Some Bounds and Conditional Maximum Bounds for RIC in Compressed Sensing

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Abstract: Compressed sensing seeks to recover an unknown sparse signal with \( p \) entries by making far fewer than \( p \) measurements. The Restricted Isometry Constants (RIC) has become a dominant tool used for such cases since if RIC satisfies some bound then sparse signals are guaranteed to be recovered exactly when no noise is present and sparse signals can be estimated stably in the noisy case. During the last few years, a great deal of attention has been focused on bounds of RIC. Finding bounds of RIC has theoretical and applied significance. In this study we obtain a bound of RIC. It improves the results. Further we discuss the problems related larger bound of RIC and give the conditional maximum bound.

Key words: Compressed sensing, \( L_1 \) minimization, restricted isometry property, sparse signal recovery

INTRODUCTION

Compressed sensing aims to recover high-dimensional sparse signals based on considerably fewer linear measurements. We consider:

\[
y = \Phi \beta + z
\]

(1)

where the matrix \( \Phi \in \mathbb{R}^{m \times n} \) with \( m < n \), \( z \in \mathbb{R}^{m} \) is a vector of measurement errors and the unknown signal \( \beta \in \mathbb{R}^{n} \). Our goal is to reconstruct \( \hat{\beta} \) based on \( y \) and \( \Phi \).

A naive approach for solving this problem is to consider \( L_1 \) minimization where the goal is to find the sparsest solution in the feasible set of possible solutions. However, this is NP hard and is computationally infeasible. It is then natural to consider the method of \( L_1 \) minimization which can be viewed as a convex relaxation of \( L_0 \) minimization. The \( L_1 \) minimization method in this context is:

\[
\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^n} \| y - \Phi \beta \|_1 \text{ subject to } \| y - \Phi \beta \|_2 \leq \epsilon
\]

(2)

This method has been successfully used as an effective way for reconstructing a sparse signal in many settings. (Donoho and Huo, 2001; Donoho, 2006, Candes and Tao, 2005; Candes et al., 2006, Candes and Tao, 2006, 2007; Cai et al., 2010a, b).

Recovery of high dimensional sparse signals is closely connected with Lasso and Dantzig selectors, (Candes and Tao, 2007; Bickel et al., 2009; Wang and Su, 2013a-c). One of the most commonly used frameworks for sparse recovery via \( L_1 \) minimization is the Restricted Isometry Property (RIP) with a RIC introduced by Candes and Tao (2005). It has been shown that \( L_1 \) minimization can recover a sparse signal with a small or zero error under various conditions on \( \delta_k \) and \( \delta_{k-1} \). For example, the condition \( \delta_k + \delta_{k+1} + \delta_{2k-1} \leq 1 \) is used in (Candes and Tao, 2005), \( \delta_k + 3\delta_{2k-2} \leq 2 \) in (Candes et al., 2006), \( \delta_k + \delta_{2k-1} \leq 1 \) in (Candes and Tao, 2007), \( \delta_{1.3k} + \delta_{1.5k} < 1 \) in (Cai et al., 2009) and \( \delta_{1.3k} + \delta_{1.5k} < 1 \) in (Cai et al., 2010b).

The RIP conditions are difficult to verify for a given matrix \( \Phi \). A widely used technique for avoiding checking the RIP directly is to generate the matrix \( \Phi \) randomly and to show that the resulting random matrix satisfies the RIP with high probability using the well-known Johnson-Lindenstrauss Lemma. (Baramiuk et al., 2008). This is typically done for conditions involving only the restricted isometry constant \( \delta_k \). Attention has been focused on \( \delta_k \) as it is obviously necessary to have \( \delta_k < 1 \) for model identifiability. In a recent study, Davies and Gribonval (2009) constructed examples which showed that if \( \delta_k \geq 0.7071 \), exact recovery of certain \( k \) sparse signals can fail in the noiseless case. On the other hand, sufficient conditions on \( \delta_k \) have been given. For example, \( \delta_k < 0.4142 \) is used in (Candes, 2008), \( \delta_k < 0.4531 \) in (Foucart and Lai, 2009), \( \delta_k < 0.4652 \) in (Foucart, 2010), \( \delta_k < 0.4721 \) in (Cai et al., 2010b), \( \delta_k < 0.4734 \) in (Foucart, 2010) and \( \delta_k < 0.4931 \) in (Mo and Li, 2011). Some sufficient conditions on \( \delta_k \) has been given. For example, \( \delta_k < 0.307 \) is used in (Cai et al., 2010c) and \( \delta_k < 0.308 \) in Ji and Peng, (2012) when \( k \) is even. In this study, \( \delta_k < 0.308 \) is given for any \( k \) and the conditional maximum bound \( \delta_k < 0.5 \) is obtained.

