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Research Article
Bayesian Study Using MCMC of Gompertz Distribution Based on Interval Censored Data with Three Loss Functions

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Abstract
Interval censored data consist of upper and lower bounds of failure time when the event cannot be observed directly but can only be determined between interval inspection times. The analyzing interval censored data has been developed because it is very common of the medicine and reliability field. The study describes estimation of the Bayesian study using Markov Chain Mote Carlo of the Gompertz distribution under interval censored data, where the full conditional distributions for the parameters, survival function and hazard function are obtained via Metropolis- Hastings algorithm with three loss functions, the Square Error loss function, the Linear Exponential loss function and General Entropy loss function. The methods are compared to maximum likelihood estimation with respect to the Mean Square Error (MSE) and absolute bias to determine the best estimating of the scale and shape parameters, survival function and hazard function of the Gompertz distribution.

Key words: Gompertz distribution, Bayesian method, interval censored data, Markov chain monte carlo, loss function

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

The Gompertz distribution can be used as a survival and hazard model in medicine science, reliability and life testing. The Gompertz distribution was first introduced by Gompertz (1825) and many researchers have contributed the distribution to the statistical model, Ahuja and Nash (1967), Makany (1991) and Franses (1994). Ananda *et al.* (1996) estimated parameters and survival function of the Gompertz distribution by using Bayesian methods. Al-Hussaini *et al.* (2000) estimated the survival and hazard functions of a finite mixture of two Gompertz distribution under type I and II censored data, using the maximum likelihood estimation and Bayesian approach. Jaheen (2003) obtained the tow parameters of the Gompertz distribution by maximum likelihood estimation and Bayesian methods under square error and Linex loss function. Soliman *et al.* (2012b) obtained for the two parameter Gompertz distribution with progressive first-failure censored data.

In the interval censoring, the Lindsey (1998) consider study for interval censoring in parametric regression models. Scallan (1999) estimated the parameters of the Weibull distribution based on interval censored failure time data. Flygare *et al.* (1985) estimated the scale and shape parameters of Weibull distribution based on interval data.

The Metropolis-Hasting algorithm is considered a general Monte Carlo Markov chain algorithm method that was developed by Hastings (1970). It can be used to acquire random samples from any type of randomly difficult target distribution with any type of dimension that is known up to a normalizing constant, Soliman *et al.* (2012a). Upadhyay and Gupta (2010) discussed some Bayes analysis via Markov Chain Monte Carlo technique for complete samples and independent vague priors for the unknown parameters.

The objective of this paper is to estimate the parameters, the survival function and the hazard function of the Gompertz distribution based interval censored data by using Bayesian approach via Markov Chain Monte Carlo technique and compared to maximum likelihood estimator by using Mean Square Error (MSE) and absolute bias to determine the best estimator under several conditions.

MATERIALS AND METHODS

Maximum likelihood estimation: The probability density function of Gompertz distribution parameters λ and β is:

$$f(x;\lambda,\beta)=\lambda e^{\beta x}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x}\right)\right)$$

The cumulative distribution function (cdf):

$$F(x;\lambda,\beta)=1-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x}\right)\right)$$

The likelihood function of Gompertz distribution based on interval censored data is shown below:

$$L(\lambda,\beta\,|\,l_i,u_i)=\prod_{i=1}^{n}[F(u_i;\lambda,\beta)-F(l_i;\lambda,\beta)]$$

$$=\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right] \quad (1)$$

See Flygare *et al.* (1985).

The logarithm of the likelihood function of Gompertz distribution is:

$$\ln L(\lambda,\beta\,|\,l_i,u_i)=\sum_{i=1}^{n}\log\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right] \quad (2)$$

Differentiating Eq. 2 partially with respect to the parameters λ and β and equaling to zero. The resulting equations are given, respectively, as:

$$\frac{\partial L(\lambda,\beta\,|\,l_i,u_i)}{\partial\lambda_i}=\sum_{i=1}^{n}\frac{1}{\beta}\left(1-e^{\beta l_i}\right)\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\frac{1}{\beta}\left(1-e^{\beta u_i}\right)\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)/D_i \quad (3)$$

$$\frac{\partial L(\lambda,\beta\,|\,l_i,u_i)}{\partial\lambda_i}=\sum_{i=1}^{n}\frac{1}{\beta}\left(1-e^{\beta l_i}\right)\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\frac{1}{\beta}\left(1-e^{\beta u_i}\right)\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)/D_i \quad (4)$$

Here:

$$D_i=\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)$$

Equation 3 and 4 cannot be solved analytically and therefore we employed Newton Raphson method to find the numerical solution.

