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Research Article
Milne's Implementation on Block Predictor-corrector Methods

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Abstract
Milne's implementation on block predictor-corrector methods for integrating nonstiff ordinary differential equations is been considered. The introduction of Milne's implementation attracts a lot of computational benefits, which guarantees step size variation, convergence criteria and error control. Existence and uniqueness for the nonstiff problems were recognized. The approach was employ Milne's implementation of the principal local truncation error on a pair of predictor-corrector method of Adams type formulas, which is implemented either in $P(EC)^m$ or $P(EC)^m E$ mode. The implementation of Milne's estimate and evaluation of the block method for nonstiff ODEs was analyzed in details. In addition, an algorithm for the implementation of the method was specified.

Key words: Step size variation, block method, predictor-corrector methods, convergence criteria, principal local truncation error

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INTRODUCTION

Several techniques have been formulated to yield global error estimation according to Dormand[1]. A distinctive approach, frequently adopted if local error control is used called tolerance reduction. This relies on the presumption of tolerance balance being correct. In solving a differential equations over the required interval, a new result is achieved employing a decreased or increased tolerance. The deviations in the result, obtained at like points can be used to approximate the global error.

Computational methods for providing solution of ODEs of initial value type are commonly divided as single-step or multistep processes. From each one has its pros and cons and many numerical analyst favours one or the other technique. Moreover, such a choice may originate from the needs of the problem being worked out. Authors viewed generally that several types of numerical methods had better be equated to the user aims[1].

The initial value problem of a first-order differential Eq. 1 of the form is consider as:

$$Y(x) = f(x, y),\ y(a) = \alpha,\ x \in [a, b] \text{ and } f: R \times R^m \rightarrow R^m \quad (1)$$

Eq. 1 is generally written as in Eq. 2:

$$\sum_{i=1}^{j} \alpha_i y_{n+i} = h \sum_{i=1}^{j} \beta_i f_{n+i} \quad (2)$$

where, h is the step size, α_i, i = 1,...j, β_j are unknown constants, which are uniquely specified such that the equation is of order j as discussed[2].

It is assumed that $f \in R$ is sufficiently differentiable on $x \in [a, b]$ and satisfies a global Lipschitz condition, i.e., there is a constant $L \geq 0$ such that:

$$|f(x, y) - f(x, \bar{y})| \leq L|y - \bar{y}|, \forall y, \bar{y} \in R$$

Under this presumption, Eq. 1 assured the existence and uniqueness defined on $x \in [a, b]$ as well as viewed to fulfill the Weierstrass theorem[3-5].

Where, a and b are finite and $y^{(i)}\ [y^{(i)}_1, y^{(i)}_2, ..., y^{(i)}_n]^T$ for i = 0 (1) 3 and $f = [f_1, f_2, ..., f_n]^T$, originate in real life applications for problems in science and engineering, such as fluid dynamics and motion of rocket as presented by Mehrkanoon *et al.*[6].

However, Adesanya *et al.*[7], Ehigie *et al.*[8], Fatunla[9], Ismail *et al.*[10], James *et al.*[11] and Ken *et al.*[12] proposed block multistep methods, which were employed in predictor-corrector mode. Block multistep methods have the vantage of evaluating simultaneously at all points with the integration interval, thereby reducing the computational burden when evaluation is required at more than one point within the grid. Again, Taylor series expansion is used to provide the initial values in order to compute the corrector.

Scholars have suggested block predictor-corrector methods for the numerical solution of nonstiff and mildly stiff ODEs in the simple form of Adams type combined with $P(EC)^m$ or $P(EC)^m E$ mode implemented using variable step size appear[5,13-16]. Nevertheless, their implementation was geared towards Backward Differentiation Formula (BDF). This study presents Milne's implementation on block predictor-corrector method for solving nonstiff ODEs of Eq. 1 founded on variable step size technique implemented in $P(EC)^m$ or $P(EC)^m E$ mode. This technique comes with many numerical advantages as expressed in the abstract.

