Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 13 (15): 2924-2928, 2013
ISSN 1812-5654 / DOI: 10.3923/jas.2013.2924.2928

An Improved Recursive Formula for Computing Normal Forms of Multi-dimensional Dynamical Systems

[1,2]Lei Guo, [3]Qunhong Li and [4]Zigen Song
[1]School of Management, Fudan University, Shanghai, 200433, China
[2]School of Business Administration, Shanghai Finance University, Shanghai, 201209, China
[3]College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi, 530004, China
[4]College of Information Technology, Shanghai Ocean University, Shanghai, 201306, China

Abstract: In this study, an improved explicit recursive formula of normal forms under nonlinear near-identity transformations is firstly introduced and the associated proof is also given out. Compared with traditional method, the improved method can get the specific expression of the higher order terms of dynamical systems after nonlinear transformations. So the normal forms of the original dynamical systems can be obtained by a series of nonlinear transformations but without changing its topology structure, also by solving a series of algebra equations with the aid of computing software Maple, not only the coefficients of the jth order normal form and the associated nonlinear transformations but also higher (>j) order terms of the original equations can be obtained. The application of the improved method will greatly simplify the computation during the research of dynamical systems.

Key words: Normal form, recursive formula, maple, differential equations

INTRODUCTION

Normal form theory dates back to the days of Poincare and Dulac, is very important tool to simplify the analysis of nonlinear dynamical systems near an equilibrium (Nayfeh, 1993). The basic idea of normal form theory is to employ a series of nonlinear near-identity transformations to simplify form of the original differential equations without changing the topology structure of the system in the vicinity of equilibrium. Before normal form theory, center manifold theory is usually applied to get a locally invariant small dimensional manifold in the vicinity of an equilibrium, called centre manifold (Carr, 1981). Theoretically it is possible to compute the coefficients of the normal form and the associated nonlinear transformations for any a given system. Symbolic computation using symbolic computer language such as Maple and Methematica have been introduced in computing normal forms. However it seems that even with a symbolic manipulator the computation of normal forms is still limited to lower order approximations especially the dimension is less than 3. This is because algebraic manipulations becomes involved and executing a program of computing a normal form usually runs out of computer memory quickly as the order of normal form increases.

In the past few decades, many researchers have obtained great achievements in the study of the normal form. Up to now, the five basic methods for the computation of the normal form have been introduced and developed: The matrix representation theory (Arnold, 1983; Guckenheimer and Holmes, 1993; Chow *et al*., 1983); the adjoint operator method (Elphick *et al*., 1987); the representation theory of Lie algebra (Carr, 1981; Yu, 1998); the method of more Lie brackets (Chua and Kokubu, 1998, 1988a, b; Ushiki, 1984) and the perturbation method (Yu, 1998; Yu and Leung, 2003).

The normal form for most two-dimensional dynamical systems have already been computed, now many researchers pay more attention on its reduction based on conventional normal form. It is well known that first study on this direction is Given by Ushiki (1984) using the method of Lie algebra (Baider and Sanders, 1992) specified nilpotent vector fields in dimension 2 into three categories and they have gave unique normal forms for the first two categories but such a unique normal form are not the simplest normal form and thus we prefer to use the simplest normal form, which means that the total numbers of the normal form up to certain order cannot be further simplified by any other nonlinear transformations. Later Kokubu *et al*. (1996) introduced linear grading function method to further reduction in a special case of nilpotent

Corresponding Author: Qunhong Li, College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi, 530004, China

vector fields in dimension 2 for the third category. In the study of three-dimensional dynamical systems (Yu and Yuan, 2001a, b) gave the simplest normal forms of several cases up to certain order, the main analysis and symbolic computation four dimensional dynamical systems on the condition of double Hopf bifurcation.

The adjoint operator method developed by Zhang *et al*. (2004) has successfully got the 3rd order normal forms of 4-dimensional dynamical systems. In Zhang *et al*. (2004) they failed to consider the influence of non-linear transformations on the higher order terms of original dynamical systems, sometimes the higher order terms of dynamical should to be considered forfurther analysis. In this study, a recursive formula is proposed to compute the kth order normal form of dynamical systems with k(k = 2) being an integer number. We further developed the adjoint operator method introduced in Zhang *et al*. (2004) by solving a series of algebra equations. The recursive formula can get not only the coefficients of the jth order normal forms and the associated nonlinear transformations but also higher order terms of original equations for which the order of terms is bigger than j but less than the certain specified number. The associated proof is also given in the following section.