The estimate of the survival function and hazard function of Gompertz distribution are:

$$\hat{S}_M(t) = \exp\left(\frac{\hat{\lambda}_M}{\hat{\beta}_M}\left(1-e^{\hat{\beta}_M l_i}\right)\right) - \exp\left(\frac{\hat{\lambda}_M}{\hat{\beta}_M}\left(1-e^{\hat{\beta}_M u_i}\right)\right) \quad (5)$$

$$\hat{h}_M(t) = \hat{\lambda}_M\left(e^{\hat{\beta}_M l_i} - e^{\hat{\beta}_M u_i}\right) \quad (6)$$

Here, $\hat{\lambda}_M$ is the scale parameter of Gompertz distribution estimated by maximum likelihood estimator and the $\hat{\beta}_M$ is the shape parameter of Gompertz distribution estimated by maximum likelihood estimator.

Bayesian estimations: We consider the scale and shape parameters are unknown and we compute the Bayesian estimation of the scale and shape parameters. It is assumed that λ and β each have independent gamma priors as follows:

$$g_1(\lambda/a, b) = \lambda^{a-1}\exp(-b\lambda)$$

$$g_2(\beta/c, d) = \beta^{c-1}\exp(-d\beta)$$

The posterior of Gompertz distribution based on interval censored data is given as:

$$\prod{}_1(\lambda,\beta \mid l_i, u) = \frac{L(\lambda,\beta \mid l_i, u_i)\, g_1(\lambda/a,b)\, g_2(\beta/c,d)}{\int_0^\infty\int_0^\infty L(\lambda,\beta \mid l_i,u_i)\, g_1(\lambda/a,b)\, g_2(\beta/c,d)\, d\lambda d\beta}$$

$$= \frac{1}{J_1}\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right) - \exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta)) \quad (7)$$

Here:

$$J_1 = \int_0^\infty\int_0^\infty\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right) - \exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))\, d\lambda d\beta$$

Loss functions: A wide variety of loss functions have been found in literature review to describe various types of loss structures. In this study, we describe three loss functions: The symmetric loss function, square error loss function and the asymmetric loss functions are Linear Exponential loss (LINEX) and General Entropy loss function.

Square error loss function: The square error loss function was used to estimate the scale and shape parameters of Gompertz distribution as given, respectively, below:

$$\hat{\lambda}_S = \frac{1}{J_1}\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\lambda\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right) - \exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (8)$$

$$\hat{\beta}_S = \frac{1}{J_1}\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\beta\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right) - \exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (9)$$

The Bayesian estimates for the survival and hazard functions under squared error loss function are given as:

$$\hat{S}_S(t) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta t}\right)\right)\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right) - \exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]$$
$$\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (10)$$

$$\hat{h}_S(t) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\lambda e^{\beta t}\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right) - \exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right] \quad (11)$$
$$\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta$$

Equation 8-11, we can't solve it analytical for that we used Metropolis Hastings Algorithm to estimate the scale and shape parameters and survival and hazard functions of Gompertz distribution.

Linear exponential loss function (LINEX): The Linear Exponential loss function is under the assumption that the minimal loss occurs at $\hat{\lambda} = \lambda$ and is expressed as:

$$L(\Delta) = \exp(r\Delta) - r\Delta - 1,\ r \neq 1 \quad (12)$$

with $\Delta = (\hat{\lambda}-\lambda)$, $\hat{\lambda}$ is an estimate of λ. Of r>1 means overestimation and underestimation of c<1. For c close to zero the Linear Exponential loss function approximated the square error loss function.