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There are several benefits for improving the bound on \( \delta_k \). Firstly, it allows more measurement matrices to be used in compressed sensing. Secondly, for the same matrix \( \Phi \), it allows \( k \) to be larger, that is, it allows recovering a sparse signal with more nonzero elements. Furthermore, it gives better error estimation in a general problem to recover noisy compressible signals.

**PRELIMINARIES**

Let \(|u|\) be the number of nonzero elements of vector \( u = (u_i) \in \mathbb{R}^n \). \( u \) is called \( k \)-sparse if \(|u| \leq k \). For an \( n \times p \) matrix \( \Phi \) and an integer \( 1 \leq k \leq p \), the \( k \) restricted isometry constant \( \delta_k(\Phi) \) is the smallest constant such that:

\[
\sqrt{1 - \delta_k(\Phi)} |u|_2 \leq \| \Phi u \|_2 \leq \sqrt{1 + \delta_k(\Phi)} |u|_2
\]

for every \( k \)-sparse vector \( u \). If \( k + k' \leq p \), the \( k \)- restricted orthogonality constant \( \delta_{k,k'}(\Phi) \) is the smallest number that satisfies:

\[
|\langle \Phi u, \Phi u' \rangle| \leq \delta_{k,k'}(\Phi) |u|_2 |u'|_2
\]

for all \( u \) and \( u' \) such that \( u \) and \( u' \) are \( k \)-sparse and \( k' \)-sparse, respectively, and have disjoint supports. For notational simplicity we shall write \( \delta_k \) for \( \delta_k(\Phi) \) and \( \delta_{k,k'} \) for \( \delta_{k,k'}(\Phi) \) hereafter.

The following monotone properties can be easily checked:

\[
\delta_k \leq \delta_k', \text{ if } k \leq k' \leq p
\]

\[
\delta_{k,k'} \leq \delta_{k',k}, \text{ if } k \leq k' \leq j \text{ and } j + j' \leq p
\]

Candes and Tao (2005) showed that the constants and are related by the following inequalities:

\[
\delta_{k,k'} \leq \delta_{k,k}^* \leq \delta_{k,k'} + \max(\delta_{k,k'}, \delta_{k',k})
\]

Cai et al. (2010b) showed that for any \( a \geq 1 \) and positive integers \( k, k' \) such that \( ak' \) is an integer, then:

\[
\delta_{a,k'} \leq \sqrt{a} \delta_{k,k'}
\]

Cai et al. (2010c) showed that for any \( x \in \mathbb{R}^n \):

\[
\|x\|_1 \leq \frac{\|x\|_2}{\sqrt{2}} \leq \frac{\sqrt{n}}{4} (\|x\|_1 - \min \|x\|_1)
\]

Where:

\[
\|x\|_1 = \max_{i \in [n]} |x_i| \text{ and } \|x\|_\infty = \min_{i \in [n]} |x_i|
\]

**NEW RIC BOUNDS OF COMPRESSED SENSING MATRICES**

In this section, we consider new RIP conditions for sparse signal recovery. Suppose:

\[
y = \Phi \psi + z
\]

with \( \|z\|_1 \leq c \). Denote \( \hat{\beta} \) the solution of the following \( L_1 \) minimization problem:

\[
\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^p} \|\beta\|_1 \text{ subject to } \|y - \Phi \beta\|_2 \leq \epsilon
\]

(10)

The following is one of our main results of the study.