METHODS AND THEORIES

Consider differential equation:

$$\dot{x} = Ax + f^2(x) + \ldots + f^k(x) + O(|x|^k), \tag{1}$$

where, A is n×n Jordan canonical matrix $f^j(x) \in H_n^i$ (j = 2...k) and O ($|x|^k$) represents higher terms of system (1), here H_n^j denotes the vector space of homogeneous polynomials of order j in n variables. The origin X = 0 is assumed to be a singularity, then X (0) = 0. The nonlinear terms of system (1) can be simplified by a series of near-identity nonlinear transformations, given by:

$$x = y + P^j(y), \; P^j(y) \in H^j_n i, \; j = 2, \ldots, k \tag{2}$$

Where:

$$P^j(y) = (P_1^j(y), \; P^j_2(y), \ldots, P_n^j(y)) \tag{3}$$

Then we have:

$$\dot{x} = (I + DP^j(y))\, \dot{y} \tag{4}$$

and:

$$(I+DP^j(y))^{-1} = I - DP^j(y) + (DP^j(y))^2 + \ldots + (-1)p\,(Dp^j(y))^p \tag{5}$$

Theorem 1: The recursive equation for computing the coefficients of jth order normal form and the higher order terms of original systems under the jth order nonlinear transformation is given by:

$$\begin{aligned} \dot{y} &= Ay + f^2(y) + \cdots + f^{j-1}(y) \\ &+ f^j(y) - DP^j(y)Ay \\ &+ \sum_{\substack{(j-1)m+p=j+1 \\ m\geq 0, p\geq 1}} (-1)^m (DP^j(y))^m \tilde{f}^p(y); \end{aligned} \tag{6}$$

Where:

$$\tilde{f}^p(y) = \sum_{i\leq[p/j]} \frac{1}{i!} D^i f^{p-(j-1)i}(y)(P^j(y))^i,$$
$$j = 2, \ldots, k \quad p = j+1, \ldots, k;$$

and f^j, P^j, respectively denote the jth order vector homogenous polynomials of y (where y has been dropped for simplicity), f^j represents the jth order terms of the original system, P^j denotes the th order nonlinear transformation.

From above theorem one can see that the jth nonlinear transformation just have influence on jth order and higher order term and have no influence on the terms which order is less than jth.

If j>[k/2] the less order terms in the last past of Eq. 6 will not need to be considered, because under this condition the lowest order of any term in the past is bigger than k but at this paper just kth or lower order items are need to be considered.

Next, the procedure of computing the jth order normal form with the aid of the adjoint operator method is introduced, a definition of linear operator was given as:

$$ad^j_A : H^j_n \to H^j_n \;\; ad^j_A P^j(y) = DP^j(y)Ay - AP^j(y) \tag{7}$$

where, ad^k_A is called a homological operator. Let R^j be the image of ad^j_A, that is, R^j = $amad^j_A$ and C^j be any complementary subspaces of R^j in H_n^j, i.e.

$$H_n^j = R^j \oplus C^j$$

Assume f^j, = $h^j + g^j$, where $h^j \in R^j$, $g^j \in C^j$. We may choose P^j (y) such that:

$$ad^j_A P^j(y) = h^j(y) = f^j(y) - g^j(y) \tag{8}$$

Equation 8 is called a homological equation, Using a series of near identity non-linear transformations, then Eq. 1 can be changed after replacing by into:

$$x = Ax+g^2(x)\ g^3(x)+g^k(x),\quad x\in R^n \tag{9}$$

where, $g^j(x)\in C^j$. for We refer to Eq. 9 as the normal form up to kth order of non-linear systems (1). The goal to compute its normal form is to choose a series of near identity non-linear transformations $g(x) = (g^2(x), g^3(x), ..., g^k(x)^T$ such that non-linear systems (1) will be simplified as simple as possible. From above analysis it is known that the key step of computing normal form is to find of C^j for $j = 2,...,k$. The following analysis shows how to find corresponding basis of vertical complementary subspace of $Imad^j_A$ in H^j_n for $j = 2,...,k$.

Assume that is a finite dimensional inner product space, is L a linear operator in V and L* is the adjoint operator of L. So we have:

$$\begin{aligned}&(1)KerL^* = (ImL)^{\perp};\\&(2)V= Im\,L \oplus KerL^*\end{aligned} \tag{10}$$

where, KerL* is the null space of L*. The above results can be found in literatures (Taylor and Lay, 1980). As we known that if the adjoint operator $(ad^j_A)^*$ of the linear operator ad^k_A may be found $Ker\ (ad^j_A)^*$ is a vertical complementary subspace of $Imad\ ad^j_A$ in H^j_n. From Refs (Chow *et al*., 1983; Elphick *et al*., 1987), the operator ad^j_A is the adjoint operator of ad^j_A, that is, $ad^j_{A^*} = (ad^j_A)$, where $A^* = A\overline{A}^T$ is the adjoint transposed matrix of A. Then, we have:

$$ad^j_{A^*}\,P^j(x) = DP^j(x)A^*x-A^*P^j(x) \tag{11}$$

Thus, Ker $ad^j_{A^*}$ is a vertical complementary subspace of $Imad^j_A$ iH^j_n, i.e., $H_n^j = Im\ ad^j_A\oplus \ker ad^j_{A^*}$. Moreover Ker $ad^j_{A^*}$ is a subspace which consists of all jth order vector polynomial solutions in n variables for linear partial differential equation:

$$ad^j_{A^*}\,P^j(x) = 0 \qquad P^j(x)\in H_n^j \tag{12}$$

or:

$$DP^j(x)\,A^*x-A^*P^j(x) = 0 \tag{13}$$

In order to obtain the normal form of non-linear systems(1), we have to solve a series of partial differential equations Eq. 13 for j = 2,..., k.

