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Free ΓM-Modules

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Abstract: The characterizations of free ΓM-modules are developed. The cardinality of the basis of the free ΓM-modules is studied. At last we have studied the invariant rank property of free ΓM-modules.

Key words: Group, ring, modules, invariant property, mapping, liner property and rank

INTRODUCTION

Gamma ring: Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M\times\Gamma\times M\rightarrow M$ (sending (x, α, y) into xαy) such that

- $(x+y)\alpha z = x\alpha z+y\alpha z$
  $x(\alpha+\beta)z = x\alpha z+x\beta z$
  $x\alpha(y+z) = x\alpha y+x\alpha z$
- $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where x, y, z∈M and α, β∈Γ. Then M is called a Γ-ring. This definition is due to Barnes[1].

Ideal of Γ-rings: A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c\in M, \alpha\in\Gamma, a\in A\}$ $(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.

If A and B are both left (respectively right or two sided) ideals of M, then $A+B = \{a+b \mid a\in A, b\in B\}$ is clearly a left (respectively right or two sided) ideal, called the sum of A and B. We can say every finite sum of left (respectively right or two sided) ideal of a Γ-ring is also a left (respectively right or two sided) ideal.

It is clear that the intersection of any number of left (respectively right or two sided) ideal of M is also a left (respectively right or two sided) ideal of M.

If A is a left ideal of M, B is a right ideal of M and S is any non empty subset of M, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i\gamma_i s_i \mid a_i\in A, \gamma_i\in\Gamma, s_i\in S$, n is a positive integer$\}$ is a left ideal of M and SΓB is a right ideal of M. AΓB is a two sided ideal of M.

If a∈M, then the principal ideal generated by a denoted by ⟨a⟩ is the intersection of all ideals containing a and is the set of all finite sum of elements of the form $na+x\alpha a+a\beta y+u\gamma a\mu v$, where n is an integer, x, y, u, v are elements of M and α, β, γ, μ are elements of Γ. This is the smallest ideal generated by a. Let a∈M. The smallest left (right) ideal generated by a is called the principal left (right) ideal ⟨a| (| a⟩).

Division gamma ring: Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non zero ideal is itself.

Zorn's lemma: Let A be a nonempty partially ordered set in which every totally ordered subset has an upper bound in A. Then A contains at least one maximal element.

ΓM-module: Let M be a Γ-ring and let (P, +) be an abelian group. Then P is called a left ΓM-module if there exists a Γ-mapping (Γ-composition) from M×Γ×P to P sending (m, α, p) to mαp such that

- $(m_1+m_2)\alpha p = m_1\alpha p+m_2\alpha p$
- $m\alpha(p_1+p_2) = m\alpha p_1+m\alpha p_2$
- $(m_1\alpha m_2)\beta p = m_1\alpha(m_2\beta p)$,
  for all $p, p_1, p_2\in P$, $m, m_1, m_2\in M$, $\alpha, \beta\in\Gamma$.

If in addition, M has an identity 1 and $1\gamma p = p$ for all p∈P and some γ∈Γ, then P is called a unital ΓM-module.

Sub ΓM-module: Let M be a Γ-ring. Let P be a left ΓM-module. Let (Q, +) be a subgroup of (P, +). We call Q, a sub left ΓM-module of P if mγq∈Q for all m∈M, q∈Q and γ∈Γ.

Quotient ΓM-module: Let M be a Γ-ring and P be left ΓM-module. Let Q be a sub left ΓM- module of P. Then the set $\{p+Q \mid p\in P\}$ is called the quotient ΓM-module of P by Q. It is denoted by P/Q, where $m\gamma(p+Q) = m\gamma p+Q$ for all m∈M, p∈P and γ∈Γ and $(p_1+Q)+(p_2+Q) = (p_1+p_2)+Q$ for all $p_1, p_2\in P$.

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ΓM-homomorphism: Let M be a Γ-ring. Let P and Q be two left ΓM-modules. Let φ be a map of P into Q. Then φ is called a ΓM-homomorphism if and only if $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(m\gamma x) = m\gamma\varphi(x)$ for all $x, y \in P$, $m \in M$ and $\gamma \in \Gamma$. If φ is one-one and onto, then φ is a ΓM-isomorphism and is denoted by $P \cong Q$. If φ is a ΓM-homomorphism of P into Q, then kernel of φ, i.e., $\ker\varphi = \{x \in P \mid \varphi(x) = 0\}$, which is a left sub ΓM-module of P and image of φ i.e., $\text{Im}\varphi = \{y \in Q \mid y = \varphi(x) \text{ for some } x \in P\}$ is a left sub ΓM-module of Q.

Let M be a Γ-ring and A is an ideal of M. Since every ideal A is a ΓM-module, then the homomorphism between two ideals are the same as that of given above.

