ISSN 1996-3343

Asian Journal of **Applied** Sciences



http://knowledgiascientific.com

ට OPEN ACCESS

Asian Journal of Applied Sciences

ISSN 1996-3343 DOI: 10.3923/ajaps.2017.96.101



Research Article A Variable-step-size Block Predictor-corrector Method for Ordinary Differential Equations

¹J.G. Oghonyon and ²S.A. Okunuga and ¹S.A. Bishop

¹Department of Mathematics, College of Science and Technology, Covenant University, P.M.B. 1023, Ota, Ogun State, Nigeria ²Department of Mathematics, University of Lagos, Yaba, Akoka, Lagos State, Nigeria

Abstract

Background and Objective: Over the years, block predictor-corrector method has been limited to predicting and correcting methods without further use. Predictor-corrector method possesses other attributes that utilize the Principal Local Truncation Error (PLTE) to design a suitable step size, tolerance level and control error. This study examined a variable-step-size block predictor-corrector method for solving first-order Ordinary Differential Equations (ODEs). **Materials and Methods:** The combination of Newton's backward difference interpolation polynomial and numerical integration methods were applied and evaluated at some selected grid points to formulate the block predictor-corrector method. Nevertheless, this process advances to generate the PLTE of the block predictor-corrector method after establishing the order of the method. **Results:** The numerical results were shown to demonstrate the performance of the variable step-size block predictor-corrector method in solving first-order ODEs. The complete results were incurred with the aid of Mathematica 9 kernel for Microsoft windows (64 bit). **Conclusion:** Numerical results showed that the variable step-size block predictor-corrector method is more effective and perform better than existing methods in terms of the maximum errors at all tested tolerance levels as well as designing a suitable step size to control error.

Key words: Predictor-corrector method, tolerance level, maximum errors, principal local truncation error

Received: October 10, 2016

Accepted: February 18, 2017

Published: March 15, 2017

Citation: J.G. Oghonyon, S.A. Okunuga and S.A. Bishop, 2017. A variable-step-size block predictor-corrector method for ordinary differential equations. Asian J. Applied Sci., 10: 96-101.

Corresponding Author: J.G. Oghonyon, Department of Mathematics, College of Science and Technology, Covenant University, P.M.B. 1023, Ota, Ogun State, Nigeria

Copyright: © 2017 J.G. Oghonyon *et al.* This is an open access article distributed under the terms of the creative commons attribution License, which permits unrestricted use, distribution and reproduction in any medium, provided the original author and source are credited.

Competing Interest: The authors have declared that no competing interest exists.

Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

Predictor-corrector method is very essential for finding a suitable step-size¹. This study is concerned with approximating the solution of Initial Value Problems (IVPs) for first-order ODEs of the form¹:

$$y' = f(x, y)$$
, for $a \le x \le b$, with $y'(a) = \alpha$ (1)

The aim was to formulate a variable step-size block predictor-corrector method. This technique of continuing in variable step-size predictor-corrector method started with Milne and it is referred to as Milne's device¹⁻⁴. Other researchers proposed block predictor-corrector method for computing the solution of ODEs in the simple form of Adams type as sited⁵⁻¹². Gear's method known for stiff problems is the Backward Differentiation Formula (BDF) as stated previously¹³⁻¹⁵. In addition, this study possesses a lot of computational advantages as discussed previously^{3,4}. Predictor-corrector techniques constantly provide two estimates at each step, thirdly, they are an efficient device for error-control adaptation which has been reported previously⁷. To present this process, a variable step-size block predictor-corrector method apply the explicit Adams-Bashforth K-Step method as a predictor and the implicit Adams-Moulton K-1-Step method as a corrector of the same order^{1,10,11}.

Furthermore, it is speculated by Adesanya *et al.*^{5,6}, Anake *et al.*⁷, Bakoji *et al.*⁸, James and Adesanya⁹ and Voss and Abbas¹² that block predictor-corrector method(s) is a faster method than other non-block predictor-corrector method(s) with better results and as such, ensure convergence. Again, Adesanya *et al.*^{5,6}, Anake *et al.*⁷, Bakoji *et al.*⁸, James and Adesanya⁹ and Voss and Abbas¹² suggested that solving block predictor-corrector method(s) simultaneously using fixed step size is sufficient enough to guarantee maximum errors, while others proposed Backward Differentiation Formula (BDF) to provide the solution for stiff ODEs. This study is motivated by the fact that block predictor-corrector method can be extended using the variable step-size technique to solve nonstiff and mildly stiff ODEs.