IMPROVED FORMULA AND PROOF

In the following, we suppose that the (j-1)th order normal forms of Eq. 1 can be get by using nonlinear transformations from 2nd to (j-1)th order. Before we consider the jth order nonlinear transformations, we still rewrite the obtained (j-1)th jorder normal forms and higher order terms of Eq. 1 in the form as:

$$x = Ax+f^2(x)+...+f^k(x) \tag{14}$$

In above expression, Eq. 14 denotes different meanings comparing to Eq. 1. Substituting Eq. 2 and 5 into Eq. 1 yields:

$$(I+DP^j(y))y = A(y+P^j(y))+f^2(y+P^j(y))+...+f^k(y+P^j(y)) \tag{15}$$

namely:

$$y = (I+DP^j(y))^{-1}\{A(y+P^j(y))+f^2(y+P^j(y))+...+f^k(y+P^j(y))\} \tag{16}$$

using Taylor expansions, we have:

$$\begin{aligned}&(I+DP^j(y))^{-1}=I-DP^j(y)+(DP^j(y))^2\\&+...+(-1)^{[\frac{k}{j-1}]+1}(DP^j(y))^{[\frac{k}{j-1}]+1}\\&=I-DP^j(y)+\sum_{m=2}^{[\frac{k}{j-1}]+1}(-1)^m(DP^j(y))^m\end{aligned} \tag{17}$$

and:

$$\begin{aligned}&A(y+P^j(y))+f^2(y+P^j(y))+\cdots+f^k(y+P^j(y))\\&=Ay+f^2(y)+...+f^j(y)+AP^j(y)\\&+\sum_{i\le[(j+1)/j]}\frac{1}{i!}D^if^{((j+1)-(j-1)i)}(y)(P^j(y))^i\\&+\sum_{i\le[(j+2)/j]}\frac{1}{i!}D^if^{((j+2)-(j-1)i)}(y)(P^j(y))^i\\&+\sum_{i\le[k/j]}\frac{1}{i!}D^if^{(k-(j-1)i)}(y)(P^j(y))^i\\&=\tilde{f}^1(y)+\tilde{f}^2(y)+\cdots+\tilde{f}^j(y)+\cdots+\tilde{f}^k(y)\end{aligned} \tag{18}$$

where, $\tilde{f}^1(y)=Ay,\ldots\ \tilde{f}^{j-1}(y)=f^{j-1}(y)$

$\tilde{f}^j(y)=f^j(y)+AP^j(y)$

$$\tilde{f}^p(y)=\sum_{i\le[p/j]}\frac{1}{i!}D^if^{(p-(j-1)i)}(y)(P^j(y))^i,$$
$$(p=j+1,\cdots,k)$$

So, Eq. 16 can be expanded up to order k as:

$$
\begin{aligned}
\dot{y} &= (I - DP^j(y) + \sum_{m=2}^{[\frac{k}{j-1}]+1} (-1)^m (DP^j(y))^m) \\
&\quad (A(x) + \tilde{f}^2(x) + \ldots + \tilde{f}^k(x)) \\
&= \tilde{f}^1(x) + \ldots + \tilde{f}^j(x) - DP^j(y)\tilde{f}^1(x) \\
&\quad + \sum_{\substack{(j-1)m+p=j+1 \\ m\geq 0, p\geq 1}} (-1)^m (DP^j(y))^m \tilde{f}^p(y) \\
&\quad + \sum_{\substack{(j-1)m+p=j+2 \\ m\geq 0, p\geq 1}} (-1)^m (DP^j(y))^m \tilde{f}^p(y) \\
&\quad + \cdots + \sum_{\substack{(j-1)m+p=k \\ m\geq 0, p\geq 1}} (-1)^m (DP^j(y))^m \tilde{f}^p(y) \\
&= \tilde{f}^1(x) + \ldots + \tilde{f}^j(x) - DP^j(y)\tilde{f}^1(x) \\
&\quad + \sum_{\substack{(j-1)m+p=j+1 \\ m\geq 0, p\geq 1}}^{k} (-1)^m (DP^j(y))^m \tilde{f}^p(y) \\
&= Ay + \ldots + f^j(y) + AP^j(y) - DP^j(y) \\
&\quad * A(y) + \sum_{\substack{(j-1)m+p=j+1 \\ m\geq 0, p\geq 1}}^{k} (-1)^m (DP^j(y))^m \tilde{f}^p(y)
\end{aligned} \tag{19}
$$

Where:

$$
\tilde{f}^p(y) = \sum_{i\leq [p/j]} \frac{1}{i!} D^i f^{(p-(j-1)i)}(y)(P^j(y))^i,
$$

$$
(p = j+1, \cdots, k)
$$

From above analysis derived from normal form theory and the adjoint operator method, the following idea can be obtained.