Γ-ring homomorphism: Let M and N be two Γ-rings. Let φ be a map from M to N. Then φ is a Γ-ring homomorphism if and only if $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(x\gamma y) = \varphi(x)\gamma\varphi(y)$ for all $x, y \in M$ and some $\gamma \in \Gamma$. If φ is one-one and onto, then φ is Γ-ring isomorphism. If φ is a Γ-ring homomorphism of M into N, then kernel of φ, i. e., $\varphi^{-1}(0) = \{x \in M \mid \varphi(x) = 0\}$ which is also an ideal of M. More generally, if B is a left (right, two sided) ideal of N, then $\varphi^{-1}(B) = \{x \in M \mid \varphi(x) \in B\}$ is also a left (respectively right or two sided of M. Similarly, if φ is a Γ-ring homomorphism of M onto N and A is any left (right, two sided) ideal of M, then $\varphi(A) = \{\varphi(a) \mid a \in A\}$ is a left (right, two sided) ideal of N.

Equivalent sets: A set A is called equivalent to a set B, written A~B if There exists a function $\varphi: A \to B$ which is one-one and onto. Clearly two finite sets are equivalent if and only if they contain the same number of elements.

Cardinality of sets: If A is equivalent to B, that is, $A \sim B$, then we say that A and B have the same cardinality or cardinal number. We write $|A|$ for the cardinality or cardinal number of A. So $|A| = |B|$ if and only if $A \sim B$.

Theorem (schroeder-bernstein theorem): If $|A| = |B|$ and $|B| = |A|$, then $|A| = |B|$. For the above preliminaries we refer to[2-5].

In this study, free ΓM-modules are considered. We have defined free ΓM-modules and some of its properties are developed. We also study invariant rank properties of these modules.

Our results are the generalizations of the results due to[6].

Basic notions of free ΓM-modules

Definition: Let P be a Γ-module over a Γ-ring M. Then for any $\gamma \in \Gamma$ a subset X of P is said to be linearly γ-independent or simply γ-independent over M if there exist distinct elements $x_1, x_2, \ldots, x_n$ in X and elements $m_1, m_2, \ldots, m_n$ in M all of which are zero, such that $m_1\gamma x_1 + m_2\gamma x_2 + \ldots + m_n\gamma x_n = 0$.

If X is linearly γ-independent for every $\gamma \in \Gamma$, then X in called linearly Γ-independent or simply Γ-independent.

Again for any $\gamma \in \Gamma$, a subset X of P is said to linearly γ-dependent or simply γ-dependent if there exist distinct elements $x_1, x_2, \ldots, x_n$ in X and elements $m_1, m_2, \ldots, m_n$ not of all which are zero, such that $m_1\gamma x_1 + m_2\gamma x_2 + \ldots + m_n\gamma x_n = 0$.

If X is linearly γ- dependent for every $\gamma \in \Gamma$, then X is said to be linearly Γ-dependent or simply Γ-dependent.

If $X = \{x_i \mid i \in \Lambda\}$ is a set of distinct elements of a left ΓM-module P, then for every $\gamma \in \Gamma$, an expression $\sum_{i \in \Lambda} m_i\gamma x_i$, where $m_i \in M$ and at most finitely many $m_i \neq 0$, is called a linear Γ-combination of $\{x_i \mid i \in \Lambda\}$. Infact, whenever we write $x = \sum_{i \in \Lambda} m_i\gamma x_i$, we mean that x is a linear Γ-combination of $\{x_i \mid i \in \Lambda\}$.

Definition: Let P be a unital left ΓM-module and let $\{x_i \mid i \in \Lambda\}$ be a subset of P such that each element $p \in P$ can be written in at least one way in the form

$$p = m_1\gamma x_{i_1} + m_2\gamma x_{i_2} + \ldots\ldots\ldots + m_n\gamma x_{i_n}$$

where $m_i \in M$, all $\gamma \in \Gamma$ and $i_j \in \Lambda$;

$\{x_i \mid i \in \Lambda\}$ is called a set of generators of P. If each element $p \in P$ can be written in only one way in this form, then $\{x_i \mid i \in \Lambda\}$ is a basis for P. A unital left ΓM-module P is said to be a free left ΓM-module if it has a basis (finite or infinite) , that is , if each element of P can be written is precisely one way as

$$p = m_1\gamma x_{i_1} + m_2\gamma x_{i_2} + \ldots\ldots\ldots + m_n\gamma x_{i_n}$$

$m_i \in M$, all $\gamma \in \Gamma$ and $x_{i_j} \in \{x_i \mid i \in \Lambda\}$, then P is a free left ΓM-module on the basis.

Definition: Let M be Γ-ring. A left ΓM-module P is called finitely generated if P can be generated by finite set of elements, that is, P is finitely generated if and only if there exist finitely many elements $x_1, x_2, \ldots, x_n \in P$ such that each $p \in P$ can be expressed as a linear Γ-combination $p = \sum_{i=1}^{n} m_i\gamma x_i$ of the x_i with coefficients $m_i \in M$ and all $\gamma \in \Gamma$.