Definition 1: b-block, r-point method. If r denotes the block size and h is the step size, then block size in time is rh. Let m = 0, 1, 2,... represent the block number and let n = mr, then the b-block, r-point method can be written in the following general form in Eq. 2:

$$Y_{\mu} = \sum_{s=1}^{b} A_{s} Y_{\mu-s} + \sum_{s=0}^{b} B_{s} F_{\mu-s}$$
(2)

Where:

$$Y_{\mu} = [y_{n+1}, ..., y_{n+i}, ..., y_{n+r}]^{T}$$

$$F_{\mu} = [f_{n+1}, ..., f_{n+i}, ..., f_{n+r}]^T$$

 A_s and B_s are r×r coefficients matrices¹³.

MATERIALS AND METHODS

Formulation of the block predictor-corrector method: Newton's backward difference formula was used to formulate the block predictor-corrector method.

Suppose f(x) has a continuous kth derivative, $t_m = t_0 + mh$, $f_m = f(t_m)$ and backward differences are presented by Eq. 3:

$$\nabla^{q+1} \mathbf{f}_{m} = \nabla^{q} \mathbf{f}_{m} \cdot \nabla^{q} \mathbf{f}_{m-1}$$

where, $\nabla^q f_m = f_m$, then:

$$f(t) = f_{m} + \left(\frac{t - t_{m}}{h}\right) \nabla f_{m} + (t - t_{m})(t - t_{m} - 1) \frac{\nabla^{2} f_{m}}{2!h^{2}} + \dots + (t - t_{m}) \dots (t - t_{m-k+2}) \frac{\nabla^{k-1} f_{m}}{(k-1)!h^{k-1}} + (t - t_{m}) \dots (t - t_{m-k+1}) \frac{f^{(k)}(\epsilon)}{k!}$$
(3)

where, $f^{(k)}(\varepsilon)$ is the kth derivative of f appraised at some point in an interval having t, t_{m-k+1} and t_m . Assume to fix $s = \frac{(t - t_{n-1})}{h}$ and m = n-1, (1.3) yields:

$$f\left(t\right) = \binom{-s}{0} f_{n-1} - \binom{-s}{1} \nabla f_{n-1} + \ldots + \left(-1\right)^{k-1} \binom{-s}{k-1} \nabla^{k-1} f_{n-1} + \left(-1\right)^k h^k \binom{-s}{k} f^{(k)}(\epsilon)$$

Where:

$$\binom{s}{k} = \frac{s(s-1)\dots(s-q+1)}{q!} \text{ and } \binom{s}{0} = 0$$

Subbing the above in:

$$y(t_{n}) = y(t_{n-1}) + \int_{t_{n-1}}^{t_{n}} f(t) dt$$

to get:

$$y(t_{n}) = y(t_{n-1}) + \int_{t_{n}-1}^{t_{n}} \left[\sum_{j=0}^{k-1} (-1)^{j} {\binom{-s}{j}} \nabla^{j} f_{n-1} + (-1)^{k} h^{k} {\binom{-s}{k}} y^{(k+1)}(\epsilon) \right] dt$$

$$y(t_{n}) = y(t_{n-1}) + h \sum_{j=0}^{k-1} \gamma^{j} \nabla^{j} f_{n-1} + (-1)^{k} h^{k} \int_{0}^{1} {\binom{-s}{k}} y^{(k+1)}(\varepsilon) ds \qquad (4)$$

Whenever the last term in Eq. 4 is disregarded, the left behind will be called k-step Adams-Bashforth formula which is shown in Eq. 5:

$$y_{n} = y_{n-1} + h \sum_{j=0}^{k-1} \gamma^{j} \nabla^{j} f_{n-1}$$
 (5)

Expressing the backward differences in terms of the values at continuing points by:

$$\nabla^{q} \mathbf{f}_{n-1} = \sum_{i=0}^{q} (-1)^{i} {\binom{q}{i}} \mathbf{f}_{n-1-1}$$

Thus, Eq. 5 can be rewritten as:

$$y_{n} = y_{n-1} + h \sum_{j=0}^{k-1} \beta_{ki-i} f_{n-i}$$
(6)

Continuing with Eq. 6 to generate the block k-step Adams-Bashforth Formula⁸.