A block-by-block method is a method for computing vectors Y_0, Y_1, in sequence. Let the r-vector (r is the number of points within the block) Y_μ, F_μ and G_μ for n = mr, m = 0, 1,. . . be given as $Y_w = (y_{n+1}, ..., y_{n+r})^T$, $F = (f_{n+1}, ..., f_{n+r})^T$ then the l-block r-point methods for Eq. 1 are given by:

$$Y_w = \sum_{i=1}^{j} A^{(i)} Y_{w-i} + h \sum_{i=1}^{j} B^{(i)} F_{w-i}$$

where, $A^{(i)}$, $B^{(i)}$, i = 0,..., j are r by r matrices[2,9,17].

Thus, from the above explanation, a block method has the numerical advantage that in each practical application, the solution is estimated at more than one point concurrently. The number of points depends on the construction of the block method. Hence, employing these methods can give quicker and faster solutions to the problem, which can be managed to generate the desired accuracy[6,15]. Therefore, the objective of this study is to propose Milne's implementation of block predictor-corrector methods for solving nonstiff and mildly stiff ODEs implemented in $P(EC)^m$ or $P(EC)^m E$ mode adopting variable step size technique. This technique possess the following vantages like designing a suitable step size/changing the step size, specifying the convergence criteria (tolerance level) and error control/minimization as well as addressing the gaps stated above.

MATERIALS AND METHODS

In this section according to Akinfenwa *et al.*[2], the main aim is to derive the principal implicit block method of the Eq. 2. It is proceed forward by seeking an approximation of the exact solution y(x) by assuming a continuous solution Y(x) of the Eq. 3:

$$Y(x) = \sum_{i=0}^{q+k-1} d_i\ {}_i(x) \tag{3}$$

where, $x \in [a, b]$, d_i are unknown coefficients and $\vartheta_i(x)$ are polynomial basis functions of degree q+k-1, where, q is the number of interpolation point and k is the collocation points, respectively chosen to satisfy $q = j \geq 3$ and $k>1$. The integer $j \geq 1$ denotes the step number of the method. Thus, it is construct a j-step implicit block multistep method with $\vartheta_i(x) = \left(\frac{x - x_i}{h}\right)^i$ by imposing the following Eq. 4 and 5:

$$\sum_{i=0}^{q} d_i \left(\frac{x - x_i}{h}\right)^i = y_{n-i}, i=0,\dots,q-1 \tag{4}$$

$$\sum_{i=0}^{q} d_i^i \left(\frac{x - x_i}{h}\right)^{i-1} = f_{n+i},\ i \in Z \tag{5}$$

where, y_{n+1} is the approximation for the exact solution $y(x_{n+1})$, $f_{n+1} = f(x_{n+1}, y_{n+1})$, n is the grid index and $x_{n+1} = x_n+ih$. It should be observed that Eq. 4 and 5 leads to a system of q+1 Eq. 6 of the AX = B where:

$$A = \begin{vmatrix} x_n^0 & x_n^1 & x_n^2 & x_n^3 & x_n^4 & \cdots & x_n^q \\ & & \cdots & \cdots & & & \\ x_{n-k}^0 & x_{n-k}^1 & x_{n-k}^2 & x_{n-k}^3 & x_{n-k}^4 & & x_{n-k}^q \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n-k}^3 & 0 & 0\ 0\ 0 & 0 \\ & & & \cdots & \cdots & & \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n-k}^3 & k(k-1)(k-2)x_{n-k}^4 & \cdots & k(k-1)(k-2)x_{n-k}^q \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n+1}^3 & k(k-1)(k-2)x_{n+1}^4 & \cdots & k(k-1)(k-2)x_{n+1}^q \\ & & & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n+k}^3 & k(k-1)(k-2)x_{n+k}^4 & \cdots & k(k-1)(k-2)x_{n+k}^q \end{vmatrix}$$

$$\begin{aligned} X &= [x_0, x_1, x_2, x_3\dots, x_k]^T \\ U &= [f_n, f_{n-1}, \dots, f_{n-k-1}, f_{n+1}, f_{n+2}, \dots, f_{n+k}, y_n, y_{n-1}, \dots, y_{n-k-1}]^T \end{aligned} \tag{6}$$

Solving Eq. 6 using mathematica, we get the coefficients of d_i and substituting the values of d_i into Eq. 4 and after some algebraic computation, the implicit block multistep method is obtain Eq. 7 as:

$$\sum_{i=0}^{q-1} \alpha_i y_{n-i} = h\left[\sum_{i=0}^{q-1} \beta_i f_{n-1} + \sum_{i=0}^{q-1} \beta_i f_{n+1}\right] \tag{7}$$

where, α_i and β_i are continuous coefficients.

