A Characterization of Alternating Group $A_{28}$ by Conjugate Class Sizes

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ABSTRACT
For a group, let $N(G)\{n\}$ conjugate class sizes of order $n$ in $G$. The groups $A_{27}$, $A_{24}$, $A_{23}$ are characterized by $N(G)$ only. If $5 \mid p$ belong to the set of element orders of $G$, then whether are the alternating groups $A_{23}$ characterized by $N(G)$. In this study, finite simple classification theorem and the properties of the set $N(G)$ was used to characterize alternating group $A_{28}$, namely, we will prove that if $G$ is a finite group with trivial center and $N(G) = N(A_{28})$, then $G$ is isomorphic to $A_{28}$.

Key words: Element order, alternating group, thompson’s problem, conjugate class size, simple group

INTRODUCTION
All groups in this study considered are finite and simple groups mean simple non-abelian groups. Denote the alternating and symmetric groups of degree $n$ by $A_n$ and $S_n$, respectively. Set $\text{Aut}(G)$ denotes the automorphism group of a group $G$. Let $\text{o}(G)$ denote the set of element order of $G$. Denote the set of nonidentity orders of conjugate classes of elements in $G$ by $N(G)$. The other notations are standard (Conway et al., 1985).

With respect to $N(G)$, Thompson gave the following known conjecture. Thompson’s Conjecture (Mazurov and Kukhro, 2010). If $L$ is a finite simple non-Abelian group, $G$ is a finite group with trivial center and $N(G) = N(L)$, then $G = L$.

For a finite group $G$, we set $\pi(G) = \pi(|G|)$. Let, $\text{GK}(G)$ be a graph with vertex set $\pi(G)$ such that two primes $p$ and $q$ in $\pi(G)$ are joined by an edge if $G$ has an element of order $pq$. We set $s(G)$ denote the number of connected components of the prime graph $\text{GK}(G)$. A classification of all finite simple groups with disconnected prime graph was obtained in Kondrat’ev (1989) and Williams (1981) study. Based on these results, Thompson’s conjecture was proved valid for all finite simple groups with $s(G) \leq 2$ (Griyun, 1996; Chen, 1999). So whether there is a group with connected prime graph for which Thompson’s conjecture would be true? Recently, the groups $A_{10}$, $A_{16}$, and $A_{22}$ were proved valid for this conjecture (Vasil’ev, 2009; Gorskhov, 2012, Xu, 2013). Whether is there a group $A_{25}$ characterized by $N(G)$ in connection to Thompson’s Conjecture? In this study, we give an example for $A_{27}$, namely, it will be proved that if $G$ is a finite group with trivial center and $N(G) = N(A_{27})$, then $G$ is isomorphic to $A_{27}$.

MATERIAL AND METHODS
Some preliminary results are given in this section.

Lemma 1: Let $x$, $y \in G$, $(|x|, |y|) = 1$ and $xy = yx$. Then:
1. $C_0(xy) \subseteq C_0(x) \cap C_0(y)$
2. $|x^p|$ divides $|xy|^p$
3. If $|x| = |(xy)^p|$, then $C_0(x) \subseteq C_0(y)$

Proof: See Lemma 1.2 of Vasil’ev (2009) and Lemma 2.3 of Ahanjideh and Ahanjideh (2013).

Lemma 2: If $P$ and $H$ are finite groups with trivial centers and $N(P) = N(H)$, then $\pi(P) = \pi(H)$.


Lemma 3: Suppose that $G$ is a finite group with trivial center and $p$ is a prime from $\pi(G)$ such that $p^2$ does not divide $|x^p|$ for all $x$ in $G$. Then a Sylow $p$-subgroup of $G$ is elementary abelian.

Lemma 4: Let, K be a normal subgroup of G and $\bar{G} = G/K$:

- If $\bar{x}$ is the image of an element x of G in $\bar{G}$. Then $|\bar{x}|$ divides $|x|$
- If $(|a|, |K|) = 1$, then $C_G(\bar{x}) = xK/K$
- If $y \in K$, then $|y|$ divides $|y^0|$


Lemma 5: Let $L = A_{20}$. Then the following hold:

- $|L| = 2^{24}.3^{11}.5^6.7^4.11^2.13^2.17.19.23$
- The following numbers from N(L) are maximality with respect to divisibility:
  - 219.31.5^6.7^4.11^2.13.17.23.19.23, 219.31.5^6.7^4.11^2.13^2.17.23.19.23
  - 219.31.5^6.7^4.11^2.13^2.17.19.23, 219.31.5^6.7^4.11^2.13^2.17.23.19.23
  - 219.31.5^6.7^4.11^2.13^2.17.19.23, 219.31.5^6.7^4.11^2.13^2.17.23.19.23

23'-numbers in N(L)\{1\} are:

- 24.3^3.5^6.7^4.11^2.13.17.19.23, 24.3^5.7.13, 3^3.5^3.7.13, 2^5.3^2.7.13

19'-numbers in N(L)\{1, 23'-number\} are:

- 24.3^2.5^6.7^4.11^2.13.17.23.19.23, 24.3^2.5^6.7^4.11^2.13^2.17.23

23'-numbers in N(L)\{1\} are:

- 24.3^2.5^6.7^4.11^2.13^2.17.23, 24.3^2.5^6.7^4.11^2.13^2.17.23

17'-numbers in N(L)\{1, 19'-number, 23'-number\} are:

- 23.3^3.5^6.7^4.11^2.13.17.23, 23.3^3.5^6.7^4.11^2.13^2.17.23

Step 1: Let $p \in \{17, 19, 23\}$. Then the Sylow p-subgroup S of G is of order $p$. There is no elements of order 17.19, 19.23 and 19.23.

Proof: By Lemma 2, we have that $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$. The desired result was obtained by first showing that there is no elements of order 17.19, 19.23 and 19.23 see proving that if K is a maximal normal subgroup of G, then K is a [2,3]-group, in particular, G is insoluble and third getting that G is isomorphic to $A_{20}$.

RESULTS AND DISCUSSION

In this section, the main theorem and its proof is given.

Theorem: Let, G be a finite group with trivial center and N(G) = N(A_{20}). Then G is isomorphic to A_{20}.

Proof: By Lemma 2, we have that $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$.

The desired result was obtained by first showing that there is no elements of order 17.19, 19.23 and 19.23 see proving that if K is a maximal normal subgroup of G, then K is a [2,3]-group, in particular, G is insoluble and third getting that G is isomorphic to $A_{20}$.
In these four cases, \(2^{20} \cdot 3^{13} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23 \), which contradicts Lemma 5.  

Set 23 \(|x|\). Then we write \(|x| = 23\ m\). Since \(S\) is elementary abelian, then the numbers 23 and \(m\) are coprime. Let \(u = x^{23}\) and \(v = x^{m}\). Then \(x = uv\) and \(C_{d}(x) = C_{d}(u) \circ C_{d}(v)\).  

Therefore \(|v||x| = 2^{20} \cdot 3^{13} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23\). On the other hand, \(|v| = 23\) and so by Lemma 5, \(|v|^{3}\) is equal to \(2^{20} \cdot 3^{13} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23\). It follows that \(13|\langle x \rangle|\), a contradiction.  

Assume that \(p = 19\) and \(|S| \geq 19^{2}\). Then exists an element \(x\) of \(G\) such that \(|x| = 2^{20} \cdot 3^{13} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 17^{19}\) by Lemma 5.  

Set 19\(|x|\). Let \(y\) be an element of \(C_{d}(x)\) having order 17. Since \(S\) is elementary abelian and \(|y| = 17\), then \(|y|\) is a 17-number. Therefore \(|y|\) equals to:  

\[
2^{20} \cdot 3^{13} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23.
\]

Therefore \(13|\langle y \rangle|\), a contradiction.  

Set 19\(|x|\). Then let \(|x| = 19\ m\). Since \(S\) is elementary abelian, then the numbers 19 and \(m\) are coprime. Let \(u = x^{19}\) and \(v = x^{m}\). Then \(x = uv\) and \(C_{d}(x) = C_{d}(u) \circ C_{d}(v)\) and \(|v| = 19\). By Lemma 5, \(|v|\) equals to:  

\[
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23.
\]

It follows that \(13|\langle y \rangle|\), a contradiction.  

Assume that \(p = 17\) and \(|S| \geq 17^{2}\). Then by Lemma 5, there exists an element \(x\) of \(G\) such that \(|x| = 2^{20} \cdot 3^{13} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 17^{19} \cdot 23\).  

Set 17\(|x|\). Let \(y\) be an element of \(C_{d}(x)\) having order 17. Since \(S\) is elementary abelian and \(|y| = 17\), then \(|y|\) is a 17-number. By Lemma 5, \(|y|\) equals to:  

\[
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23.
\]

We also have \(13|\langle y \rangle|\), a contradiction.  

Set 17\(|x|\). Then let \(|x| = 17\ m\). Since \(S\) is elementary abelian, then the numbers 17 and \(m\) are coprime. Let \(u = x^{17}\) and \(v = x^{m}\). Then \(|v| = 17\), \(x = uv\) and \(C_{d}(x) = C_{d}(u) \circ C_{d}(v)\). Therefore \(|u|^{3} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23\). On the other hand, \(|v|\) is 17-number and so \(|v|\) is equal to:  

\[
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23, \\
2^{20} \cdot 3^{5} \cdot 5^{5} \cdot 7^{4} \cdot 11^{2} \cdot 13^{2} \cdot 17^{19} \cdot 23.
\]
2. 3. 4. 5. 7. 11. 13. 17. 23. 2. 3. 13. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.
2. 3. 5. 7. 11. 13. 17. 19. 23. 2. 3. 5. 7. 11. 13. 17. 19. 23.