The posterior expectation of Linear Exponential loss function (LINEX) in Eq. 12 is:

$$E_\lambda\left(L(\hat{\lambda}-\lambda)\right) \propto \exp(r\hat{\lambda})\, E_\lambda\left(\exp(r\hat{\lambda})\right) - r\left(\hat{\lambda} - E_\lambda(\lambda)\right) - 1 \quad (13)$$

Therefore, the Bayesian estimation of scale parameter of Gompertz distribution under LINEX loss function is given as follows:

$$\hat{\lambda}_L = -\frac{1}{r}\ln E_r\left(\exp(-r\lambda)\right)$$

$$E_r\left(\exp\,(-r\lambda)\right) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\exp\,(-r\lambda)\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (14)$$

The Bayesian estimation of shape parameter of Gompertz distribution under LINEX loss function is:

$$\beta_L = -\frac{1}{r}\ln E_r\left(\exp\,(-r\beta)\right)$$

$$E_r\left(\exp\,(-r\beta)\right) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\exp\,(-r\beta)\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp\,(-(b\lambda+d\beta))d\lambda d\beta \quad (15)$$

The survival function under LINEX loss function is shown below:

$$\hat{S}_L(t) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\exp\left(-r\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta t}\right)\right)\right)\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (16)$$

The hazard function is:

$$\hat{h}_L(t) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\exp\left(-r\lambda e^{\beta t}\right)\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp\,(-(b\lambda+d\beta))d\lambda d\beta \quad (17)$$

Equation 14-17 under LINEX loss function can't solve it analytical for that we used Metropolis Hastings Algorithm.

Therefore, the Algorithm used to generate MCMC sample under LINEX loss function to estimate the parameters, survival function and hazard function of Gompertz distribution.

General entropy loss function: The second asymmetric loss function is the General Entropy loss function which is a generalization of the entropy loss and as shown:

$$L(\hat{\lambda}-\lambda) \propto \left(\frac{\hat{\lambda}}{\lambda}\right)^k - k\ln\left(\frac{\hat{\lambda}}{\lambda}\right) - 1$$

Therefore, the Bayesian estimation of scale parameter of Gompertz distribution under General Entropy loss function is given as follows:

$$\hat{\lambda}_{BG} = \left(E_\lambda(\lambda^{-k})\right)^{-\frac{1}{k}}$$

$$E_\lambda(\lambda^{-k}) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\lambda^{-k}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (18)$$

The Bayesian estimation of shape parameter of Gompertz distribution under General Entropy loss function is given as:

$$\hat{\beta}_{BG} = \left(E_\beta(\beta^{-k})\right)^{-\frac{1}{k}}$$

$$E_\lambda(\beta^{-k}) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\prod_{i=1}^{n}\beta^{-k}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (19)$$

The Bayesian estimates for the survival and hazard functions under General Entropy loss function are given as:

$$\hat{S}_B(t) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\left[\exp\left(-k\frac{\lambda}{\beta}\left(1-e^{\beta t}\right)\right)\right]^{-k}\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (20)$$

$$\hat{h}_B(t) = \frac{1}{J_1}\int_0^\infty\int_0^\infty\left(\lambda e^{\beta t}\right)^{-k}\prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))d\lambda d\beta \quad (21)$$

As show in the Eq. 19-21 of the scale and shape parameters it's not follow any close distribution for that we suggest to use the Metropolis- Hastings algorithm to generate MCMC sample.

Metropolis-hastings algorithms: The full conditional of the posterior density function using gamma prior of λ and β given the data are combining the gamma prior with likelihood as given below:

$$\Pi_S(\lambda,\beta \mid l_i,u_i) \propto \lambda^{a-1}\beta^{c-1} \prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right]\exp(-(b\lambda+d\beta)) \quad (22)$$

From Eq. 22 we can get the conditional posterior of the scale parameter λ as follows:

$$\Pi_S(\lambda \mid \beta;l_i,u_i) \propto \lambda^{a-1}\exp(-b\lambda) \prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right] \quad (23)$$

The conditional posterior of the shape parameter β is given below:

$$\Pi_S(\beta \mid \lambda;l_i,u_i) \propto \beta^{c-1}\exp(-d\beta) \prod_{i=1}^{n}\left[\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta l_i}\right)\right)-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta u_i}\right)\right)\right] \quad (24)$$

As show in Eq. 23 and 24 the conditional posterior of the scale and shape parameters it's not follow any close distribution therefore we suggest to use the Metropolis Hastings algorithm to generate MCMC sample as shown in Algorithm 1.