If P is finitely generated, among all generating sets, then there are those with a minimum number of elements. The number of elements in a minimal generating set is called the rank of P. It is denoted by rankP.

Theorem: Let P be a non zero left Γ-module over a Γ-ring M. A non empty subset B of P is a basis of P if and only if every element of P can be uniquely written as a linear Γ-combination of the elements of B.

Proof: Let $B = \{x_i \mid i \in \Lambda\}$ be a basis of P. Let $x \in P$, then x can be written as a linear Γ-combination of the elements of B. Suppose that $x = \sum_{i\in\Lambda} m_i\gamma x_i$ and also $x = \sum_{i\in\Lambda} s_i\gamma x_i$ where $m_i, s_i \in M$ and all $\gamma \in \Gamma$ and are non zero finitely many indices $i \in \Lambda$.

$$\text{Then } \sum_{i\in\Lambda} m_i\gamma x_i = \sum_{i\in\Lambda} s_i\gamma x_i$$
$$\Rightarrow \sum_{i\in\Lambda} m_i\gamma x_i - \sum_{i\in\Lambda} s_i\gamma x_i = 0$$
$$\Rightarrow \sum_{i\in\Lambda} (m_i - s_i)\gamma x_i = 0.$$

Since B is a linearly Γ-independent, then $m_i - s_i = 0$ for all $i \in \Lambda$. Hence $m_i = s_i$ for all $i \in \Lambda$. Therefore every element of P can be expressed uniquely as a linear Γ-combination of the elements of B.

Conversely, suppose that every element of P can be expressed uniquely as a linear Γ-combination of the elements of B. Then clearly B generates P. If B is linearly Γ- dependent, then there exist distinct elements $x_1, x_2, \ldots\ldots, x_n$ of B and $m_1, m_2, \ldots\ldots, m_n \in M$ not all zero, such that $m_1\gamma x_1 + m_2\gamma x_2 + \ldots\ldots + m_n\gamma x_n = 0$ for all $\gamma \in \Gamma$. Also $0 = 0\gamma x_1 + 0\gamma x_2 + \ldots\ldots + 0\gamma x_n$. This is a contradiction, as 0 can now be expressed in more than one way as a linear Γ-combination of the elements of B. Hence B is linearly Γ-independent. Therefore B is a basis of P. Thus the theorem is proved.

Theorem: A left ΓM-module P is free if and only if it is isomorphic to a direct sum of copies of the left ΓM-module M^M, where M^M is a left ΓM-module over itself.

Proof: Suppose that P is a free left ΓM-module. Let $B = \{x_i \mid i \in \Lambda\}$ be a basis of P. Then by Theorem 2.4, we have $P = \bigoplus_{i\varepsilon\Lambda} M\gamma x_i$ for all $\gamma \in \Gamma$. Now consider a mapping $\varphi: M^M \to M\gamma x_i$ defined by $\varphi(m) = m\gamma x_i$. Let $m_1, m_2 \in M$, then $\varphi(m_1) = m_1\gamma x_i$ and $\varphi(m_2) = m_2\gamma x_i$. Therefore $\varphi(m_1 + m_2) = (m_1 + m_2)\gamma x_i$

$$= m_1\gamma x_i + m_2\gamma x_i$$
$$= \varphi(m_1) + \varphi(m_2)$$

Let $m \in M$, then $\varphi(m\gamma m_1) = (m\gamma m_1)\gamma x_i$

$$= m\gamma(m_1\gamma x_i)$$
$$= m\gamma\varphi(m_1).$$

Hence φ is a ΓM- homomorphism.

Let $\varphi(m_1) = \varphi(m_2)$

$\Rightarrow \quad m_1\gamma x_i = m_2\gamma x_i$

$\Rightarrow \quad m_1\gamma x_i - m_2\gamma x_i = 0$

$\Rightarrow \quad (m_1 - m_2)\gamma x_i = 0$

$\Rightarrow \quad m_1 - m_2 = 0$, since $x_i \neq 0$.

Thus $m_1 = m_2$. Hence φ is a one-one. Clearly φ is onto. Therefore $M^M \cong M\gamma x_i$. Thus P is isomorphic to a direct sum of copies of the left ΓM-module M^M.

Conversely, let $P \cong \bigoplus_{i\in\Lambda}$, where $M_i = M^M$ and let $B = \{e_i \mid i \in \Lambda\}$, where $e_i(j) = \delta_{ij}$ for $j \in \Lambda$. Hence δ_{ij} is the Kronicker delta function. Then B is a basis of P. Since if $x \in P$, then $x = \sum_{i\in\Lambda} m_i\gamma e_i$ and if $\sum_{i\in\Lambda} m_i\gamma e_i = 0$, then $(\sum_{i\in\Lambda} m_i\gamma e_i)(j) = 0$, that is, $m_j = 0$ for all $j \in \Lambda$. Hence P is a free left ΓM-module. Thus the theorem is proved.