It follows that \( |V_0^p| = |x^p| \), a contradiction.

Therefore, the Sylow \( p \)-subgroup is of order \( p \) for \( p \in \{17, 19, 23\} \).

There is no element of order 17. 19. 23 and 17. 23.

**Step 2:** Let \( \pi = \{2, 3, 5\} \). Then \( \text{O}_{\pi}(\text{G}) = \text{O}_{\pi}(\text{G}) \). In particular, \( \text{G} \) is insoluble.

Let \( K = \text{O}_{\pi}(\text{G}) \), \( \text{G} = \text{G}/\text{K} \) and denote by \( x^k \) and \( y^k \) the images of an element \( x \) and a subgroup \( H \) of \( \text{G} \) in \( \text{G} \), respectively. If the result is invalid, then there exists \( p \in \pi \) for which \( \text{O}_p(\text{G}) \) is not 1.

Let \( \text{O}_p(\text{G}) = 1 \) for \( p \in \{17, 19, 23\} \). Then \( \text{G} \) contains a Hall \( \{p, q\}-\text{subgroup of order } p \), where \( p \in \{17, 19, 23\} \) and \( q \in \{p\} \). This subgroup must be cyclic and so there is an element of order \( p \) for which contradicts Step 1.

Let \( \text{P} \) be a Sylow 13-subgroup of \( \text{G} \). If \( \text{O}_{13}(\text{G}) = 1 \), then \( \text{A} = \text{Z}(\text{O}_{13}(\text{G})) \) is a nontrivial normal subgroup of \( \text{G} \). Let \( x^k \) be an element of order 23 in \( \text{G} \). Then \( |(x^k)^p| \) is a divisor of \( 23^4 \). 13. 5. 7. 11. 13. 17. 19. 23. 3. 5. 7. 13, or 2. 3. 7. 13. It is easy to get \( \text{A} = \text{C}_A(x^k) \). Hence the index of \( \text{C}_A(x^k) \) is at most 13. Obviously \( n = 11 \) is the least number with that \( 23|13^2 \) and so \( |A, x^k| \) is abelian. It means that \( |A, x^k| = 1 \) and \( A = \text{C}_A(x^k) \). Let \( z^k \) be a nontrivial element of \( (p) \text{n}\text{A} \), then \( 23 \mid |C_p(z^k)| \). By Lemma 4, the preimage \( z \) in \( G \) lies in the center of the Sylow 13-subgroup, contradicting Step 1. So \( \text{O}_{13}(\text{G}) = 1 \).

Let \( \text{P} \) be a Sylow 7-subgroup of \( \text{G} \). If \( \text{O}_{7}(\text{G}) = 1 \), then \( \text{A} = \text{Z}(\text{O}_{7}(\text{G})) \) is a nontrivial normal subgroup of \( \text{G} \). Let \( x^k \) be an element of order 23 in \( \text{G} \). Then \( |(x^k)^p| \) is a divisor of \( 23^4 \). 3. 5. 7. 11. 13. 17. 19. 23. 3. 5. 7. 13, or 2. 3. 7. 13. It is easy to get \( \text{A} = \text{C}_A(x^k) \). Hence the index of \( \text{C}_A(x^k) \) is at most 13. Obviously \( n = 11 \) is the least number with that \( 23|13^2 \) and so \( |A, x^k| \) is abelian. It means that \( |A, x^k| = 1 \) and \( A = \text{C}_A(x^k) \). Let \( z^k \) be a nontrivial element of \( (p) \text{n}\text{A} \), then \( 23 \mid |C_p(z^k)| \). By Lemma 4, the preimage \( z \) in \( G \) lies in the center of the Sylow 7-subgroup, contradicting Step 1. So \( \text{O}_{7}(\text{G}) = 1 \).

Therefore \( \text{O}_{13}(\text{G}) = \text{O}_{7}(\text{G}) \). In particular, \( \text{G} \) is insoluble.

**Step 3:** \( G \) is isomorphic to \( \text{A}_{\pi} \).

By Lemma 2, \( \pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\} \) and \( \text{O}_{\pi}(\text{G}) \) is a maximal normal soluble subgroup of \( \text{G} \) with \( \pi = \{2, 3, 5\} \).