Algorithm 1:

- Start with initial value
- The current value and generate the candidate value from arbitrary distribution uniform (0, 1)
- The next value of is given below as:

$$\lambda_{i+1} = \begin{cases} \lambda^* \text{ with probability p} \\ \lambda_i \text{with probability 1-p} \end{cases}$$

Where:

$$p = \min\left\{1,\ \frac{\Pi(\lambda^* \mid \beta;l_i,u_i)}{\Pi(\lambda_i \mid \beta;l_i,u_i)}\frac{q(\lambda^* \mid \beta;l_i,u_i)}{q(\lambda_i \mid \beta;l_i,u_i)}\right\}$$

- Generate u from uniform (0, 1) and accept λ* with probability p if λ*<p and return to step 2, otherwise accept λ_i
- The next value of is given below as:

$$\beta_{i+1} = \begin{cases} \beta^* \text{with probability p} \\ \beta_i \text{with probability 1-p,} \end{cases}$$

Where:

$$p = \min\left\{1,\ \frac{\Pi(\beta^* \mid \lambda;l_i,u_i)}{\Pi(\beta_i \mid \lambda;l_i,u_i)}\frac{q(\beta^* \mid \lambda;l_i,u_i)}{q(\beta_i \mid \lambda;l_i,u_i)}\right\}$$

- Generate u from Uniform (0, 1) and accept β* with probability p if β*<p and return to step 2, otherwise accept β and return to step 2
- The Bayesian based interval censored data of the scale and shape parameters under the squared error loss function is given as:

$$\hat{E}_{S1}(\lambda \mid \beta;l_i,u_i) = \frac{1}{m}\sum_{i=1}^{m}\lambda_i$$

$$\hat{E}_{S2}(\beta \mid \lambda;l_i,u_i) = \frac{1}{m}\sum_{i=1}^{m}\beta_i$$

- The Bayesian based interval censored data of the scale and shape parameters under the LINEX loss function is given as:

$$\hat{E}_{r1}(\lambda \mid \beta;l_i,u_i) = -\frac{1}{r}\ln\left(\frac{\sum_{i=1}^{m}\exp(-r\lambda_i)}{m}\right)$$

$$\hat{E}_{r2}(\beta \mid \lambda;l_i,u_i) = -\frac{1}{r}\ln\left(\frac{\sum_{i=1}^{m}\exp(-r\beta_i)}{m}\right)$$

- The Bayesian based interval censored data of the scale and shape parameters under the General Entropy loss function is given as follow:

$$\hat{E}_{g1}(\lambda \mid \beta;l_i,u_i) = \left(\frac{\sum_{i=1}^{m}\lambda_i^{-k}}{m}\right)^{-\frac{1}{k}}$$

$$\hat{E}_{g2}(\beta \mid \lambda;l_i,u_i) = \left(\frac{\sum_{i=1}^{m}\beta_i^{-k}}{m}\right)^{-\frac{1}{k}}$$

The survival and hazard functions are as follows

$$\hat{S}_S^*(t) = \int_0^\infty\int_0^\infty S(t)\,\Pi_1(\lambda,\beta \mid l_i,u)\,d\lambda d\beta$$

$$= \frac{1}{N}\sum_{i=1}^{N}\exp\left(\frac{\hat{\lambda}_S}{\hat{\beta}_S}\left(1-e^{\hat{\beta}_M l_i}\right)\right)-\exp\left(\frac{\hat{\lambda}_S}{\hat{\beta}_S}\left(1-e^{\hat{\beta}_M u_i}\right)\right)$$

$$\hat{h}_S^*(t) = \int_0^\infty\int_0^\infty h(t)\,\Pi_1(\lambda,\beta \mid l_i,u)\,d\lambda d\beta$$

$$= \frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_S\left(e^{\hat{\beta}_S l_i}-e^{\hat{\beta}_S u_i}\right)$$

Simulation study: To compare the difference between the four methods: Maximum Likelihood Estimation (MLE), Bayesian under square error loss function, Linear Exponential loss (LINEX) function and General Entropy loss function a Monte Carlo experiment were conducted, The samples size are n = 25, 50 and 100 to show small medium and large sample size. Five thousands replicate for each result with the initial value of the scale parameter λ are 1.5 and 3 and the shape parameter β are 0.5 and 1.5. The following steps are employed:

