Transmission Angle Analysis in a Type of Manipulators

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Abstract: This study presents a novel common approach to analyze the transmission angle of a type of manipulators. Here, two types of transmission angles are defined to evaluate the kinematic performance of parallel manipulators, with which the direct and indirect singularities can be identified in a straightforward manner. The relationships between the angles and condition number of Jacobian which is a vital performance index for parallel manipulators are investigated throughout the workspace by taking both the Diamond and Delta parallel manipulators as an example. A simple method for determining the workspace bounded by the angle constraints is proposed which make less time be cost in the optimization or any other design procedure. Considering the effects of transmission angles, a global and comprehensive index is presented. The kinematic optimization of Diamond manipulator has been taken as an example to illustrate the effectiveness of this approach.

Key words: Manipulator, transmission angle, singular configuration, optimal design

INTRODUCTION

Due to some advantages in acceleration, speed, payload capability, stiffness, dynamic behaviors compared with serial counterparts, parallel manipulators have aroused great attention in the fields of high speed machines, high speed pick-and-place applications, micro-motion manipulators, motion simulators, robotic end-effector and so on (Gosselin and Angeles, 1988). Such as Delta/Diamond (Gosselin and Angeles, 1989) robot and Sparacino and Hervé proposed the Y-STAR and H-ROBOT parallel robot (Gosselin and Angeles, 1991), Tsai (Miller, 2002) presented the 3-UPU manipulator (P stands for prismatic joint). This study attempts to deal with definition of transmission/pressure angle in a type of parallel manipulators which can realize translations in space with limbs having same topological structures with that using in Delta/Diamond robot. And, kinematic models and Jacobian matrix are presented. In what follows, through the analysis of Jacobian matrix, with which the direct and indirect singularities can be identified in a straightforward manner. In which a comprehensive performance index composed by condition number and transmission angle is firstly investigated. Notably, the number of limbs may exceed the needed numbers for actuating the platform since it has only 3-DOF translations at most in space which can make the manipulator to be actuation redundancy. Therefore, the methods are still suitable for implementing the kinematic design of redundant Delta-like robot mentioned in.

KINEMATIC ANALYSIS

The type of parallel manipulators that we investigated has one rigidity platform which has only translation motion in space and kinematic chains consist of three joints and two moving links. Two kinds of kinematic chains which are accordance with the earlier conditions are shown in Fig. 1.

For the purpose of analysis, the coordinate systems are defined as shown in Fig. 2. The coordinate system O-xyz is attached to the fixed base and another moving coordinate frame Puvw is located on the moving platform. As illustrated in Fig. 2, the closed-loop position equation associated with the ith kinematic chain can be written as:

\[ r + c = a_i + b_i e_i l_i u_i + l_i w_i \quad i = 1, 2, 3, \ldots, n - 2 \]

(1)

where, \( r, c, a, b, e, u \) and \( w \) denote the vector OP, the vector PC, the vector OD, the length of \( DA_i \), the unit vector along \( DA_i \), the unit vector along the active link \( AB_i \), the unit vector along the passive link \( BC_i \) and the joint \( A_i \) can be revolute or prismatic joint. So the inverse position solution can be achieved by:

\[ l_i = \sqrt{(r + c - a_i - b_i e_i - l_i u_i)^2 + (r + c - a_i - b_i e_i - l_i w_i)^2} \]

(2)

Jacobian analysis: Taking the derivative of Eq. (1) with respect to time yields:

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4863
Fig. 1(a-b): Two kinds of kinematic chains of the type of parallel manipulators, (a) Actuated by revolute joint and (b) Actuated by prismatic joint

\[ \mathbf{v} = b_1 \mathbf{e} + l_2 \omega_2 \times \mathbf{u}_1 + l_2 \omega_2 \times \mathbf{w}_i \]  

where \( \mathbf{v}, \omega_2, \omega_2 \) denote the linear velocity of the moving platform, the angular velocity of the link \( \mathbf{AB}_i \) and the angular velocity of the link \( \mathbf{BC}_i \).

If the joint \( \mathbf{A}_i \) is a revolute joint, the Eq. 3 can be written as:

\[ \mathbf{v} = l_2 \omega_2 \times \mathbf{u}_1 \]  

where, \( \mathbf{v}_1 \) is the unit vector along the direction of the angular velocity of \( \mathbf{AB}_i \).