**Theorem 1:** Suppose \( \beta \) is \( k \)-sparse with \( k > 1 \). Then under the condition:

\[
\delta_k < 0.308
\]

the constrained \( L_1 \) minimizer \( \hat{\beta} \) given in (10) satisfies:

\[
\|\beta - \hat{\beta}\|_2 \leq \frac{6}{0.308 - \delta_k}
\]

In particular, in the noiseless case \( \hat{\beta} \) recovers \( \beta \) exactly.

This theorem improves \( \delta_k < 0.307 \) in Cai et al., 2010c to \( \delta_k < 0.308 \) and \( k \) is even in Ji and Peng, 2012 to any \( k \). The proof of the theorem is very long but elementary.

**Proof:** Let \( s, k \) be positive integers, \( 1 \leq s < k \) and:

\[
t = \frac{k}{k-s} \leq \frac{1 + \sqrt{5}}{4}
\]

Then from Theorem 3.1 in Cai et al., 2010c, under the condition \( \delta_k + \delta_{k,s} < 1 \), we have:

\[
\|\beta - \hat{\beta}\|_2 \leq \frac{\sqrt{5 + 1 - \delta_k}}{1 - \delta_k - \delta_{k,s}}
\]

(8)

By (8):

\[
\delta_{k,s} = \delta_{k,s}^{(k-s)} \leq \sqrt{\frac{k}{k-s} - \delta_k} \leq \sqrt{\frac{k}{k-s} \delta_k}
\]

(9)

We show below that:

\[
\|x\|_1 - \max_{i \in [n]} |x_i| \text{ and } \|x\|_\infty - \min_{i \in [n]} |x_i|
\]
\[
\sqrt{\frac{k}{k-s}} - \left( \sqrt{\frac{k}{k}} + 4\sqrt{k} \right) = 1 - \frac{5}{4} + \frac{\sqrt{\pi}}{4} \neq f(x)
\]

(12)

\[
f\left( \frac{4}{5} \right) = f\left( \frac{k-1}{k+1} \right) = f(0)
\]

(15)

Where:

\[
x = \frac{s}{k-s}
\]

The proof is of elementary trigonometric functions, but it is very clever.

Let \( s = k \sin^2 \alpha, \alpha \in (0, \frac{\pi}{2}) \), then \( k-s = k \cos^2 \alpha \)

So:

\[
\sqrt{\frac{k}{k-s}} - \left( \sqrt{\frac{k}{k}} + 4\sqrt{k} \right) = \frac{1}{\sin \alpha} \left( \frac{\sqrt{k}}{\sqrt{4}} + \frac{s}{4} \right)
\]

\[
= \frac{1}{\sin \alpha} \left( \frac{s}{4} + \frac{5}{4} \sin \alpha \right) = \frac{1}{\sqrt{k}} \left( \frac{5}{4} + \frac{\sqrt{s}}{4} \right)
\]

It is easy to see \( f(x) \) is increasing when:

\[
x \geq \frac{4}{5}
\]

and decreasing when:

\[
x \leq \frac{4}{5}
\]

Thus \( f(x) \) obtains the minimum value:

\[
f\left( \frac{4}{5} \right) = \sqrt{\frac{5}{2}}
\]

(18)

That is, if \( k = 0(\text{mod} 9) \), let

\[
s = \frac{4}{9} k
\]

then under the condition \( \delta_k < 0.309 \) we have, see (Cai et al., 2010c):

\[
\| \beta - \hat{\beta} \|_2 \leq \frac{e}{0.309 - \delta_k}
\]

(13)

If \( k \) is even, let \( s = \frac{k}{2} \), then:

\[
f(1) = 2.250
\]

(14)

If \( k \neq 9 \) is odd, let \( s = \frac{k-1}{2} \), then:

Since \( f(x) \) is increasing when:

\[
x \geq \frac{4}{5}
\]

When \( k = 7 \), then:

\[
f(2) = \frac{31\sqrt{3}}{24} = 2.237
\]