- Generate lifetime T with different sample sizes n = 25, 50 and 100 from Gompertz distribution with λ and β, where the scale parameter λ are 1.5 and 3 and the shape parameter β are 0.5 and 1.5
- Generate of a vector V for a set of clinic visits assuming that 20 clinic visits are possible and with different sample sizes n = 25, 50 and 100. In the Gompertz distribution, the first visit v_1 was generated from a uniform (0, z) where hyper parameter z equal to 1 and the next visit v_2 was generated from uniform (v_1, v_2+z). Subsequent generations are employed with similar approach
- Generate a set of matrix named bounds for each of the data set. To obtain the lower and upper bounds we made use of the following:

$$\text{bound}[i,1] = \begin{cases} 0: \text{ if } t[i] < V[1] \\ V[j]: \text{if } \quad V[j] < t[i] < V[j+1] \\ V[20]: \text{if } \quad t[i] > V[20] \end{cases}$$

$$\text{bound }[i,2] = \begin{cases} V[1]: \text{ if } t[i] < V[1] \\ V[j+1]: \text{if } \quad V[j] < t[i] < V[j+1] \\ n: \text{if } \quad t[i] > V[20] \end{cases}$$

- The indicator defined as shown below:

$$\text{indictor }[i] = \begin{cases} 0: \text{ if bounds}[i,2] = n \\ 1: \text{ otherwise} \end{cases}$$

- The maximum likelihood estimated the scale and the shape parameters of Gompertz distribution as using numerical method for Eq. 4 and 5 respectively, also follow by estimated the survival and hazard function in Eq. 7 and 8, respectively
- The Metropolis- Hastings Algorithm used in Eq. 8-11 for Bayesian under square error loss function to estimate the parameters, the survival function and hazard function of Gompertz distribution respectively, where hyper-parameters of gamma priors are equal to 0.0001 as a = b = c = d = 0.0001
- The Metropolis-Hastings Algorithm used in Eq. 14-17 for Bayesian under linear exponential loss function (LINEX), also in Eq. 18-21 for Bayesian under General Entropy loss function to estimate the parameters, survival function and hazard function of Gompertz distribution based on interval censoring data. The values for the loss parameter were taken to be and a detailed discussion on the choice of the loss parameter of LINEX and General Entropy loss function can be obtained from Calabria and Pulcini (1996)
- Steps 1-5 are repeated 5000 times and the Mean Square Error (MSE) and absolute bias of the parameters, the survival function and the hazard function of Gompertz distribution for each method was calculated. The results are displayed in Table 1-8 for the different choice of the parameters, loss paraeter, hyper-parameters and sample size

RESULTS AND DISCUSSION

As shown in Table 1, the estimate of the scale parameter of Gompertz distribution based on interval censored data is obtained using Maximum Likelihood Estimator (MLE), Bayesian with square error loss function (BS), Linear Exponential loss function (BL) and General Entropy loss function (BG). Also in Table 2 we estimated the shape parameter of Gompertz distribution based on interval censored data by employing the four estimators.

Table 3 and 5 the estimate of the scale parameter of Gompertz distribution was compared by Mean Squared Error (MSE) and absolute bias. The results show that, the Bayesian under LINEX loss function with (r = +0.7) is better compare to the others except when for size 50 and 100 the Maximum Likelihood Estimator (MLE) is better compare to the others. Additionally, Bayesian under square error loss function and LINEX loss function with (r = -0.7) are better than MLE when (λ = 1.5, β = 0.5) for size 25. Soliman *et al.* (2012b) and Jaheen (2003) which LINEX loss function give better results than other estimators.

As shown in Table 4 and 6 the estimate of the shape parameter of Gompertz distribution based on interval censored data was compared by Mean Squared Error (MSE) and Absolute bias, Flygare *et al.* (1985) and Ahmed (2014)

Table 1: The estimates λ of Gompertz distribution based on interval censored data

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	1.3326	1.3372	1.3410	1.6516	1.3296	1.3066
		1.5	1.3865	1.3513	1.3793	1.5089	1.3789	1.3789
	3	0.5	2.7099	2.6931	2.6867	3.2624	2.6781	2.6924
		1.5	2.7158	2.7027	2.6951	3.2117	2.6882	2.6997
50	1.5	0.5	1.3595	1.3489	1.3519	1.6499	1.3443	1.3404
		1.5	1.3956	1.3567	1.3828	1.6147	1.3804	1.3847
	3	0.5	2.7253	2.7077	2.7154	3.2216	2.6927	2.7046
		1.5	2.7451	2.7257	2.7282	3.1986	2.7002	2.7276
100	1.5	0.5	1.4041	1.3781	1.3801	1.6113	1.3768	1.3953
		1.5	1.5982	1.3878	1.3738	1.6029	1.3911	1.6019
	3	0.5	2.8148	2.7439	2.7607	3.1692	2.7429	2.7462
		1.5	2.8456	2.7709	2.8009	3.1286	2.7572	2.8286