The jth nonlinear transformation just have influence on jth and higher (>j) order terms of original systems and have no influence on lower order (<j) terms of original systems.

Each higher (>j) order terms of original systems can be expressed by the polynomial of lower (j or <j) order non-linear transformations and ith order ($2\leq i<j$) normal forms.

Each order normal form of non-linear system can be obtained by the associated non-linear transformations order by order.

CONCLUSION

The improved recursive equation for computing normal form of non-linear systems is firstly introduced in the paper and it is especially effective in computing normal form of multi-dimensional non-linear systems. In this method, starting from the lowest order non-linear transformations up to some degree, in each order non-linear transformation, we introduced the associated basis of H^j_n, changing non-linear terms of original systems into the form of matrix, then obtaining the basis of ker $ad^j_{A^*}$ in H^j_n by solving adjoint operator partial differential equations. Lastly balancing both sides of a set of the obtained homological equations, the normal form of original systems and the associated transformations are obtained. The new higher order terms of original systems can be determined after the non-linear transformations based on Theorem 1. The higher order normal forms and the associated non-linear transformations will be also gotten by the same method, step by step, we can get normal form of the original system up to the required degree. Therefore, our result extends the achievements (Zhang *et al*., 2004).

ACKNOWLEDGMENTS

This research is supported by the National Natural Science Foundation of China (No. 11302126, 11002103), Shanghai Education Research and Innovation Key Project (No. 11ZS183) and Young Teacher Training Program of Colleges and Universities in Shanghai (No. ZZhy12030).

REFERENCES

Arnold, V.I., 1983. Geometrical Methods in the Theory of Ordinary Differential Equations. Springer-Verlag, Berlin, New York.

Baider, A. and J.A. Sanders, 1992. Further reduction of the takens-bogdanov normal forms. J. Different. Equat., 99: 205-244.

Carr, J., 1981. Applications of Center Manifold Theory. Springer-Verlag, New York.

Chow, S.N., C. Li and D. Wang, 1983. Normal Forms and Bifurcation of Planar Vector Fields. Cambridge University Press, Cambridge UK.

Chua, L.O. and H. Kokubu, 1988. Normal forms for nonlinear vector fields. I. Theory and algorithm. IEEE. Trans Circuits Syst., 35: 863-880.

Chua, L.O. and H. Kokubu, 1989. Normal forms for nonlinear vector fields. II. Applications. IEEE. Trans. Circuits Syst., 36: 51-70.

Elphick, C., E. Tirapegui, M.E. Brachet, P. Coullet and G. Iooss, 1987. A simple global characterization for normal forms of singular vector fields. Phys. D: Nonlin. Phenom., 29: 95-117.

Guckenheimer, J. and P. Holmes, 1993. Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer-Verlag, New York.

Kokubu, H., H. Oka and D. Wang, 1996. Linear grading function and further reduction of normal forms. J. Different. Equat., 132: 293-318.

Nayfeh, A.H., 1993. Methods of Normal Forms. Springer-Verlag, New York.

Taylor, A.E. and D.C. Lay, 1980. Introduction to Functional Analysis. John Wiley and Sons, New York.

Ushiki, S., 1984. Normal forms for singularities of vector fields. Jpn J. Applied Math., 1: 1-37.

Yu, P. and A.Y.T. Leung, 2003. A perturbation method for computing the simplest normal forms of dynamical systems. J. Sound Vibrat., 26: 123-151.

Yu, P. and Y. Yuan, 2001. The simplest normal form for the singularity of a pure imaginary pair and a zero eigenvalue. J. Math. Res. Exp., 8: 219-249.

Yu, P. and Y. Yuan, 2001. The simplest normal forms associated with a triple zero eigenvalue of indices one and two. Nonlin. Anal., 47: 1105-1116.

Yu, P., 1998. Computation of normal forms via a perturbation technique. J. Sound Vibrat., 211: 19-38.

Zhang, W., F. Wang and J.W. Zu, 2004. Computation of normal forms for high dimensional non-linear systems and application to non-planar non-linear oscillations of a cantilever beam. J. Sound Vibrat., 278: 949-974.