When \( k = 5 \), then:

\[
f(3) = \frac{11}{24} = 2.245
\]

When \( k = 3 \), we note from the remark of Theorem 3.1 in (Cai et al., 2010c) that in these cases \( s = 1 \) and \( t = \sqrt{k} \), then:

\[
t = \frac{k}{\sqrt{k-s}} = \sqrt{\frac{5}{2}} = 2.212
\]

From (11-18) yield:

\[
\delta_k < 0.309 \Rightarrow \frac{3.25\delta_k}{\delta_k} < 1
\]

if \( k \) is even and:

\[
\delta_k + 0.1 < 0.309 \Rightarrow \frac{3.25\delta_k}{\delta_k} < 1
\]

if \( k \) is odd. With the above relations we can also get:

\[
\| \beta - \hat{\beta} \|_2 \leq \frac{2\sqrt{\frac{5}{2}}} {1 - \delta_k - 0.309 - \delta_k} \leq \frac{e}{0.309 - \delta_k}
\]

Corollary 1: Suppose \( \beta \) is \( k \) sparse with \( k \equiv 0(\text{mod} 9) \). Then under the condition \( \delta_k < 0.309 \) the constrained \( L_1 \) minimizer \( \hat{\beta} \) given in (10) satisfies:

\[
\| \beta - \hat{\beta} \|_2 \leq \frac{e}{0.309 - \delta_k}
\]

In particular, in the noiseless case \( \beta \) recovers \( \beta \) exactly.

The proof sees (11-13):
Corollary 2: Suppose β is k sparse. If k > 9 is odd, then under the condition δ_β < c_β the constrained L₁ minimizer $\hat{\beta}$ given in (10) satisfies:

$$|\beta - \hat{\beta}|_2 \leq \frac{\varepsilon}{\delta_\beta}$$

Where:

$$\delta_\beta = \frac{4\sqrt{k^2 - 1}}{4\sqrt{k^2 - 1} + 9k - 1}$$

In particular, in the noiseless case $\hat{\beta}$ recovers $\beta$ exactly. The proof sees (11-12) and (15). Note that 0.308 < $c_\beta$ < 0.309 from (15).

To the best of our knowledge, this seems to be the first result for sparse recovery with conditions that only involve $\delta_\beta$ and k. In fact, only involving $\delta_\beta$, k and only involving $\delta_\beta$ are equivalent.

THE CONDITIONAL MAXIMUM BOUND FOR RIC

Let $h = \beta - \hat{\beta}$. For any subset $Q \subset \{1, 2, ..., p\}$ we define $h_Q = h1_Q$, where $1_Q$ denotes the indicator function of the set Q, i.e., $1_Q(j) = 1$ if $j \in Q$ and 0 if $j \not\in Q$. Let T be the index set of the k largest elements (in absolute value) and let $\Omega$ be the support of $\beta$. The following fact which is based on the minimality of $\beta$, has been used widely (Candes et al., 2006):

$$|h_{\Omega}| \geq |h_T|$$

We shall show that:

$$|h_T| \geq |h_{\Omega}|$$

and:

$$|\Omega| = |T|$$

In fact:

$$|h_T| = |h_{\Omega}| + |h_{\bar{\Omega}}| = |h_{\Omega}| + |h_{\bar{\Omega}}|$$

and $T$ has the k largest elements (in absolute value) and $\Omega$ has at most k elements, so we have by (19):

$$|h_T| \geq |h_{\Omega}| \geq |h_{\Omega}|$$

And:

$$|h_{\bar{\Omega}}| \leq |h_{\Omega}|$$

Definition 1: Let $T_m$ be the index set of the m largest elements (in absolute value). The set $T_m$ is called a sparse index set, if:

$$|h_{T_m}| \geq |h_{\Omega}|$$

and $m \leq k$.

It is obvious that the sparse index set exists. In fact $T_k$ is a sparse index set since:

$$|h_{T_k}| \geq |h_{\Omega}|$$

Here we prove that any sparse index set $T_m$ instead of $T_k$, Theorem 3.1 in (Cai et al., 2010b) can be improved.