General overview of the block predictor and corrector methods: Assuming P defines the application program of the block predictor, C a block corrector application program, with E as the evaluation application program of f with respect to given values of its parameter. If $y_{n+k}^{(0)}$ is computed from the block predictor, $f_{n+k}^{(0)} \equiv f(x_{n+k}\, y_{n+k}^{(0)})$ is calculated one time and employ the corrector at one time as well to obtain $y_{n+k}^{(1)}$; this describe the computation as PEC. Further appraisal of $f_{n+k}^{(1)} \equiv f(x_{n+k}\, y_{n+k}^{(1)})$ succeeded by another application program of the corrector gives $y_{n+k}^{(2)}$ and thus, denoted by $PEC^{(2)}$. Implementing the application program of the block corrector m many times can be referred to as $PEC^{(m)}$. Since m is constant, $y_{n+k}^{(m)}$ is accepted as the computational solution at x_{n+k}. At this point, the last computational value for f_{n+k} is preferred as $f_{n+k}^{(m-1)} \equiv f(x_{n+k}\, y_{n+k}^{(m-1)})$ and this will be further decided whether or not to execute $f_{n+k}^{(m)} \equiv f(x_{n+k}\, y_{n+k}^{(m)})$. Assuming this concluding execution is done, the mode is denoted by $P(EC)^m$ or $P(EC)^m$ E. Eventually the decision clearly impacts the next step of the execution, when both predicted and corrected numerical values for y_{n+k+1} will rely on whether f_{n+k} is accepted as $f_{n+k}^{(m)}$ or $f_{n+k}^{(m-1)}$. Finally, for a given m, $P(EC)^m$ or $P(EC)^m$ E mode utilize the corrector the same number of times; only $P(EC)^m$ E requires one more evaluation per step than $P(EC)^m$ as discussed[4, 18].

Theorem 1: If the multistep method (2) is convergent for pth order equations, then the order of (2) is at least p[19].

Theorem 2: The order of a predictor-corrector method for first order equations must be ≥ 1 if it is convergent[19].

Theorems 1 and 2 draws the conclusion that the order and convergence of the method holds.

Implementation on Milne's block predictor-corrector methods: Holding to[3,4,18], the implementation in the $P(EC)^m$ or $P(EC)^m$ E mode becomes significant for the explicit (predictor) and implicit (corrector) methods if both are separately of like order and this requirement makes it necessary for the stepnumber of the explicit (predictor) method to be one step higher than that of the implicit (corrector) method. Consequently, the mode $P(EC)^m$ or $P(EC)^m$ E can be formally examined as follows for m = 1, 2,....$P(EC)^m$:

$$y_{n+j}^{[0]} + \sum_{i=0}^{j-1} \alpha_i^{\bullet} y_{n+i}^{[m]} = h\sum_{i=0}^{j-1} \beta_i^{\bullet} f_{n+i}^{[m-1]}$$

$$f_{n+j}^{[s]} \equiv f(x_{n+j}, y_{n+j}^{[s]})$$

$$y_{n+j}^{[s+1]} + \sum_{i=0}^{j-1} \alpha_i y_{n+i}^{[m]} = h\beta_j f_{n+j}^{[s]} + h\sum_{i=0}^{j-1} \beta_i f_{n+i}^{[m-1]} \} s=0,1,\dots,m-1 \tag{8}$$

$P(EC)^m E$:

$$y_{n+j}^{[0]} + \sum_{i=0}^{j-1} \alpha_i^{\bullet} y_{n+i}^{[m]} = h\sum_{i=0}^{j-1} \beta_i^{\bullet} f_{n+i}^{[m]}$$

$$f_{n+j}^{[s]} \equiv f(x_{n+j}, y_{n+j}^{[s]})$$

$$y_{n+j}^{[s+1]} + \sum_{i=0}^{j-1} \alpha_i y_{n+i}^{[m]} = h\beta_j f_{n+j}^{[s]} + h\sum_{i=0}^{j-1} \beta_i f_{n+i}^{[m]} \} s = 0, 1, \ldots, m-1$$

$$f_{n+j}^{[s]} \equiv f(x_{n+j}, y_{n+j}^{[s]})$$

Noting that as $m \to \infty$, the result of evaluating with either of the above mode will slope to those given by the mode of correcting to convergence.