If the joint \( \mathbf{A}_i \) is a prismatic joint, the Eq. 3 can be written as:

\[ \mathbf{v} = b_1 \mathbf{e} + l_2 \omega_2 \times \mathbf{w}_i \]  

Taking dot product of both sides of Eq. 4 and 5 with \( \mathbf{w}_i \) yields:

\[ l_2 \omega_2 \times \mathbf{u}_1 \cdot \mathbf{w}_i = l_2 \omega_2 \times \mathbf{w}_i \]

\[ b_1 \mathbf{e} \cdot \mathbf{w}_i = \mathbf{w}_i \]

Rewriting Eq. 6 and 7 in the matrix form yields:

\[ J_q q = J_{w} \mathbf{x} \]  

\[ J_{w} b = J_{u} \mathbf{x} \]  

Where:

\[ J_q = \text{diag}(l_1, w_1, l_2, w_2, \ldots, l_n, w_n) \]

\[ q = [\theta_1, \theta_2, \ldots, \theta_n] J_q = [w_1, w_2, \ldots, w_n] \]

\[ J_w = \text{diag}(w_1 \mathbf{e}, w_2 \mathbf{e}, \ldots, w_n \mathbf{e}) \]

\[ J_u = -J_w -[w_1, w_2, \ldots, w_n] \mathbf{b} = [-b_1, -b_2, \ldots, -b_n] \]

**TRANSMISSION ANGLE ANALYSIS**

Singlarities are one of the most significant and critical problems in the design and control of parallel manipulators. Unlike closed manipulators, the consequences of venturing close to a singularity can be catastrophic for parallel manipulators. Three different types of singularities (Cervantes-Sanchez, 2000) for closed kinematic chains are defined based on the
where, \( \dot{q} \) is a vector with the active joint rates and \( \dot{x} \) is a velocity vector of the moving platform. The three singularity types are:

Observing the matrix \( J_q \) and \( J_q \) derived from Eq. 8 and 9, the component of the two matrices has a common feature that it can be evaluated as trigonometric values of an angle as shown in Fig. 3.

According to the vector multiplication algorithm in mathematics, the angle depicted in Fig. 3 can be expressed as:

\[
\cos \gamma_i = \frac{w_i \cdot w_i}{|w_i|} \tag{11}
\]

\[
\cos \gamma_i = w_i \cdot \hat{w} \tag{12}
\]

It is obvious that when \( \alpha \sim 90^\circ \) and \( \gamma \sim 90^\circ \) we can obtain \( \det (J_q) = 0 \) and \( \det (J_q) = 0 \). Then, the direct kinematic singularities will occur. Here, we define the angles which are related to the direct kinematic singularities as Inner Transmission Angle (ITA) of the kinematic chain. It is noted that the matrix \( J_q \) or \( J_q \) have the same form which are both \( n \times 3 \) (3 \times 2) matrix since the moving platform has only translations in the space, the moving platform possesses three degrees of freedom at most. If the number of kinematic chain is beyond the minimum necessary to actuate the manipulator, the parallel manipulator is called redundantly actuated manipulator. The Jacobian matrix of this kind of parallel manipulator which maps an \( m \)-dimensional velocity vector \( \dot{x} \) in the operation space into an \( n \)-dimensional joint velocity vector \( \dot{q} \) is not a square matrix any more. It is known that singularities of redundant parallel manipulators take place when the rank of \( J_q \) or \( J_q \) is lower than the degrees of freedom of the end effector (number of rows of \( J_q \)). In other words, when the determinant of \( J_q^T \) is equal to zero. Thus a redundant parallel manipulator is inverse kinematic singularity when any of the minor \( 3 \times 3 \) Jacobian matrices extracted from the \( n \times 3 \) \( J_q \) are singular.

\[
J_q = \left[ \begin{array}{ccc}
\sum_{i=1}^{3} w_i \cdot w_i & \sum_{i=1}^{3} w_i \cdot \dot{w}_i & \sum_{i=1}^{3} w_i \\
\sum_{i=1}^{3} \dot{w}_i \cdot w_i & \sum_{i=1}^{3} \dot{w}_i \cdot \dot{w}_i & \sum_{i=1}^{3} \dot{w}_i \\
\sum_{i=1}^{3} \dot{w}_i \cdot w_i & \sum_{i=1}^{3} \dot{w}_i \cdot \dot{w}_i & \sum_{i=1}^{3} \dot{w}_i \\
\end{array} \right] \tag{13}
\]