Similarly, the implicit multistep methods-the Adams-moulton method can be derived by setting m = n in Eq. 3 and putting into:

$$y(t_n) = y(t_{n-1}) + \int_{t_{n-1}}^{t_n} f(t) dt$$

we get:

$$y(t_{n}) = y(t_{n-1}) + \int_{t_{n-1}}^{t_{n}} \left[\sum_{j=0}^{k-1} (-1)^{j} \binom{-s+1}{j} \nabla^{j} f_{n} + (-1)^{k} h^{k} \binom{-s+1}{k} y^{(k+1)}(\epsilon) \right] ds$$

Ignoring the error term, gives the method as Eq. 7:

$$y_n = y_{n-1} + h \sum_{j=0}^{k-1} \gamma_j^* \nabla^j f_n$$
 (7)

Replacing for $\nabla^{j}f_{n}$ in terms of f_{n} , f_{n-1} , f_{n-2} ,..., yields the form:

$$y_n = y_{n-1} + h \sum_{i=0}^{k-1} \beta_{ki}^* f_{n-i}$$
 (8)

By continuing with Eq. 8, the block k-1-step Adams-moulton method can be generated¹⁶.

Practical application of block predictor-corrector method: Assuming P defines the application program of the block predictor, C defines block corrector application program, with E as the evaluation application program of f with respect to given values of its parameter. If $y_{n+k}^{(0)}$ is computed from the block predictor, $f_{n+k}^{(0)} \equiv f(x_{n+k}, y_{n+k}^{(0)})$ is calculated one time and employ the corrector at one time as well to obtain $y_{n+k}^{(1)}$, this describes the computation as PEC. Further appraisal of $f_{n+k}^{(1)} \equiv f(x_{n+k}, y_{n+k}^{(1)})$ succeeded by another application program of the corrector gives $y_{n+k}^{(2)}$ and thus, denoted by PEC⁽²⁾. Implementing the application program of the block corrector m many times can be referred to as PEC^(m). Since m is constant, $y_{n+k}^{(m)}$ is accepted as the computational solution at X_{n+k} . At this point, the last computational value for \boldsymbol{f}_{n+k} is preferred as $f_{n+k}^{(m-l)} \equiv f(x_{n+k}, y_{n+k}^{(m-l)})$ and this will be further decided whether or not to execute $\ f_{_{n+k}}^{_{(m)}}\ \equiv \ f(x_{_{n+k}},\ y_{_{n+k}}^{_{(m)}})$. Assuming this concluding execution is done, the mode is denoted by P(EC)^m or P(EC)^mE. Eventually the decision clearly impacts the next step of the execution, when both predicted and corrected numerical values for y_{n+k+1} will rely on whether f_{n+k} is accepted as $f_{n+k}^{(m)}$ or $f_{n+k}^{(m-1)}$. Finally, for a given m, P(EC)^m or P(EC)^mE mode utilize the corrector the same number of times; only P(EC)^m E requires one more evaluation per step than $P(EC)^m$ as expressed^{3,4}.

Theorem 1: If the multistep method 2 is convergent for pth order equations, then the order of 2 is at least p^{16} .

Theorem 2: The order of a predictor-corrector method for first order equations must be >1 if it is convergent¹⁶.

Theorem 1 and 2 draw the conclusion that the order and convergence of the method hold.

