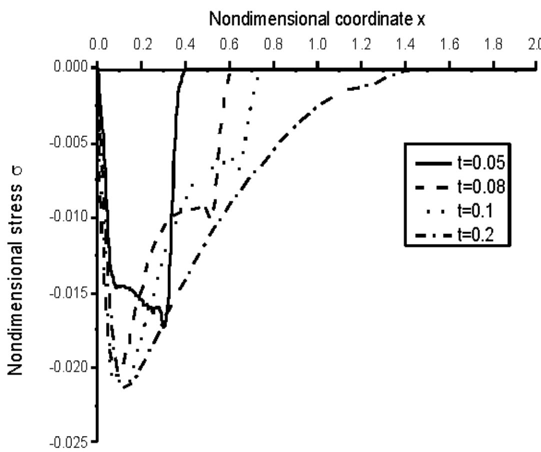

Fig. 4: Distributions of non-dimensional stress along axis- at different times when $\alpha = 0$

and serve as a basis to understand the effect of temperature-dependent properties on the considered variables, and the last three to demonstrate the effect of temperature dependent properties on the considered variables. From Fig. 5-7, it can be seen the non-dimensional temperature, displacement, and stress are sensitive to the temperature dependence of material properties, and the effect of the temperature-dependent properties on the considered variables is reducing their magnitudes. This indicates in the investigation of generalized thermoelastic transient problems taking the temperature dependence of material properties into

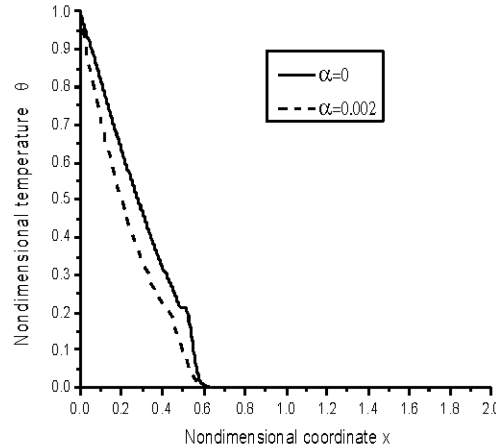

Fig. 5: Distributions of non-dimensional temperature along axis-x at t = 0.08

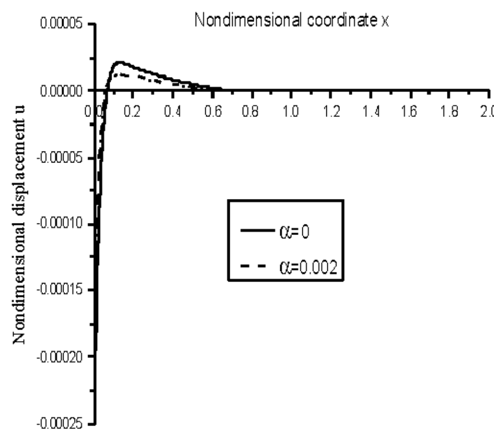

Fig. 6: Distributions of non-dimensional displacement along axis-x at t = 0.08

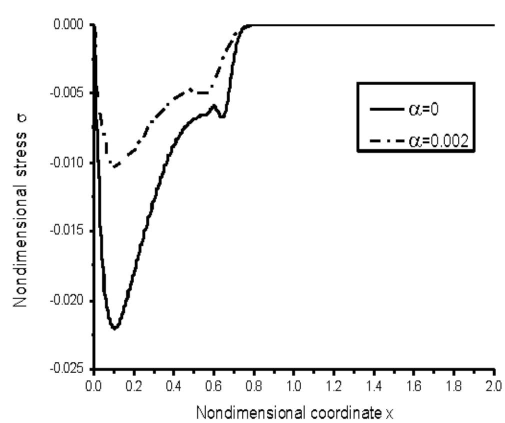

Fig. 7: Distributions of non-dimensional stress along axis-x at t = 0.08

account is very necessary and important for accurately predicting the thermoelastic behaviors.

CONCLUSIONS

In the context of L-S theory, the generalized thermoelastic coupled problem with temperature-dependent material properties is formulated. The results show that (1) The non-zero values of all the considered variables are only in a bounded region; (2) The temperature-dependent properties act to reduce the magnitudes of the considered variables; (3) Taking the temperature-dependence of material properties into account in the investigation of generalized thermoelastic problems is very necessary and practical for accurately predicting the thermoelastic behavior; (4) The approach presented here can also be adopted to solve two-dimensional problems as well as other problems formulated in the context of generalized thermoelastic theories.

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REFERENCES

Aouadi, M., 2006. Generalized thermo-piezoelectric problems with temperature-dependent properties. Int. J. Solids Struct., 43: 6347-6358.

Biot, M., 1956. Thermoelasticity and irreversible thermodynamics. J. Applied Phys., 27: 240-253.

Catteneo, C., 1958. A form of heat conduction equation which eliminates the paradox of instantaneous propagation. Comput. Rendus, 247: 431-433.

Chen, T.C. and C.I. Weng, 1988. Generalized coupled transient thermoelastic plane problems by Laplace transform-finite element method. J. Appl. Mech., 55: 377-382.

Chen, T. and C. Weng, 1989. Coupled transient thermoelastic response in an axisymmetric circular cylinder by Laplace transform-finite element method. Comput. Struct., 33: 533-542.

Dhaliwal, R.S. and J.G. Rokne, 1989. One-dimensional thermal shock problem with two relaxation times. J. Thermal Stresses, 12: 259-279.

Ezzat, M.A., A.S. El-Karamany and A.A. Samaan, 2004. The dependence of the modulus of elasticity on reference temperature in generalized thermoelasticity with thermal relaxation. Applied Math. Comput., 147:169-189.

Green, A.E. and K.A. Lindsay, 1972. Thermoelasticity. J. Elasticity, 2: 1-7.

Lord, H. and Y. Shulman, 1967. A generalized dynamical theory of thermoelasticity. J. Mech. Phys. Solids, 15: 299-309.

Rishin,V.V., B.A. Lyashenko, K.G. Akinin and G.N. Nadezhdin, 1973. Temperature dependence of adhesion strength and elasticity of some heat-resistant coatings. Strength Mater., 5: 123-126.

Sherief, H.H. and M.N. Anwar, 1994. State-space approach to two-dimensional thermoelasticity problems. J. Thermal Stresses, 17: 567-590.

Sherief, H.H. and R.S. Dhaliwal, 1981. Generalized one-dimensional thermal shock problem for small times. J. Thermal Stresses, 4: 407-420.

Tian, X., Y. Shen, C. Chen and T. He, 2006. A direct finite element method study of generalized thermoelastic problems. Int. J. Solids Struct., 43: 2050-2063.

Vernotte, P., 1961. Some possible complications in the phenomenon of thermal conduction. Comput. Rendus, 252: 2190-2191.