Theorem 2: Suppose $\beta$ is k-sparse and $T_m$ is sparse index set. Let $k_i$, $k_o$ be positive integers such that $k_o \leq m$ and $8(k_o - m) < k_i$. Let:

$$t = \frac{k_o}{k_i} + \frac{1}{4} \frac{k_o}{k_i}$$

Then under the condition $\delta_{k_i} + t\delta_{k_o} < k_i$ the L₁ minimizer defined in (10) satisfies:

$$|h_{\Omega}| \leq \frac{2\sqrt{\delta_{k_i} + \delta_{k_o}}}{1 - \delta_{k_i} - \delta_{k_o}}$$

In particular, in the noiseless case where $y = \Phi \beta$, L₁ minimization recovers $\beta$ exactly.

Proof: For any sparse index set $T_m$, let $S_k \supseteq T_m$ be the index set of the k largest elements (in absolute value). Rearrange the indices of $S_k$ if necessary according to the descending order of $|h_i|$, i.e., $S_k \supseteq \cdots$. Partition $S_k$ into:

$$S_k = \sum_{l=1}^{E_k} S_l$$

where $|S_k| = k_i$, the last $S_k$ satisfies $|S| \leq k_i$, If $h_{\Omega} = 0$, then the theorem is trivially true. So here we assume that $h_{\Omega} \neq 0$. Then it follows from (9) that:

$$\sum_{l=1}^{E_k} |h_l| \leq \frac{1}{\sqrt{k_i}} \sum_{l=1}^{E_k} |h_l| + \frac{1}{2} \sum_{l=1}^{E_k} \left(|h_l| - |h_{\Omega}|\right)$$

$$\leq \frac{1}{\sqrt{k_i}} \sum_{l=1}^{E_k} |h_l| + \frac{1}{2} \sum_{l=1}^{E_k} |h_{\Omega}|.$$
\[
\begin{aligned}
&= \frac{1}{\sqrt{K_{1}}} \| h_{b} \| + \frac{\sqrt{K_{2}}}{4} \| h_{e} \|

&= \frac{1}{\sqrt{K_{1}}} \left( \| h_{b} \| - \| h_{e} \| \right) + \frac{\sqrt{K_{2}}}{4} \| h_{e} \|

&= \frac{1}{\sqrt{K_{1}}} \left( \| h_{b} \| - 2 \| h_{e} \| \right) + \frac{\sqrt{K_{2}}}{4} \| h_{e} \|

&= \frac{1}{\sqrt{K_{1}}} \left( \| h_{b} \| - 2(k_{1} - m) \| h_{e} \| \right) + \frac{\sqrt{K_{2}}}{4} \| h_{e} \|

&= \frac{1}{\sqrt{K_{1}}} \left( \sqrt{K_{2}} \frac{2(k_{1} - m)}{\sqrt{K_{1}}} \| h_{e} \| \right) + \frac{\sqrt{K_{2}}}{4} \| h_{e} \|

&\leq \left( \frac{\sqrt{K_{1}}}{\sqrt{K_{2}}} + \frac{\sqrt{K_{2}}}{4\sqrt{K_{1}}} - \frac{2(k_{1} - m)}{\sqrt{K_{2}}} \right) \| h_{e} \| - 1 \| h_{e} \|^{2}

\end{aligned}
\]

Note that:
\[
\| \Phi \|_{2} \leq \| \Phi \lambda_{1} - y \|_{2} + \| \Phi \lambda_{1} - y \|_{2} \leq 2\varepsilon
\]

\[
\| \Phi \lambda_{1} - y \|_{2} \leq \| \Phi \lambda_{1} \| \| \lambda_{1} \|_{2} \leq 2\varepsilon \| \lambda_{1} \|_{2}
\]

Also the next relation:
\[
\| h_{e} \|_{2} \leq \| h_{e} \|_{2} \leq \| h_{e} \|_{2} \leq \| h_{e} \|_{2} \leq \| h_{e} \|_{2}
\]

implies:
\[
\| h_{e} \|_{2} \leq \| h_{e} \|_{2} \leq \| h_{e} \|_{2} \leq \| h_{e} \|_{2} \leq \| h_{e} \|_{2}
\]

Putting them together we get:
\[
\| \Phi \|_{2} \leq \frac{2\sqrt{\| \lambda \|_{2}}}{\sqrt{\| \lambda \|_{2}}} \leq 2\varepsilon \| \lambda \|_{2}
\]

If let \( m = k \), then Theorem 2 is Theorem 3.1 in (Cai et al., 2010c).