Implementation of block predictor-corrector method: Concurring to Jain *et al.*¹⁷ and Lambert^{3,4}, the implementation in the P(EC)^m or P(EC)^m E mode becomes substantial for the explicit (predictor) and implicit (corrector) methods if both are separate of like order and this requirement makes it indispensable for the step number of the explicit (predictor) method to be one step higher than that of the implicit (corrector) method. Consequently, the mode P(EC)^m or P(EC)^m E can be formally examined in Eq. 9 for m = 1, 2,...; P(EC)^m:

$$\begin{split} y_{n+j}^{[0]} + \sum_{i=0}^{j-l} \alpha_i^{\bullet} y_{n+i}^{[m]} = h \sum_{i=0}^{j-l} \beta_i^{\bullet} y_{n+i}^{[m-l]}, \, 0 \\ \\ f_{n+j}^{[s]} \equiv f \Big(x_{n+j}, \, y_{n+j}^{[s]} \Big) \end{split}$$

$$\{y_{n+j}^{[s+1]} + \sum_{i=0}^{j-1} \alpha_i y_{n+i}^{[m]} = h\beta_j f_{n+j}^{[s]} + h \sum_{i=0}^{j-1} \beta_i y_{n+i}^{[m-1]} \}, s = 0, 1, ..., m-1$$
(9)

P(EC)^mE:

$$\begin{split} y_{n+j}^{[0]} + \sum_{i=0}^{j-1} \alpha_i^{\star} y_{n+i}^{[m]} &= h \sum_{i=0}^{j-1} \beta_i^{\star} y_{n+i}^{[m]} \\ f_{n+j}^{[s]} &\equiv f\left(x_{n+j}, \ y_{n+j}^{[s]}\right) \Big\} \\ y_{n+j}^{[s+1]} + \sum_{i=0}^{j-1} \alpha_i y_{n+i}^{[m]} &= h \beta_j f_{n+j}^{[s]} + h \sum_{i=0}^{j-1} \beta_i f_{n+i}^{[m]} \}, \ s = 0, \ 1, ..., m-1 \\ f_{n+j}^{[m]} &\equiv f\left(x_{n+j}, \ y_{n+j}^{[m]}\right) \Big\} \end{split}$$

Remarking that as $m \rightarrow \infty$, the result of evaluating with either of the above mode will slope to those given by the mode of correcting to convergence.

Moreover, predictor and corrector pair based on method 2 can be implemented. The mode $P(EC)^m$ or $P(EC)^mE$ specified by Eq. 9, where h is the step size. Since the predictor and corrector both have the same order p.

Theorem 3 demonstrates adequate condition for the convergence of $P(EC)^m$ or $P(EC)^mE$.

Theorem 3: Let $\{y_{n+1}^{[m]}\}\$ be a sequence of approximations of y_{n+1} obtained by a PECE method. If:

$$\left|\frac{\partial f}{\partial y}\big(_{X_{n+l}},\,y\big)\right| \leq L$$

(for all y near y_{n+1} including $y_{n+1}^{[0]}, y_{n+1}^{[1]}...$) where, L satisfies the condition $\ L < \frac{1}{\left|h\beta_{0}\right|}$, then the sequence $\{y_{n+1}^{[m]}\}$ converges to y_{n+1} .

Proof: The numeric solution satisfies the equation:

$$y_{_{n+1}} = \sum_{_{i \ = \ 0}}^{^{j-1}} \! \alpha_{_i} y_{_{n+i}} + h \beta_{_0} \ f(x_{_{n+1}}, y_{_{n+1}}) + h \sum_{_{i \ = \ 0}}^{^{j-1}} \! \beta_{_i} f_{_{n+i}}$$

The corrector satisfies the equation:

$$y_{n+1}^{(m+1)} = \sum_{i=0}^{j-1} \alpha_i y_{n+i} + h\beta_0 f(x_{n+1}, y_{n+1}^{(m)}) + h \sum_{i=0}^{j-1} \beta_i f_{n+i}$$

Subtracting these two equations, we obtain:

$$y_{n+1} - y_{n+1}^{(m+1)} = h\beta_0 \left[\left| f(x_{n+1}, y_{n+1}) - f(x_{n+1}, y_{n+1}^{(m)}) \right| \right]$$