Moreover, predictor and corrector pair based on Eq. 1 can be applied. The mode $P(EC)^m$ or $P(EC)^m E$ specified by Eq. 8, where, h is the step size. Since the predictor and corrector both have the same order p, Milne's device is applicable and relevant.

The following theorem demonstrate adequate condition for the convergence of $P(EC)^m$ or $P(EC)^m E$.

Theorem 1: Let $\{y_{n+1}^{[m]}\}$ be a sequence of approximations of y_{n+1} obtained by a PECE... method. If:

$$\left| \frac{\partial f}{\partial y}(x_{n+1}, y) \right| \le L$$

(for all y near y_{n+1} including $y_{n+1}^{[0]}, y_{n+1}^{[1]}$) where, L is satisfies the condition $L < \frac{1}{|h\beta_0|}$, then the sequence $\{y_{n+1}^{[m]}\}$ converges to y_{n+1}.

Proof: The numeric solution satisfies the equation:

$$y_{n+1} = \sum_{i=0}^{j-1} \alpha_i y_{n+i} + h\beta_0 f(x_{n+1}, y_{n+1}) + h\sum_{i=0}^{j-1} \beta_i f_{n+i}$$

The corrector satisfies the equation:

$$y_{n+i}^{(m+1)} = \sum_{i=0}^{j-1} \alpha_i y_{n+i} + h\beta_0 f(x_{n+1}, y_{n+i}^{(m)}) + h\sum_{i=0}^{j-1} \beta_i f_{n+i}$$

Subtracting these two equations, it is obtain:

$$y_{n+1} - y_{n+1}^{(m+1)} = h\beta_0 [|f(x_{n+1}, y_{n+1}) - f(x_{n+1}, y_{n+1}^{(m)})|]$$

Applying the Lagrange mean value theorem to arrive at:

$$y_{n+1} - y_{n+1}^{(m+1)} = h\beta_0 (y_{n+1} - y_{n+1}^{(m)}) \frac{\partial f}{\partial y}(x_{n+1}, y^*)$$

where, $y_{n+1}^{(m)} \le y^* \le y_{n+1}$. Thus,

$$\left|y_{n+1} - y_{n+1}^{(m+1)}\right| \le |h\beta_0| \left|y_{n+1} - y_{n+1}^{(m)}\right| \left|\frac{\partial f}{\partial y}(x_{n+1}, y)\right|$$

$$\le hL|\beta_0| \left|y_{n+1} - y_{n+1}^{(m)}\right|$$

$$\le [hL|\beta_0|]^m \left|y_{n+1} - y_{n+1}^{(0)}\right|$$

Now,

$$\lim_{m \to \infty} \left| y_{n+1} - y_{n+1}^{(m+1)} \right| \to 0, \text{ if}$$

$$hL|\beta_0| < 1 \text{ or } L < \frac{1}{h|\beta_0|}$$

This means that the conclusion of theorem 1 holds[3]. In cases, where, C_{p+1}, C^*_{p+1} are the computed error constant of the predictor-corrector method, respectively. The following consequence holds.

Proposition: Suppose the predictor method have order p* and the corrector method have order p. Then: If $p^* \ge p$ (or $p^* < p$ with $m > p-p^*$), then the predictor-corrector methods possesses the same order and the same PLTE as the corrector. If $p^* < p$ and $m = p-p^*$), then the predictor-corrector method possesses the same order as the corrector, but different PLTE. If $p^* < p$ and $m \le p-p^*-1$, then the predictor-corrector method possesses the same order equal to p^*+m (thus less than p).

Specifically, it is observe that suppose the predictor has order p-1 and the corrector has order p, the PEC answers to get a method of order p. Moreover, the $P(EC)^m$ or $P(EC)^m E$ scheme has always the same order and the same PLTE[4,20].