From Step 2, \( \text{G} \) is insoluble and so there is a normal series \( K \leq H \leq G \) such that \( H/K \) is a direct product of nonabelian simple groups isomorphic to the groups listed in (Zavarnitsine, 2003), namely, \( H/K = S_8 \times S_8 \times \cdots \times S_8 \). We know that \( G \) cannot contain a Hall \( \{17, 19, 23\}-\text{subgroup, the numbers } 17, 19 \) and 23 divide the order of exactly one of these groups and so assume that \( 17, 19, 23 \mid |\text{S}_8| \). Therefore, \( S_8 \leq G \). If \( K \geq 1 \), then there is a Sylow 7-subgroup \( P_7 \) of \( \text{G}/\text{S}_8 \). Let \( M = H/K \) and \( Z = Z(P_7) \). Then \( Z/M \leq \text{S}_8 \) is nontrivial. Consider an element \( x \) of \( S_8 \times \cdots \times S_8 \) such that its image in \( G \) lies in \( Z \). Since \( x \) centralizes \( S_8 \), then \( x \in Z \) and so centralizes an element of order 23, a contradiction.

So, we have \( H/K = S_8 \). We know that \( H/K \cup \leq \text{Aut}(H/K) \). Since \( 17, 19, 23 \mid |\text{G}| \), then \( H/K \) is isomorphic to \( A_{\pi} \), with \( n = 23, 24, 25, 26, 27, 28 \). Thus \( A_{23} \leq G \). By Lemma 4, \( \pi = \{2, 3, 5\} \), a contradiction.

Let \( H/K = A_{\pi} \). Then there exists an element \( x^k \) of order 23 in \( H/K \) such that \( |(x^k)^p| = 2^{22} \times 3^4 \times 5^2 \times 7^4 \times 11^2 \times 13^2 \times 17 \times 19 \) and \( |C_{23}(x^k)| = 3 \times 23 \). Let \( x \) be an element of order 23.
in H corresponding to $x^*$, by Lemmas 4 and 5, $|x^*| |x| = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ and $|C_{G}(x)| = 5.23 \cdot 5.23 |C_{H}(x)|$. Therefore by Lemma 4:

$$3.23 \geq \frac{5.23}{|K \cap C_{H}(x)|}$$

a contradiction.

Let $H/K = A_{23}$ or $S_{23}$. Then there exists an element $x^*$ of order 23 in $H/K$ such that $|x^*| = 2^{11} \cdot 3^{11} \cdot 5^{11} \cdot 7^{11} \cdot 11^{11} \cdot 13^{11} \cdot 17^{11} \cdot 19$ and $|C_{H}(x^*)| = 2^{11} \cdot 3^{11}$. Let $x$ be an element of order 23 in H corresponding to $x^*$, by Lemmas 4 and 5, $|x| = 2^{11} \cdot 3^{11} \cdot 5^{11} \cdot 7^{11} \cdot 11^{11} \cdot 13^{11} \cdot 17^{11} \cdot 19$ and $|C_{G}(x)| = 5.23 \cdot 5.23 |C_{H}(x)|$. Therefore by Lemma 4:

$$2^{11} \cdot 23 \geq \frac{5.23}{|K \cap C_{H}(x)|}$$

It follows that $5|K \cap C_{H}(x)|$, in particular $x$ centralizes an element of order 23 in $H/K$, a contradiction.

Let $H/K = S_{23}$. Then there is an element $x^*$ of order 23 such that $|x^*| = 2^{11} \cdot 3^{11} \cdot 5^{11} \cdot 7^{11} \cdot 11^{11} \cdot 13^{11} \cdot 17^{11} \cdot 19$, a contradiction. Hence $H/K$ is isomorphic to $A_{23}$. Now we prove $K = 1$. If $G = A_{23}$, then we refine the normal series $1 < K < G$ into the chief ones. Let $K$ be a nontrivial p-group with $p \in \{2, 3, 5\}$.

Let $p = 2$. Then $G = K \cdot A_{23}$. If the former, there is an element of order 23 such that $2^{11} \cdot 3^{11} \cdot 5^{11} \cdot 7^{11} \cdot 11^{11} \cdot 13^{11} \cdot 17^{11} \cdot 19$, a contradiction. If the latter, then by Lemma 2.7 of Jiang et al. (2011) also there is an element of order 23 such that $2^{11} \cdot 3^{11} \cdot 5^{11} \cdot 7^{11} \cdot 11^{11} \cdot 13^{11} \cdot 17^{11} \cdot 19$, a contradiction.

For $p = 3, 5$, we also can similarly rule out the case $p = 2$. Therefore $K = 1$ and $G = A_{23}$.

This completes the proof.

CONCLUSION

Using the properties of $N(A_{23})$, we proved that the group $A_{23}$ is characterizable by the set of its conjugate classes sizes. As it was proved that the group $A_{10}$ can be characterized by its order and two special conjugacy classes sizes. Then obviously, we also have the following result.

Corollary 7. Let $G$ be a finite group with trivial center. Assume that $N(G) = N(A_{23})$ and $|G| = |A_{23}|$. Then $G = A_{23}$.

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