Applying the Lagrange mean value theorem to arrive at:

$$\boldsymbol{y}_{n+1} - \boldsymbol{y}_{n+1}^{(m+1)} = h \, \beta_0 \, \big(\, \boldsymbol{y}_{n+1} - \boldsymbol{y}_{n+1}^{(m)} \big) \frac{\partial f}{\partial \boldsymbol{y}} \Big(\, \boldsymbol{x}_{n+1} , \boldsymbol{y}^* \Big)$$

where, $y_{n+1}^{(m)} \le y^* \le y_{n+1}$. Thus:

$$\begin{split} \left| y_{n+1} - y_{n+1}^{(m+1)} \right| &\leq \left| h\beta_0 \right| \left| y_{n+1} - y_{n+1}^{(m)} \right| \left| \frac{\partial f}{\partial f}(x_{n+1}, y) \\ \\ &\leq hL \left| \beta_0 \right| \left| y_{n+1} - y_{n+1}^{(m)} \right| \\ \\ &\leq \left[hL \left| \beta_0 \right| \right]^m \left| y_{n+1} - y_{n+1}^{(0)} \right| \end{split}$$

Now:

$$\begin{split} &\lim_{m \to \infty} \left| y_{n+1} - y_{n+1}^{(m+1)} \right| \to 0, \text{ if} \\ & hL \left| \beta_0 \right| < 1 \text{ or } L < \frac{1}{h \left| \beta_0 \right|} \end{split}$$

This means that the conclusion of Theorem 3 holds as seen¹⁷.

In cases, where C_{p+1} , C_{p+1}^* are the computed error constant of the predictor-corrector method, respectively. The following consequence holds.

Proposition: Suppose the predictor method have order p^* and the corrector method have order p. Then: If $p^* \ge p$ (or $p^* < p$ with $m > p - p^*$), then the predictor-corrector methods possesses the same order and the same PLTE as the corrector.

If $p^* < p$ and $m = p - p^*$, then the predictor-corrector method possesses the same order as the corrector, but different PLTE.

If $p^* < p$ and $m \le p - p^* - 1$, then the predictor-corrector method possesses the same order equal to $p^* + m$ (thus less than p).

Specifically, it is observed that, suppose the predictor has order p-1 and the corrector has order p, the PEC answers to get a method of order p. Moreover, the P(EC)^m or P(EC)^mE scheme has always the same order and the same PLTE as discussed^{3,4}.

Combining¹⁻⁴, Milne's device stated that it is viable to estimate the principal local truncation error of the explicit and implicit (predictor-corrector) method without estimating higher derivatives of y(x). Assuming that p = *, where p^* and p defines the order of the explicit (predictor) and implicit (corrector) methods with the same order. Directly, for a method of order p, the principal local truncation errors can be written as Eq. 10 and 11:

$$C_{p+1}^{*} h^{p+1} y^{(p+1)}(x_{n}) = y(x_{n+j}) - W_{n+j} + O(h^{p+2})$$
(10)

Also:

$$C_{p+1}h^{p+1} y^{(p+1)}(x_n) = y(x_{n+j}) - C_{n+j} + O(h^{p+2})$$
(11)

where, W_{n+j} and C_{n+j} are called the predicted and corrected approximations are given by the method of order p while C_{p+1}^* and C_{n+1} are independent of h.

Neglecting terms of degree p+2 and above, it is easy to make estimates of the principal local truncation error of the method as Eq. 12:

$$C_{p+1}h^{p+1}y^{(p+1)}(x_{n}) = \frac{C_{p+1}}{C_{p+1}^{*} - C_{p+1}} |W_{n+j} - C_{n+j}| < \epsilon$$
(12)

Noting the fact that $C_{p+1} \neq C_{p+1}^*$ and $W_{n+j} \neq C_{n+j}$.

However, the estimate of the principal local truncation error (12) is used to determine whether to accept the results of the current step or to reconstruct the step with a smaller step size. The step is accepted based on a test as prescribed by Eq. 12¹⁸. Equation 12 is the convergence criteria otherwise called Milne's estimate for correcting to convergence. Furthermore, Eq. 12 ensures the convergence criterion of the method during the test evaluation¹⁸.

Problem tested: Three test problems are employed. These problems are implemented using variable-step-size block predictor-corrector method.