Let \( m\leq m \) be smallest positive integer so that:
\[
\| h_{e} \|_{2} \leq \frac{\| h_{e} \|_{2}}{m}
\]

Then we have:

**Theorem 3**: Suppose \( \beta \) is \( k \)-sparse. Let \( k_{1}, k_{2} \) be positive integers such that \( k_{1} \geq k \geq m \) and \( 8(k_{1} - m) \leq k_{2} \). Let:
\[
t = \frac{k_{1}}{\sqrt{K_{1}}} + \frac{1}{4\sqrt{K_{1}}} - \frac{2(k_{1} - m)}{\sqrt{K_{2}}} \frac{k_{2}}{K_{2}}
\]

Then under the condition \( \delta_{k} + t \delta_{k_{1}, k_{2}} < 1 \) the \( L_{1} \) minimizer defined in (10) satisfies:
\[
\| \beta - \hat{\beta} \|_{2} \leq \frac{2\sqrt{2\| \beta \|_{2} + \delta_{k}}}{1 - \delta_{k} - t \delta_{k_{1}, k_{2}}}
\]

In particular, in the noiseless case where \( y = \Phi \beta \), \( L_{1} \) minimization recovers \( \beta \) exactly.

The proof is similar to that of Theorem 2.

Note that \( k_{1} \) is independent of \( h \) but \( m \) and \( m_{1} \) are dependent on \( h \), i.e., \( m = m(h) \) and \( m_{1} = m_{1}(h) \).

The following is one of our main results of the study. It is the consequence of Theorem 2.

**Theorem 4**: Suppose \( \beta \) is \( k \)-sparse with \( k \geq 1 \). If \( k \equiv 0 \pmod{5} \) and \( T_{5} \) is sparse index set, then under the condition \( \delta_{k} < 0.5 \) the constrained \( L_{1} \) minimizer \( \hat{\beta} \) given in (10) satisfies:
\[
\| \beta - \hat{\beta} \|_{2} \leq \frac{\sqrt{\| \beta \|_{2} + \delta_{k}}}{0.5 - \delta_{k}}
\]

In particular, in the noiseless case \( \hat{\beta} \) recovers \( \beta \) exactly.

**Proof**: If \( k \equiv 0 \pmod{5} \) and \( T_{5} \) is sparse index set, then in Theorem 2, set:
\[
k_{1} = \frac{k}{5}, \quad k_{2} = \frac{4k}{5}
\]

Then:
\[
t = \frac{k_{1}}{\sqrt{K_{1}}} + \frac{1}{4\sqrt{K_{1}}} - \frac{2k_{1}}{\sqrt{K_{2}}} \frac{k_{2}}{K_{2}} = 1
\]

Thus:
\[
t = \frac{k_{1}}{\sqrt{K_{1}}} + \frac{1}{4\sqrt{K_{1}}} - \frac{2k_{1}}{\sqrt{K_{2}}} \frac{k_{2}}{K_{2}} = 1
\]
\[
\delta_0 + \theta \frac{\theta_0}{\delta_0} < 1
\]

we have:

\[
\|\beta - \hat{\beta}\|_2 \leq \frac{2\sqrt{2} \sqrt{\delta_0 + \theta}}{1 - \delta_0 \frac{\theta_0}{\delta_0}} \frac{1}{\sqrt{5}}
\]

By (5) and (7) we get:

\[
\delta_0 + \theta \frac{\theta_0}{\delta_0} \leq 2\delta_0 < 1
\]