Combining[4,18,21], Milne's device stated that it is viable to estimate the principal local truncation error of the explicit and implicit (predictor-corrector) method without estimating higher derivatives of y(x). Assuming that $p = p^*$, where, p* and p defines the order of the explicit (predictor) and implicit (corrector) methods with the same order. Directly, for a method of order p, the principal local truncation errors can be written as in Eq. 9:

$$C^*_{p+1}\, h^{p+1}\, y^{(p+1)}\,(x_n) = y(x_{n+j})\text{-}W_{n+j}\text{+}O(h^{p+2}) \quad (9)$$

Also in Eq. 10:

$$C_{p+1}\, h^{p+1}\, y^{(p+1)}\,(x_n) = y(x_{n+j})\text{-}C_{n+j}\text{+}O(h^{p+2}) \quad (10)$$

where, W_{n+j} and C_{n+j} are called the predicted and corrected approximations given by method of order p while, C^*_{p+1} and C_{p+1} are independent of h.

Neglecting terms of degree p+2 and above, it is easy to make estimates of the principal local truncation error of the Eq. 11 as:

$$C_{p+1}\, h^{p+1}\, y^{(p+1)}(x_n) = \frac{C_{p+1}}{C^*_{p+1} - C_{p+1}}\left|W_{n+j} - C_{n+j}\right| < \varepsilon \quad (11)$$

Noting the fact that $C_{p+1} \neq C^*_{p+1}$ and $W_{n+j} \neq C_{n+j}$.

However, the estimate of the principal local truncation error Eq. 11 is used to determine, whether to accept the results of the current step or to reconstruct the step with a smaller step size. The step is accepted based on a test as prescribed by Eq. 11 as[22]. Equation 11 is the convergence criteria otherwise called Milne's estimate for correcting to convergence.

Furthermore, Eq. 11 ensures the convergence criterion of the method during the test evaluation.

Algorithm: A written algorithm that will design a new step size and evaluate the maximum errors of the predictor-corrector methods in the form of $P(EC)^m$ or $P(EC)^m$ E mode, if the mode is run m times:

Step 1: Choose a step size for h

Step 2: The order of the predictor-corrector methods must be the same

Step 3: The step number of predictor method must be one step higher than the corrector method

Step 4: State the principal local truncation errors of both the predictor-corrector methods

Step 5: Define the tolerance level (Convergence criteria)

Step 6: Input the predictor-corrector methods in any mathematical language

Step 7: Use any one step method to generate starting values in needed, if not ignore step 6 and proceed to step 7

Step 8: Implement the $P(EC)^m$ or $P(EC)^m$ E mode as m increases

Step 9: If step 7 fails to converge, repeat the process again and divide the step size (h) by 2 from step 0 or if not, proceed to step 9

Step 10: Evaluate the maximum errors after convergence is reached

Step 11: Print maximum errors

Step 12: Use this formula stated below to design a new step size after converge is reached:

$$qh = \left|\frac{\varepsilon}{2(C^*_{p+1} - C_{p+1})}\right|^{\frac{1}{4}}$$

Milne's implementation approach is a collection of Adams type of the predictor-corrector methods, which can be implemented in $P(EC)^m$ or $P(EC)^m$ E mode[1,4,13,21-23]. All of these sited above favour the implementation of Milne's approach for solving nonstiff ODEs.

Moreover, in its implementation, the predictor-corrector methods have the same order, thus, demand that the stepnumber of the predictor to be one step higher than the corrector method. The principal local truncation error of both the predictor-corrector methods are considered in the construction of Milne's implementation for the evaluation of maximum errors. Again, evaluation of Milne's implementation is achieved with aid of the convergence criteria. This convergence criteria decide, whether the result is accepted or repeated as seen in the algorithm.

Nevertheless, on block predictor-corrector methods[7-12], do not possess the same attributes of Milne's implementation approach, which includes varying the step size, deciding the convergence criteria for accepting the results, designing a suitable step size, controlling the error, implementing $P(EC)^m$ or $P(EC)^m$ E mode and lastly, Adams type of the predictor-corrector methods. These are in contradiction to the implementation of the block predictor-corrector methods.

CONCLUSION

Milne's implementation approach is seen as an extension of the block predictor-corrector methods because of certain parameters, which are utilized for the implementation. In addition, the implementation of this method comes with many computational advantages as mention previously in Milne's and Gear's implementations. Finally, the algorithm provides a vital step for the successful implementation of Milne's estimate.

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