- Test problem 1: y' (x)=xy y(0) = 1 $0 \le x \le 1$ Solution: $v(x) = e^{\frac{1}{2}x^2}$
- Test problem 2: y'(x) = x-y y(0) = 1 $0 \le x \le 1$ Solution: $y(x) = x+e^{-x}-1$
- Test problem 3: y'(x) = -10xy y(0) = 1 $0 \le x \le 10$ Solution: $y(x) = e^{-5x^2}$

RESULTS AND DISCUSSION

The numeric results to demonstrate the performance of the variable step-size block predictor-corrector method in solving first-order ODEs. The complete result supplied were incurred with the aid of Mathematica 9 Kernel for Microsoft windows (64 bit). The nomenclature utilized are listed in Table 1:

VS-SBP-CM:	Variable step-size block predictor-corrector		
	Method		
TOL:	Tolerance level		
h:	Step size		
MTH:	Method used		
MAXE:	The magnitude of the maximum errors of		
	VS-SBP-CM		
ES-TM:	Error in stormer-cowell method ² for test		
	problem 1 and 2		
1BDF:	r = 1-point BDF method ⁹ for test problem 3		
2BDF:	r = 2-point BDF method ⁹ for test problem 3		
3BDF:	r = 3-point BDF method ⁹ for test problem 3		

The process of estimating the maximum errors and determining the tolerance level are defined as follows:

$$C_{p+1}h^{p+1}y^{(p+1)}(x_n) \cong \frac{C_{p+1}}{C_{p+1}^* - C_{p+1}} |P_{n+j} - C_{n+j}| < \varepsilon$$

Table 1: Shows the numeric results of problems 1, 2 and 3 using VS-SBP-CM with	ith
comparison to existing methods	

h	Method	MAXE	TOL
10 ⁻¹	ES-TM	1.3068×10 ⁻¹⁰	10-10
	VS-SBP-CM	3.2×10 ⁻¹¹	
10 ⁻¹	ES-TM	2.8732×10^{-13}	10-13
	VS-SBP-CM	3.99841×10 ⁻¹⁴	
10 ⁻²	1BDF	1.30492 × 10 ⁻²	
	2BF	2.47600×10 ⁻²	10-2
	3BDF	3.56692×10 ⁻²	
	VS-SBP-CM	3.69796×10 ^{−3}	
10-3	1BDF	1.43966×10 ⁻³	10-3
	2BF	2.86614×10 ⁻³	
	3BDF	4.2851×10 ⁻³	
	VS-SBP-CM	2.68222×10 ⁻⁴	
10 ⁻⁴	1BDF	1.45326×10 ⁻⁴	10 ⁻⁴
	2BF	2.90520×10 ⁻⁴	
	3BDF	4.35640×10 ⁻⁴	
	VS-SBP-CM	3.9969×10 ⁻⁵	
10 ⁻⁵	1BDF	1.45462×10 ⁻⁵	10-5
	2BF	2.90911×10 ⁻⁵	
	3BDF	4.36353×10 ⁻⁵	
	VS-SBP-CM	3.19979×10 ⁻⁶	
10 ⁻⁶	1BDF	1.45478×10 ⁻⁶	10-6
	2BF	2.90951×10^{-6}	
	3BDF	4.36425×10 ⁻⁶	
	VS-SBP-CM	3.99997×10^{-7}	

Observing the fact that $C_{p+1}^* \neq C_{p+1}$ and $P_{n+j} \neq C_{n+j}$. C_{p+1}^* and C_{p+1} are independent of h.

Where, C_{p+1}^* and C_{p+1} are the estimates of the principal local truncation error of the predictor and corrector method. P_{n+j} and C_{n+j} are called the predicted and corrected approximations are given by the method of order p.

CONCLUSION

Numeric results have demonstrated the VS-SBP-CM is achieved with the aid of the tolerance level. This tolerance level criteria decide whether the result is accepted or repeated. The results likewise establish the performance of the VS-SBP-CM is remarked to be quicker than the block Stormer-Cowell method and block backward differentiation formula implemented with fixed step size. Hence, it can be resolved that the method formulated is worthy for solving non-stiff and stiff ODEs.