Table 2: Estimates β of Gompertz distribution based on interval censored data

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.4334	0.4348	0.4356	0.4396	0.4312	0.4315
		1.5	1.3744	1.3502	1.3599	1.6143	1.3476	1.3718
	3	0.5	0.4405	0.4409	0.4412	0.4509	0.4344	0.4392
		1.5	1.3792	1.3691	1.3744	1.6114	1.3625	1.3699
50	1.5	0.5	0.4408	0.4377	0.4387	0.4403	0.4352	0.4378
		1.5	1.3869	1.3682	1.3699	1.6018	1.3544	1.3665
	3	0.5	0.4589	0.4413	0.4457	0.4559	0.4384	0.4455
		1.5	1.3938	1.3746	1.3771	1.5923	1.3676	1.3713
100	1.5	0.5	0.4719	0.4535	0.4569	0.4687	0.4512	0.4473
		1.5	1.4199	1.3815	1.3898	1.5746	1.3789	1.3915
	3	0.5	0.4765	0.4593	0.4617	0.4733	0.4525	0.4555
		1.5	1.4208	1.3982	1.4047	1.5621	1.3865	1.3934

Table 3: Mean Square Error of the estimates of λ

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.2184	0.2151	0.2096	0.2098	0.2232	0.2194
		1.5	0.2275	0.2358	0.2314	0.2265	0.2393	0.2321
	3	0.5	0.1971	0.2011	0.1999	0.1963	0.2039	0.2017
		1.5	0.1876	0.1946	0.1976	0.1862	0.2002	0.2011
50	1.5	0.5	0.2042	0.2101	0.2088	0.2051	0.2139	0.2099
		1.5	0.1917	0.2173	0.1982	0.1907	0.2259	0.1987
	3	0.5	0.1924	0.1989	0.1974	0.1936	0.1993	0.1982
		1.5	0.1817	0.1943	0.1918	0.1775	0.1975	0.1901
100	1.5	0.5	0.1728	0.1877	0.1869	0.1753	0.1883	0.1872
		1.5	0.1701	0.1841	0.1821	0.1651	0.1849	0.1812
	3	0.5	0.1723	0.1815	0.1773	0.1774	0.1833	0.1787
		1.5	0.1704	0.1793	0.1756	0.1644	0.1815	0.1772

Table 4: Mean Square Error of the estimates of β

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.1341	0.1338	0.1329	0.1315	0.1344	0.1341
		1.5	0.1346	0.1359	0.1356	0.1307	0.1361	0.1358
	3	0.5	0.1339	0.1337	0.1327	0.1297	0.1341	0.1340
		1.5	0.1334	0.1344	0.1341	0.1289	0.1352	0.1343
50	1.5	0.5	0.1291	0.1317	0.1315	0.1297	0.1333	0.1337
		1.5	0.1289	0.1305	0.1304	0.1262	0.1349	0.1326
	3	0.5	0.1269	0.1294	0.1273	0.1271	0.1321	0.1311
		1.5	0.1241	0.1279	0.1258	0.1226	0.1301	0.1303
100	1.5	0.5	0.1193	0.1202	0.1197	0.1195	0.1221	0.1212
		1.5	0.1185	0.1192	0.1191	0.1159	0.1208	0.1209
	3	0.5	0.1169	0.1187	0.1174	0.1176	0.1179	0.1176
		1.5	0.1086	0.1128	0.1114	0.1071	0.1167	0.1163