Our next results show that all left Γ-modules over division Γ-rings are free left Γ-modules.

Theorem: Let Δ be a division Γ-ring and let P be a left ΓΔ-module. Then P is a free left ΓΔ-module.

Proof: We apply Zorn's Lemma to prove this theorem. Let X be a generating set of P and let B_0 be any linearly Γ-independent subset of P(B_0 can be the empty set). Let R be the set of all linearly Γ-independent subset of X containing B_0. Then R is partially ordered by set inclusion. If $\{B_i \mid i \in \Lambda\}$ is a chain in R, $\bigcup_{i\in\Lambda} B_i$ then is a linearly Γ-independent subset of X containing B_0. Thus every chain in R has an upper bound. By Zorn's Lemma, R has a maximal element. Let B be a maximal element of R. Then B is a maximal linearly Γ-independent subset of X that contains B_0. Now to show that B is a basis of P, all we have to show that $P = \langle B \rangle$, that is; B generates P. For this it is sufficient to show that $X \subseteq \langle B \rangle$. If $x \in X \backslash B$, then by maximality of B, the set $B \cup \{x\}$ is linearly Γ-dependent, so there exist distinct elements $x_1, x_2, \ldots\ldots, x_n$ in B and $m_1, m_2, \ldots\ldots, m_n$ in Δ, not all zero such that $m\gamma x + \sum_{i=1}^{n} m_i\gamma x_i = 0$ for all $\gamma \in \Gamma$.

Now $m \neq 0$, otherwise $m_i = 0$ for all $i = 1, 2, \ldots\ldots, n$ as $\{x_1, x_2, \ldots\ldots, x_n\}$ is a linearly Γ-independent set. Therefore

$$m\gamma x = -\sum_{i=1}^{n} m_i\gamma x_i$$

$$m^{-1}\gamma(m\gamma x) = m^{-1}\gamma\left(-\sum_{i=1}^{n} m_i\gamma x_i\right)$$

$$(m^{-1}\gamma m)\gamma x = -\sum_{i=1}^{n} m^{-1}\gamma(m_i\gamma x_i)$$

$= -\sum_{i=1}^{n} m^{-1}\gamma m_i\gamma x_i$

Therefore $x = -\sum_{i=1}^{n} m^{-1}\gamma m_i\gamma x_i \in \langle B\rangle$.

Hence $X\subseteq\langle B\rangle$. Thus P is a free left $\Gamma\Delta$-module. Hence the theorem is proved.

Corollary: Let P be a left Γ-module over a division Γ- ring Δ. Then a maximal linearly Γ- independent sub set of P is a basis of P .

If P is a free left ΓM-module, then its basis facilates the construction of a ΓM-homomorphism from P to another left ΓM- module N.

Theorem: Let M be a Γ-ring and let P be a free left ΓM-module with basis B. If N is any left ΓM-module and $\varphi: B\to N$ is any mapping, then there exists a unique ΓM-homomorphism $\Psi: P\to N$ such that $\Psi|_B = \varphi$.

Proof: Let $B=\{x_i \mid i\in\Lambda\}$. Then any $x\in P$ can be written uniquely as

$x = \sum_{i\in\Lambda} m_i\gamma x_i$, where $m_i\in M$ and all $\gamma\in\Gamma$ and at most finitely many $m_i \neq 0$. Define

$\Psi: P\to N$ by $\Psi(x) = \sum_{i\in\Lambda} m_i\gamma\varphi(x_i)$. Let $x,y\in P$, then $x = \sum_{i\in\Lambda} m_i\gamma x_i$ and

$y = \sum_{i\in\Lambda} m_i'\gamma x_i$, where m_i, $m_i'\in M$. Then $\Psi(x) = \sum_{i\in\Lambda} m_i\gamma\varphi(x_i)$ and $\Psi(y) = \sum_{i\in\Lambda} m_i'\gamma\varphi(x_i)$. Therefore $\Psi(x)+\Psi(y) = \sum_{i\in\Lambda} m_i\gamma\varphi(x_i) + \sum_{i\in\Lambda} m_i'\gamma\varphi(x_i)$

$= \sum_{i\in\Lambda} (m_i+m_i')\gamma\varphi(x_i) = \Psi(x+y)$.

Let $m\in M$, then $\Psi(m\gamma x) = \sum_{i\in\Lambda} (m\gamma m_i)\gamma\varphi(x_i)$

$= m\gamma \sum_{i\in\Lambda} m_i\gamma\varphi(x_i) = m\gamma\Psi(x)$.

Hence Ψ is a ΓM-homomorphism. Therefore $\Psi|_B = \varphi$. Thus the theorem is proved.