Considering \( n \) is equal to 2, the parallel manipulator is inverse kinematic singularity when any of the minor \( 2 \times 2 J_q \) matrices extracted from the \( 2 \times 3 J_q \) are singular:

\[
J_q = \left[ \begin{array}{ccc}
w_1 \cdot w_1 + w_2 \cdot w_2 + w_3 \cdot w_3 & w_1 \cdot \dot{w}_1 + w_2 \cdot \dot{w}_2 + w_3 \cdot \dot{w}_3 & w_1 \cdot \dot{w}_1 + w_2 \cdot \dot{w}_2 + w_3 \cdot \dot{w}_3 \\
w_1 \cdot \dot{w}_1 + w_2 \cdot \dot{w}_2 + w_3 \cdot \dot{w}_3 & w_1 \cdot w_1 + w_2 \cdot w_2 + w_3 \cdot w_3 & w_1 \cdot w_1 + w_2 \cdot w_2 + w_3 \cdot w_3 \\
w_1 \cdot \dot{w}_1 + w_2 \cdot \dot{w}_2 + w_3 \cdot \dot{w}_3 & w_1 \cdot \dot{w}_1 + w_2 \cdot \dot{w}_2 + w_3 \cdot \dot{w}_3 & w_1 \cdot w_1 + w_2 \cdot w_2 + w_3 \cdot w_3 \\
\end{array} \right] \tag{14}
\]

In order to establish the relation with the transmission angle, the determinant of matrix in Eq. 13 and 14 can be written in another form:

\[
\det (J_q) = \sum_{i=1}^{n-3} |(w_i \times w_j) \cdot (w_i \times w_k)| \quad i \neq j \neq k \quad n \geq 3 \tag{15}
\]

\[
\det (J_q^T) = (w_i \times w_j) \cdot (w_i \times w_j) \quad n = 2 \tag{16}
\]

Since \( w_i \) is the unit vector along the passive link, for \( n = 2 \), we can define the angle as:

\[
\cos \beta_i = |w_i \cdot w_j| \tag{17}
\]

For \( n \geq 3 \), we can define the angle as:

\[
\cos \beta_i = \frac{w_i \cdot (w_j \times w_i)}{|w_i \times w_j|} \quad \cos \beta_{ij} = \frac{w_j \cdot (w_i \times w_j)}{|w_i \times w_j|} \tag{18}
\]

It can be easily to note that when \( \beta_i \sim 90^\circ \) (\( 1 \leq i \leq n \)) we can obtain \( \det (J_q^T) = 0 \) (\( \det (J_q^T) = 0 \)). So, we define the angle \( \beta_i \) related to the inverse kinematic singularity as Cross Transmission Angle (CTA) of the kinematic chain. It is well known that redundancy can improve the ability of avoiding kinematic singularities. This point can easily be observed from Eq 17 which is closed to 0 only if all the angles defined by Eq. 18 are closed to 90°.
OPTIMAL KINEMATIC DESIGN

By taking the advantage of both local and global optimal approaches, this Section presents a hybrid method that enables the optimal kinematic design to be implemented by two steps. In the first step, the optimal architecture has been used to formulate a closed-form parametric relationship for minimizing the number of variables. In the second step, the workspace bounded by the specified ITA and CTA is generated which allows a well-shaped and conditioned workspace to be achieved via optimizing a comprehensive index. The Diamond robot will be taken as an example to demonstrate the effectiveness of this method.

Referring to Ref. isotropic configurations shown in Fig. 4a have been studied deeply, by which a set of parametric relationships can be produced as follows:

\[ H = \frac{\sqrt{2}}{2} \sqrt{\delta_2} \left( \frac{\delta_1}{\delta_2} \right) \]

where, \( H \) is a distance from \( O \) to \( P \) when the manipulator is at the optimal configuration (Fig. 4a).

Let \( \delta \) be the ratio of \( l_2 \) to \( l_1 \), meaning that the parametric relationships given by Eq. 19 leaves only one independent geometric parameter, \( \delta \), to be determined as \( l_1 = 1 \). As illustrated in Fig. 4b, for a given \( \delta \), the location, shape and size of the workspace bounded by a specified index \( \xi_{\text{opt}}(\alpha, \beta_{\text{maw}}) \) can be obtained. Here, this set of workspaces is defined as \( W_{\delta} \), with each element \( W_{\delta}(\delta) \) of \( W_{\delta} \) being associated with a given \( \delta \). Given \( \xi_{\text{opt}} \) and \( \delta \), the boundary of \( W_{\delta} \) can be found using the 1-D root search algorithm, the golden-section method, for example.

It is worth pointing out, however, that for the systems with higher DOFs, it is necessary to develop an effective algorithm to find the boundary bounded by the specified \( \xi_{\text{opt}} \). From an application point of view, it is more meaningful for the example robot to possess a well-shaped workspace that can be tailored out of \( W_{\delta}(\delta) \) for a given \( \xi_{\text{opt}} \). A rectangular workspace \( W_{\delta} \) with fixed width/height ratio \( \lambda = b/h \) might be particularly the first priority. It can be seen from Fig. 4b that the shape and location boundary determined by a specified index \( \xi_{\text{opt}} \) are extremely similar.