Table 5: Absolute bias of the estimates of λ

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.3198	0.3186	0.3185	0.3137	0.3195	0.3191
		1.5	0.3169	0.3172	0.3170	0.3144	0.3185	0.3178
	3	0.5	0.3151	0.3164	0.3161	0.3131	0.3176	0.3170
		1.5	0.3099	0.3109	0.3105	0.3038	0.3128	0.3118
50	1.5	0.5	0.2994	0.3019	0.3011	0.3015	0.3113	0.3094
		1.5	0.2874	0.3006	0.2997	0.2778	0.3012	0.3008
	3	0.5	0.2677	0.2972	0.2969	0.2733	0.2987	0.2982
		1.5	0.2613	0.2892	0.2855	0.2586	0.2937	0.2885
100	1.5	0.5	0.2354	0.2693	0.2672	0.2357	0.2736	0.2703
		1.5	0.2277	0.2453	0.2439	0.2122	0.2509	0.2487
	3	0.5	0.2264	0.2401	0.2371	0.2268	0.2425	0.2392
		1.5	0.2169	0.2354	0.2349	0.2072	0.2416	0.2380

Table 6: Absolute bias of the estimates of β

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.2198	0.2191	0.2192	0.2143	0.2208	0.2201
		1.5	0.2165	0.2182	0.2181	0.2123	0.2187	0.2185
	3	0.5	0.2146	0.2135	0.2134	0.2111	0.2205	0.2202
		1.5	0.2101	0.2148	0.2132	0.1967	0.2153	0.2152
50	1.5	0.5	0.1972	0.1994	0.1981	0.1976	0.1991	0.1986
		1.5	0.1952	0.1971	0.1965	0.1943	0.1987	0.1963
	3	0.5	0.1926	0.1942	0.1934	0.1938	0.1957	0.1934
		1.5	0.1908	0.1913	0.1915	0.1896	0.1922	0.1921
100	1.5	0.5	0.1804	0.1899	0.1847	0.1807	0.1904	0.1901
		1.5	0.1799	0.1807	0.1814	0.1791	0.1826	0.1842
	3	0.5	0.1739	0.1798	0.1786	0.1741	0.1811	0.1808
		1.5	0.1713	0.1747	0.1737	0.1696	0.1781	0.1782

Table 7: Mean Square Error of the estimates of the survival function

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.0514	0.0509	0.0507	0.0505	0.0520	0.0515
		1.5	0.0508	0.0516	0.0512	0.0504	0.0517	0.0516
	3	0.5	0.0499	0.0496	0.0495	0.0489	0.0506	0.0504
		1.5	0.0486	0.0500	0.0501	0.0477	0.0501	0.0491
50	1.5	0.5	0.0469	0.0489	0.0477	0.0470	0.0490	0.0487
		1.5	0.0424	0.0433	0.0424	0.0412	0.0435	0.0428
	3	0.5	0.0417	0.0424	0.0415	0.0418	0.0428	0.0423
		1.5	0.0401	0.0416	0.0409	0.0400	0.0415	0.0411
100	1.5	0.5	0.0406	0.0412	0.0403	0.0408	0.0414	0.0413
		1.5	0.0391	0.0402	0.0392	0.0388	0.0406	0.0401
	3	0.5	0.0364	0.0389	0.0393	0.0365	0.0388	0.0392
		1.5	0.0334	0.0347	0.0338	0.0323	0.0348	0.0335

which they included the interval censored data through Gompertz distribution. The results show that, the Bayesian under linear exponential loss function with (r = +0.7) is better compare to the others except, when β = 0.5 for size 50 and 100 the Maximum Likelihood Estimator (MLE) is better compare to the others. However, Bayesian under square error loss function and LINEX loss function with (r = +0.7) is better than MLE when β = 0.5 for size 25.

Table 7 and 9 when we compared the Mean Squared Error (MSE) and absolute bias of the survival function, we found that the Bayesian under linear exponential loss function with (r = +0.7) is better compare to the others except when β = 0.5 for size 50 and 100 the Maximum Likelihood Estimator (MLE) is better compare to the others. However, it is clear from Table 7 and 9 the Bayesian under square error loss function and LINEX loss function with (r = +0.7) is better than MLE when β = 0.5 for size 25.