Let P be a left Γ-module over a Γ-ring M. If P is finitely generated, then denoted by ζ(P), the minimum number of generators of P. If P is not finitely generated, then we define $\zeta(P) = \infty$. Clearly if $P=\{0\}$, then $\zeta(P) = 0$ and $\zeta(P) = 1$ for a cyclic left ΓM-module P.

Let $\varphi: P\to N$ be a ΓM-homomorphism and let P be a finitely generated left ΓM-module. If $P = \langle x_1,x_2,\ldots\ldots,x_n\rangle$, then $\varphi(P) = \langle \varphi(x_1), \varphi(x_2),\ldots\ldots, \varphi(x_n)\rangle$. Since if $y\in P$, then $y = \varphi(x)$ for some $x\in P$ and as $x = \sum_{i=1}^{n} m_i\gamma x_i$ for some $m_i\in M$ and all $\gamma\in\Gamma$. So $\varphi(x) = \varphi(\sum_{i=1}^{n} m_i\gamma x_i) = \sum_{i=1}^{n} m_i\gamma\varphi(x_i)$.

Therefore $\zeta(\varphi(P))\leq\zeta(P)$. Thus if N is a sub ΓM-module of a finitely generated ΓM-module P, then $\zeta(P/N)\leq\zeta(P)$.

Theorem: Let P be a left Γ-module over a Γ-ring M and let N be a sub ΓM-module of P. If N and P/N are finitely generated ΓM-modules, then P is also finitely generated and $\zeta(P)\leq\zeta(N)+\zeta(P/N)$.

Proof: Let $X=\{x_1, x_2,\ldots\ldots, x_n\}$ be a minimal generating set of N and let $Y = \{y_1+N, y_2+N,\ldots\ldots, y_t+N\}$ be a minimal generating set of P/N. Now if $x\in P$, then $x+N\in P/N$, so there exist $m_1, m_2,\ldots\ldots, m_t \in M$ such that $x+N = \sum_{i=1}^{t} m_i\gamma(y_i+N)$ for all $\gamma\in\Gamma$ and so $x+N = \sum_{i=1}^{t} m_i\gamma y_i+N \Rightarrow x - \sum_{i=1}^{t} m_i\gamma y_i\in N$. Since $N = \langle x_1, x_2,\ldots\ldots, x_n\rangle$ then there exist $s_1,s_2,\ldots\ldots,s_n \in M$ such that $x - \sum_{i=1}^{t} m_i\gamma y_i = \sum_{j=1}^{t} s_j\gamma x_j$ and so $x = \sum_{i=1}^{t} m_i\gamma y_i + \sum_{j=1}^{t} s_j\gamma x_j$. This proves that $P = \langle x_1, x_2,\ldots\ldots, x_n, y_1, y_2,\ldots\ldots, y_t\rangle$ Hence $\zeta(P) \leq n+t = \zeta(N)+\zeta(P/N)$. Thus the theorem is proved.

Lemma: Let P be a free left Γ- module over a Γ- ring M. If P has an infinite basis, then no finite subset of P can generate P.

Proof: Let B be an infinite basis of P and suppose on the contrary that Y is a finite subset of P and $P = \langle Y\rangle$. Since B is a basis of P, for each $y\in Y$, there exist a finite subset $\{x_1, x_2,\ldots\ldots, x_k\}$ of distinct elements of B and $m_1, m_2,\ldots, m_k\in M\backslash\{0\}$ so that $y = m_1\gamma x_1+m_2\gamma x_2+\ldots\ldots+ m_k\gamma x_k$ for all $\gamma\in\Gamma$.

Thus there is a finite subset X of B such that every element of Y is a linear Γ-combination of the elements of X, that is, $Y\subseteq\langle X\rangle$ and so $P = \langle X\rangle$. Since X is a finite subset of B, so B X is nonempty. But then for $x\in B\backslash X$, the set $X\cup\{x\}$ is a linearly Γ-dependent subset of B; a contradiction. Hence Y is infinite. Thus the lemma is proved.

Theorem: Let M be a Γ-ring and let P be a free left ΓM-module with an infinite basis B. Then every basis of P has the same cardinality as B.

Proof: Let B′ be another basis of P. Thus by Lemma 2.10, B′ is an infinite set. Now let F(B′) be the set of all finite subset of B′. Then for each x∈B, there are uniquely determined distinct elements $y_1, y_2, \ldots, y_k$ of B′ such that $x = \sum_{i=1}^{k} m_i\gamma y_i$, where $m_1, m_2, \ldots, m_k \in M\backslash \{0\}$ and all γ∈Γ. Thus, we have a well defined mapping φ: B→F(B′) given by $\varphi(x) = \{ y_1, y_2, \ldots, y_k\}$. Note that φ(B) is an infinite set. If on the contrary φ(B) is finite, then every element of B is a linear Γ-combination of elements of φ(B), that is; $B \subseteq \langle\varphi(B)\rangle$. But then $P = \langle\varphi(B)\rangle$, a contradiction to the Lemma 2.10.