In this case:

\[
\|\beta - \hat{\beta}\|_2 \leq \frac{2\sqrt{2} \sqrt{\delta_0}}{1 - \delta_0 \frac{\theta_0}{\delta_0}} \frac{1}{\sqrt{5}} \leq \frac{\sqrt{3}}{0.5 - \delta_0}
\]

An explicitly example in (Cai et al., 2010c) is constructed in which \(\delta_0 < 0.5\), but it is impossible to recover certain \(k\) sparse signals. Therefore, the bound for \(\delta_0\) cannot go beyond 0.5 in general in order to guarantee stable recovery of \(k\) sparse signals.

**CONCLUSION**

We recognized that \(\|h_s\|_{(1)}\) may be greater than \(\|h_1\|_{(1)}\) too much. Since \(\|h_s\|_{(1)} (1 \leq s \leq k)\) all may be greater than \(\|h_1\|_{(1)}\) and \(\|h_1\|_{(1)}\) is the largest of \(\|h_s\|_{(1)} (1 \leq s \leq k)\). We want to find a \(\|h_0\|_{(1)} (1 \leq s \leq k)\) such that \(\|h_s\|_{(1)} < \|h_0\|_{(1)}\). On the other hand, the bound in (11) is function of \(\delta_0\). This makes the bound more tight since \(\delta_0\) is fixed. So we propose an idea. That is, the bound in right side hand is function of \(\delta_0\), where \(s \leq k\). From \(\Omega\) and \(T\) immediately deduce four index sets \(\Omega \cap \Omega^\top\), \(\Omega \cap T\), \(\Omega \cap T^\top\) and \(\Omega \cap \Omega^\top\). And \(m_s = |\Omega \cap \Omega^\top| = k^4|\Omega \cap T|\), \(m_2 = |\Omega \cap T\|, m_3 = |\Omega \cap T^\top|\), \(m_4 = |\Omega \cap \Omega^\top|\).

It is easy to show that the bound of Theorem 2 is tighter than the one in (Cai et al., 2010c) under special cases. See the following examples.

**Example 1**: Suppose \(\beta\) is \(k\)-sparse and \(n > 0\). Let:

\[
t_1 = \frac{\sqrt{n}}{\sqrt{n} + \frac{1}{\sqrt{n}}}
\]

If \(\Omega = T\), then under the condition \(\delta_0 + \theta_0 \frac{\theta_0}{\delta_0} < 1\) the \(L_1\) minimizer defined in (10) satisfies:

\[
\|\beta - \hat{\beta}\|_1 \leq \frac{2\sqrt{2} \sqrt{\delta_0 + \theta}}{1 - \delta_0 \frac{\theta_0}{\delta_0}} \frac{1}{\sqrt{5}} \leq \frac{\sqrt{3}}{0.5 - \delta_0} < 1
\]

In particular, in the noiseless case where \(y = \Phi \beta\), \(L_1\) minimization recovers \(\beta\) exactly.

In fact, the proof is similar to of Theorem 2 and note that:

\[
\|h_s\|_{(1)} + \|h_1\|_{(1)} \leq \|h_1\|_{(1)} < 2\|h_1\|_{(1)}
\]

**Example 2**: Suppose \(\beta\) is \(k\)-sparse and \(n \geq 0\), where \(k\) is even. Let:

\[
t_1 = \frac{\sqrt{k}}{\sqrt{n} + \frac{1}{\sqrt{n}}}
\]

If \(|\Omega \cap T| = k/2\), then under the condition:

\[
\delta_0 + \theta_0 \frac{\theta_0}{\delta_0} < 1
\]

the \(L_1\) minimizer defined in (10) satisfies:

\[
\|\beta - \hat{\beta}\|_1 \leq \frac{4\sqrt{2} \sqrt{\delta_0 + \theta}}{1 - \delta_0 \frac{\theta_0}{\delta_0}} \frac{1}{\sqrt{5}} \leq \frac{\sqrt{3}}{0.5 - \delta_0} < 1
\]

The proof is similar to of Theorem 2.

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