Table 8 and 10 when we compared the hazard estimates by Mean Squared Error (MSE) and Absolute bias we found that the Bayesian under linear exponential loss function with (r = +0.7) is better compare to the others except when β = 0.5 for size 50 and 100 the Maximum Likelihood Estimator (MLE) is better compare to the others. However, it is clear from Table 7 and 9 the Bayesian under square error loss function is better than MLE when β = 0.5 for size 25. Moreover, the Bayesian under LINEX loss function with (r = -0.7) is better

Table 8: Mean Square Error of the estimates of the hazard function

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.6412	0.6403	0.6410	0.6386	0.6415	0.6413
		1.5	0.6339	0.6408	0.6402	0.6286	0.6414	0.6409
	3	0.5	0.6332	0.6331	0.6334	0.6238	0.6374	0.6377
		1.5	0.6282	0.6156	0.6175	0.6199	0.6298	0.6288
50	1.5	0.5	0.6133	0.6196	0.6165	0.6148	0.6228	0.6227
		1.5	0.6091	0.6122	0.6139	0.6012	0.6175	0.6186
	3	0.5	0.5863	0.5914	0.5906	0.5874	0.6051	0.5956
		1.5	0.5742	0.5871	0.5767	0.5525	0.5955	0.5815
100	1.5	0.5	0.5311	0.5367	0.5318	0.5312	0.5368	0.5316
		1.5	0.5224	0.5317	0.5245	0.5161	0.5268	0.5209
	3	0.5	0.5158	0.5262	0.5246	0.5191	0.5285	0.5252
		1.5	0.5013	0.5128	0.5075	0.4997	0.5062	0.5099

Table 9: Absolute bias of the estimates of the survival function

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.1314	0.1307	0.1312	0.1310	0.1319	0.1315
		1.5	0.1308	0.1309	0.1310	0.1304	0.1314	0.1312
	3	0.5	0.1299	0.1289	0.1288	0.1291	0.1302	0.1300
		1.5	0.1286	0.1291	0.1287	0.1269	0.1296	0.1292
50	1.5	0.5	0.1269	0.1271	0.1270	0.1270	0.1281	0.1280
		1.5	0.1224	0.1234	0.1231	0.1215	0.1241	0.1235
	3	0.5	0.1217	0.1223	0.1221	0.1218	0.1231	0.1225
		1.5	0.1201	0.1217	0.1215	0.1196	0.1226	0.1216
100	1.5	0.5	0.1186	0.1204	0.1210	0.1195	0.1212	0.1205
		1.5	0.1161	0.1192	0.0129	0.1158	0.1203	0.1201
	3	0.5	0.1144	0.1161	0.1151	0.1146	0.1197	0.1195
		1.5	0.1134	0.1156	0.1161	0.1123	0.1167	0.1163

Table 10: Absolute bias of the estimates of the hazard function

n	λ	β	MLE	BS	BL (r = -0.7)	BL (r = +0.7)	BG (k = -0.7)	BG (k = +0.7)
25	1.5	0.5	0.5768	0.5763	0.5761	0.5755	0.5770	0.5768
		1.5	0.5732	0.5744	0.5742	0.5730	0.5749	0.5747
	3	0.5	0.5630	0.5629	0.5639	0.5622	0.5655	0.5651
		1.5	0.5537	0.5551	0.5536	0.5517	0.5561	0.5558
50	1.5	0.5	0.5587	0.5618	0.5623	0.5594	0.5633	0.5627
		1.5	0.5575	0.5602	0.5601	0.5512	0.5612	0.5608
	3	0.5	0.5457	0.5542	0.5551	0.5459	0.5548	0.5539
		1.5	0.5404	0.5451	0.5449	0.5388	0.5463	0.5454
100	1.5	0.5	0.5213	0.5311	0.5322	0.5218	0.5325	0.5318
		1.5	0.5156	0.5201	0.5219	0.5143	0.5213	0.5211
	3	0.5	0.5092	0.5112	0.5111	0.5093	0.5115	0.5113
		1.5	0.5025	0.5110	0.5107	0.5010	0.5113	0.5111

than MLE when ($\lambda = 1.5$, $\beta = 0.5$) for size 25. From Table 1-10, once the sample size n increases the mean squared error and absolute bias decreases for all cases of the scale and shape parameter, the survival function and hazard function of Gompertz distribution.

CONCLUSION

In this study we have considered Bayesian under three loss functions: The symmetric loss function is square error loss function and the asymmetric loss functions are LINEX loss function and General Entropy loss function problems of the Gompertz distribution based on interval censored data to estimate the parameters, the survival function and the hazard function. Comparisons are made between the Bayesian under three loss functions and maximum likelihood estimators based on simulation study and we observed that the parameters and survival and hazard functions of the Gompertz overall are better estimated by Bayesian under LINEX loss function when the value for the loss parameter is positive.

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