Next, we show that for every X∈φ(B), the set $\varphi^{-1}(X)$ is finite. If $x\in \varphi^{-1}(X)$, then by definition of φ, we have $x\in\langle X\rangle$. Since X is a finite subset of B′, then there is a finite subset Y of B so that $X\subseteq\langle Y\rangle$. Therefore $x\in\langle X\rangle$ and it implies that either x∈Y or x is a linear Γ- combination of the elements of Y. In the later case, $Y\cup\{x\}$ is a linearly Γ- dependent subset of B, a contradiction. Therefore x∈Y and so $\varphi^{-1}(X)\subseteq Y$. Hence $\varphi^{-1}(X)$ is a finite set.

Now consider the collection of sets $\{\varphi^{-1}(X) \mid X\in F(B)\}$. Clearly $\bigcup_{x\in F(B)} \varphi^{-1}(X) = B$. We claim that $\varphi^{-1}(X) \cap \varphi^{-1}(Y)$ is non empty, whenever $\varphi^{-1}(X)\neq\varphi^{-1}(Y)$. Suppose that $x\in\varphi^{-1}(X) \cap \varphi^{-1}(Y)$. Since $\varphi^{-1}(X) \cap\varphi^{-1}(Y)\subseteq B$ and B is a basis of P, so x≠0. Let $\varphi^{-1}(X) = \{ x_1, x_2, \ldots, x_t\}$ and $\varphi^{-1}(Y) = \{ y_1, y_2, \ldots, y_n \}$. Then $x\in \varphi^{-1}(X)$ implies that $x = \sum_{i=1}^{t} m_i\gamma x_i$, where $m_1, m_2, \ldots, m_t\in M$ and all γ∈Γ and $x\in\varphi^{-1}(Y)$ implies that $x = \sum_{i=1}^{n} s_j\gamma x_j$, where $s_1, s_2, \ldots, s_n\in M$ and all γ∈Γ. But then $\sum_{i=1}^{t} m_i\gamma x_i = \sum_{j=1}^{n} s_j\gamma x_j$. So $\sum_{i=1}^{t} m_i\gamma x_i - \sum_{j=1}^{n} s_j\gamma x_j = 0$. Thus $m_i = 0$, i = 1,2, …., t and $s_j = 0$, j = 1, 2, ………., n. Thus x = 0, a contradiction.

Hence, the sets $\varphi^{-1}(X)$, X∈φ(B) form a partition of B. Now for each X∈φ(B), order the elements of $\varphi^{-1}(X)$, say $x_1, x_2, \ldots, x_n$ and define a mapping g_X: $\varphi^{-1}(X)\to\varphi(B)$ by $g_X(x_k)=(X, k)$. Let $x_k, x_{k'}\in \varphi^{-1}(X)$, then $g_X(x_k) = (X, k)$ and $g_X(x_{k'}) = (X, k')$.

Let $g_X(x_k) = g_X(x_{k'})$

$\Rightarrow$ (X, k) = (X, k′)

$\Rightarrow$ k = k′.

Thus $x_k = x_{k'}$. Hence g_X is one-one. It now follows that the mapping g: $B\to\varphi(B) \times Z^+$ defined by $g(x) = g_X(x)$, where $x\in\varphi^{-1}(X)$.

Let x = x′. Then $g_X(x) = g_X(x')$. So g(x) = g(x′). Hence g is well defined.

Thus $|B| \leq |\varphi(B)\times Z^+|$

$= |\varphi(B)||Z^+| = |\varphi(B)|N_0$, where N_0 is the cardinality of Z^+

$\leq |\varphi(B)| \leq |F(B')| = |B'|$. Hence $|B| \leq |B'|$.

Interchanging the role of B and B′, we get $|B'| \leq |B|$. Hence by Theorem 1.1, |B| = |B′|. Thus every basis of P has the same cardinality. Hence the theorem is proved.

Invariant rank property of free left γm-modules

Definition: Let P be a free left Γ-module over a Γ- ring M such that any two bases of P have same cardinality. Then the cardinality of a basis of P is also called the rank of P over M and we can write rankP = | B |, where B is a basis of P. We say that a Γ- ring M has an invariant rank property if for every free left ΓM-module P, the rank of P over M is defined, that is, any two bases of P have the same cardinality.

We have shown in Theorem 2.6, that a left Γ-module over a division Γ-ring is a free left Γ- module. Now we prove that the rank of such a Γ-module is defined.

Theorem: If P is a Γ-module over a division Γ- ring Δ, then any two bases of P have same cardinality.