SIGNIFICANCE STATEMENTS

The significance statements of this study are to:

- Extend the block predictor-corrector method
- Introduce the tolerance level otherwise referred to as convergence criteria
- Design a suitable step size
- Control the error with the aid of a suitable step size

ACKNOWLEDGMENT

The authors would like to thank Covenant University for providing financial support through grants during the study period.

REFERENCES

- 1. Dormand, J.R., 1996. Numerical Methods for Differential Equations: A Computational Approach. CRC Press, Boca Raton, FL., ISBN: 9780849394331, pp: 10, 142-143, 231.
- 2. Faires, J.D. and R.L. Burden, 2012. Initial-value problems for ODEs, variable step-size multistep methods. Dublin City University, Dublin, Republic of Ireland, pp: 2-32.
- Lambert, J.D., 1973. Computational Methods in Ordinary Differential Equations. John Wiley and Sons, New York, USA., ISBN: 9780471511946, pp: 87-88.
- Lambert, J.D., 1991. Numerical Methods for Ordinary Differential Systems: The Initial Value Problem. 1st Edn., John Wiley and Sons, New York, USA., ISBN: 9780471929901, pp: 103-105.

- Adesanya, A.O., M.O. Udo and A.M. Alkali, 2012. A new block-predictor corrector algorithm for the solution of y'''=f (x, y, y', y''). Am. J. Comput. Math., 2: 341-344.
- Adesanya, A.O., M.R. Odekunle, M.A. Alkali and A.B. Abubakar, 2013. Starting the five steps stormer-cowell method by Adams-Bashforth method for the solution of the first order ordinary differential equations. Afr. J. Math. Comput. Sci. Res., 6: 89-93.
- Anake, T.A., A.O. Adesanya, J.G. Oghonyon and M.C. Agarana, 2013. Block algorithm for general third order ordinary differential equation. ICASTOR J. Math. Sci., 7: 127-136.
- 8. Bakoji, A.M., A.M. Bukar and M.I. Bello, 2014. Formulation of predictor-corrector methods from 2-step hybrid Adams methods for the solution of initial value problems of ordinary differential equations. Int. J. Eng. Applied Sci., 5: 9-13.
- 9. James, A.A. and A.O. Adesanya, 2014. A note on the construction of constant order predictor corrector algorithm for the solution of y = f(x, y). Br. J. Math. Comput. Sci., 4:886-895.
- Oghonyon, J.G., S.A. Okunuga and O.O. Agboola, 2015. K-step block predictor-corrector methods for solving first order ordinary differential equations. Res. J. Applied Sci., 10: 779-785.
- Oghonyon, J.G., S.A. Okunuga and S.A. Iyase, 2016. Milne's implementation on block predictor-corrector methods. J. Applied Sci., 16: 236-241.
- 12. Voss, D. and S. Abbas, 1997. Block predictor-corrector schemes for the parallel solution of ODEs. Comput. Math. Applic., 33: 65-72.
- Ibrahim, Z.B., K.I. Othman and M. Suleiman, 2007. Implicit r-point block backward differentiation formula for solving first-order stiff ODEs. Applied Math. Comput., 186: 558-565.
- 14. Mehrkanoon, S., Z.A. Majid and M. Suleiman, 2010. A variable step implicit block multistep method for solving first-order ODEs. J. Comput. Applied Math., 233: 2387-2394.
- Ibrahim, Z.B., K.I. Othman and M. Suleiman, 2007. Variable step block backward differentiation formula for solving first order stiff ODES. Proceedings of the World Congress on Engineering, Volume 2, July 2-4, 2007, London, UK., pp: 2-6.
- Gear, C.W., 1971. Numerical Initial Value Problems in ODEs (Automatic Computation). Prentice-Hall, New Jersey, USA., pp: 184-186.
- Jain, M.K., S.R.K. Iyengar and R.K. Jain, 2007. Numerical Methods for Scientific and Engineering Computation. 5th Edn., New Age International (Pvt.) Ltd., New Delhi, India, ISBN-13: 978-8122420012, Pages: 816.
- Ascher, U.M. and L.R. Petzold, 1998. Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations. SIAM, Philadelphia, USA., ISBN-13: 9780898714128, Pages: 314.