As shown in Fig. 4b, \( W_{\delta} \) can be defined in such a way that it is tangential to the upper bound of \( W_{\delta}(\delta) \) at the points of \( P_1 \) and \( P_2 \) and intersects with the lower bound of \( W_{\delta}(\delta) \) at the corners \( Q_1 \) and \( Q_2 \), respectively. However, though we can make use of an algorithm method mentioned in to search the tailored workspace, it is obvious that a lot of time will be cost for determining the position of the rectangular workspace \( W_{\delta}(\delta) \) with each given \( \delta \). Therefore, it is better to find some general principle to decide the position of \( W_{\delta}(\delta) \). Let \( \mu \) be the ratio of \( H' \) to \( H \). Given \( \lambda = 4 \), Fig. 5 shows the variations of

![Fig. 4(a-b): (a) Isotropic configurations and (b) 1-D search for the maximum rectangular workspace in b versus \( \delta \) and \( \mu \) bounded by \( [a_{\text{maw}}] = 60^\circ \) and \( [\beta_{\text{maw}}] = 50^\circ \). It can be seen that the changes of \( \delta \) have little effect on the variations of the length of the rectangular workspace. Then, for the purpose of eliminating computable time, a specified \( \mu \) can be given to achieve the maximum value of \( b \). As shown in Fig. 5, \( \mu = 0.6017 \) with \( b = 1.3 \) can be an appropriate choice versus changes in \( \delta \) under the earlier conditions. Then, the position of \( W_{\delta}(\delta) \) can directly be decided by a given \( \delta \) and eventually, the area of the workspace also has been obtained via a specified \( \lambda \).

**Global performance index:** Since the global kinematic performance could not be guaranteed simply by conducting local optimum design, the global optimal method (Ali et al., 2002) has been well accepted and widely employed for the evaluation of the kinematic performance of existing parallel manipulators. The typical work in this phase was proposed by Gosselin and Angeles (Lafourcade et al., 2002) in which a global conditioning...
Fig. 5(a-b): (a) Variations of vs. and (b) Contours of vs. and index represented by the mean value of the reciprocal of conditioning number of the Jacobian was used as a cost function for maximization. In this case, a global and comprehensive performance index for determining δ is proposed as an objective function for maximization. This index can be represented by:

\[ \varepsilon = \sqrt{\frac{1}{\kappa^2} + (\eta \cos \bar{\alpha}_{nm})^2} \]  \hspace{1cm} (20)

where, \( \frac{1}{\kappa} \) is the mean value of the conditioning index \( 1/\kappa \) evaluated in \( W_2(\delta) \), \( \bar{\alpha}_{nm} \) is the mean value of maximum \( \Gamma \) among two limbs in \( W_2(\delta) \); \( \eta \) is the weight being placed upon the ratio of \( \frac{1}{\kappa} \) and \( \cos \bar{\alpha}_{nm} \).

In computer implementation, \( \frac{1}{\kappa} \) and \( \bar{\alpha}_{nm} \) can be numerically calculated by:

\[ \frac{1}{\kappa} = \frac{1}{M \times N} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{\kappa_{mn}}, \quad \bar{\alpha}_{nm} = \frac{1}{M \times N} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{\kappa_{mn}} \] \cos \bar{\alpha}_{nm} \hspace{1cm} (21)

Fig. 6(a-c): Distributions of \( \frac{1}{\kappa} \), \( \alpha_{nm} \) and \( \beta_{nm} \) in a normalized rectangular workspace

where, \( \kappa_{mn} \) and \( (\alpha_{nm})_{mn} \) are the value of \( \kappa \) and \( \alpha_{nm} \) at node (mn) of \( (M-1) \times (N-1) \) equally meshed \( W_2(\delta) \), respectively.

**Example:** Given \( \lambda = 4 \), one case associated with \( [\alpha_{min}] = 60^\circ \) and \( [\beta_{max}] = 50^\circ \) has been considered as examples for the optimum kinematic design of the manipulator. Fig. 6 shows the variations of \( \varepsilon, \frac{1}{\kappa} \) and \( \cos \bar{\alpha}_{nm} \) versus \( \delta \) with \( \eta = 0.8 \). It can be seen that the
With kinematic singularity analysis of translational parallel robot having limbs which is composed just like that used in the Diamond and Delta parallel robot, two kinds of angles have been proposed. This allows evaluating the kinematic performance of parallel robot in a visible manner.

A quickly and simple method for achieving the position of $W_i(\delta)$ has been presented by taking the Diamond robot as an example. By which, the time consumed by searching the maximum tailored workspace has been extremely decreased. To some extent, this method can be very meaningful in the practical design.

A modified global index has been introduced. Maximization this index enable one to achieve a parallel robot which possesses not only a better kinematic performance, but also a good force/motion transmission ratio.

**REFERENCES**