Proof: Let B and B′ be two bases of P. If either B or B′ is infinite, then by Theorem 2.11, | B | = | B′|. Therefore, we assume that B and B′ are finite. Let $B = \{ x_1, x_2, \ldots, x_t\}$ and $B' = \{ y_1, y_2, \ldots, y_n\}$. Without any loss we may assume that n = t. We write $y_n = m_1\gamma x_1+m_2\gamma x_2 + \ldots + m_t\gamma x_t$, where $m_i\in M$ and all γ∈Γ. Let k be the first index such that $m_k\neq 0$, so

$m_k\gamma x_k = y_n - m_1\gamma x_1 - m_2\gamma x_2 - \ldots - m_{k-1}\gamma x_{k-1} - m_{k+1}\gamma x_{k+1} - \ldots - m_t\gamma x_t$. Then

$m_k^{-1}(m_k\gamma x_k) = m_k^{-1} \gamma(y_n - m_1\gamma x_1 - m_2\gamma x_2 - \ldots - m_{k-1}\gamma x_{k-1} - m_{k+1}\gamma x_{k+1} - \ldots - m_t\gamma x_t)$

$\Rightarrow (m_k^{-1} m_k)\gamma x_k = m_k^{-1} \gamma y_n - m_k^{-1} \gamma m_1\gamma x_1 - m_k^{-1} \gamma m_2\gamma x_2 - \ldots - m_k^{-1} \gamma m_{k-1}\gamma x_{k-1} - m_k^{-1} \gamma m_{k+1}\gamma x_{k+1} - \ldots - m_k^{-1} \gamma m_t\gamma x_t$

$\Rightarrow 1\gamma x_k = m_k^{-1} \gamma y_n - m_k^{-1} \gamma m_1\gamma x_1 - m_k^{-1} \gamma m_2\gamma x_2 - \ldots - m_k^{-1} \gamma m_{k-1}\gamma x_{k-1} - m_k^{-1} \gamma m_{k+1}\gamma x_{k+1} - \ldots - m_k^{-1} \gamma m_t\gamma x_t$

Therefore $x_k = m_k^{-1} \gamma y_n - m_k^{-1} \gamma m_1\gamma x_1 - m_k^{-1} \gamma m_2\gamma x_2 - \ldots - m_k^{-1} \gamma m_{k-1}\gamma x_{k-1} - m_k^{-1} \gamma m_{k+1}\gamma x_{k+1} - \ldots - m_k^{-1} \gamma m_t\gamma x_t$

Thus the set $\{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m, y_n\}$ generates P. In particular

$y_{n-1} = s\gamma y_n + \sum_{i=1}^{t} s_i\gamma x_i$. Now as $\{y_n, y_{n-1}\}$ is a linearly Γ-independent set, then

$y_{n-1} - s\gamma y_n \neq 0$. Let j be the first index such that $s_j\neq 0$; then x_j is a linear Γ- combination of y_{n-1}, y_n and x_i, i≠j, k and so $\{ y_{n-1}, y_n\}\cup\{x_i | i\neq j,k\}$ generates P. In particular y_{n-2} is a linear Γ- combination of y_n, y_{n-1} and x_i, i≠j, k. The above process of adding an element of B′ and deleting an element of B may be repeated. If n < t, then after n steps, we conclude that $\{y_n, y_{n-1}, \ldots, y_{t-n+1}\}$ generates P. In particular y_{t-n} is a linear

Γ-combination of y_n, y_{n-1}, ..., y_{t-n+1} and this contradicts the linearly Γ-dependence of B′. Therefore t = n. Hence B and B′ have same cardinality. Thus the theorem is proved.

Let P be a left Γ-module over a Γ-ring M and let A be an ideal of M .If

$A\Gamma P = \{ \sum_{i=1}^{k} a_i\gamma x_i \mid a_i \in A, x_i \in P$ and all $\gamma \in \Gamma\}$, then it is easy to verify that AΓP is a sub ΓM- module of P. Also P/AΓP is a Γ-M/A-module with the action of M/A or P/AΓP given by $(m+A)\gamma(x+A\Gamma P) = m\gamma x + A\Gamma P$, where $m \in M$, $x \in P$ and all $\gamma \in \Gamma$. This is a well defined operation; if $m+A = m'+A$ and $x+A\Gamma P = x'+A\Gamma P$, then $m\gamma x - m'\gamma x' = m\gamma x - m\gamma x' + m\gamma x' - m'\gamma x' = m\gamma(x - x') + (m - m')\gamma x' \in A\Gamma P$ and so $m\gamma x + A\Gamma P = m'\gamma x' + A\Gamma P$.

Lemma: Let M be a Γ-ring, let A be a proper ideal of M and let P be a free left ΓM-module with a basis B. Then P/AΓP is a free left Γ-M/A-module with basis Π(B) and | B | = | Π(B) |, where Π:P→P/AΓP is a canonical ΓM-epimerphism of ΓM-modules.

Proof: Let $B = \{x_i \mid i \in \Lambda\}$ be a basis of P. Then $\Pi(B) = \{x_i + A\Gamma P \mid x_i \in B\}$. We now prove this lemma in steps:

Step 1: Π(B) generates P/AΓP. If $x+A\Gamma P \in P/A\Gamma P$ October 19, 2006, then as B is a basis of P, so, $x = \sum m_i\gamma x_i$, where $m_i \in M$, all $\gamma \in \Gamma$ and $m_i \neq 0$ for finitely many $i \in \Lambda$. Thus

$x+A\Gamma P = \sum m_i\gamma x_i + A\Gamma P$
$= \sum (m_i+A)\gamma(x_i+A\Gamma P)$
$= \sum (x_i+A)\gamma \Pi(x_i)$.

Hence Π(B) generates P/AΓP.

Step 2: | B | = |Π(B)|. Let x+AΓP and x′+ AΓP be elements of Π(B) such that $x \neq x'$ and x+AGP = x′+AGP. Then $x - x' \in A\Gamma P$. So $x - x' = \sum_{i=1}^{n} a_j\gamma y_j$,

where $a_j \in A\backslash\{0\}$, $y_j \in P$, for j = 1,2,3, ... , n.

Now writing each y_j as a linear Γ-combination of elements of B over M, we conclude that x – x′ is a linear Γ-combination of elements of B with coefficient from A. Since B is a basis of P, on equating the coefficient of x, we get $1 \in A$, a contradiction as $A \neq M$. Hence | B | = |Π(B)|.

Step 3: Π(B) is a linearly Γ-independent set over M. Let $x_1+A\Gamma P$, $x_2+A\Gamma P$, ... , $x_k+A\Gamma P$ be k distinct elements of Π(B) and let m_1+A, m_2+A, ... , m_k+A be elements of M/A so that

$$\sum_{i=1}^{k} (m_i+A)\gamma(x_i+A\Gamma P) = A\Gamma P$$

$$\Rightarrow \sum_{i=1}^{k} (m_i\gamma x_i + A\Gamma P) = A\Gamma P$$

$$\Rightarrow \sum_{i=1}^{k} m_i\gamma x_i + A\Gamma P = A\Gamma P$$

$$\Rightarrow \sum_{i=1}^{k} m_i\gamma x_i \in A\Gamma P.$$

If $\sum_{i=1}^{k} m_i\gamma x_i = 0$, then Γ- independence of $\{x_1,x_2, ... ,x_k\}$ implies that $m_1 = m_2 = . . . = m_k = 0$. Therefore $m_1+A = m_2+A = ... = m_k+A = A$.

Hence $x_1+A\Gamma P$, $x_2+A\Gamma P$, ... , $x_k+A\Gamma P$ are linearly Γ- independent over M/A. If $\sum_{i=1}^{k} m_i\gamma x_i \neq 0$, then $\sum_{i=1}^{k} m_i\gamma x_i = \sum_{i=1}^{k} a_i\gamma y_i$, where each $a_i \in A\backslash\{0\}$ and $y_i \in P$. Since each y_i is a linear Γ- combination of elements of B over M, we have $\sum_{i=1}^{k} m_i\gamma x_i = \sum_{j=1}^{t} b_j\gamma z_j$, where $z_j \in B$ and $b_j \in A\backslash\{0\}$. Thus, $\sum_{i=1}^{k} m_i\gamma x_i - \sum_{j=1}^{t} b_j\gamma z_j = 0$. If $x_i = z_j$ for some j, then the coefficients of x_i is $m_i - b_j$ and then Γ-independence of B implies that $m_i - b_j = 0$, so $m_i = b_j \in A$. If $x_i \neq z_j$ for any j = 1, 2, 3, ... , t, then $m_i = 0$. Thus, in any case $m_i+A = A$ for all i = 1, 2, 3, ..., k. Hence $\{ x_1+A\Gamma P, x_2+A\Gamma P, ... , x_k+A\Gamma P\}$ is a linearly Γ-independent set over M. This shows that every finite subset of Π(B) is linearly Γ-independent over M. Therefore Π(B) is linearly Γ-independent over M/A. Hence Π(B) is a basis of left Γ-M/A-module P/AΓP. Therefore P/AΓP is a free left Γ-M/S-module. Thus the lemma is proved.

Theorem: Let φ: M→S be a nonzero Γ- ring epimorphism. If S has the invariant rank property, then M has the invariant rank property.

Proof: Let P be a free left ΓM-module with basis B and B′ and K = kerφ. Note that $K \neq M$ as φ is a nonzero Γ- ring homomorphism. If K = {0}, then φ is a Γ- ring isomorphism. Now P can be viewed as a ΓS-module with scalar multiplication given by $s\gamma x = \varphi^{-1}(s)\gamma x$ for all $s \in S$, $x \in P$ and $\gamma \in \Gamma$. Clearly B and B′ are also basis of ΓS-module P. Hence | B | = | B′|.

Now let $K \neq \{0\}$. By 1st isomorphism Theorem of Γ-rings, $M/K \cong S$.

Therefore M/K has invariant rank property. Also by Lemma 3.3, P/KΓPis a free left Γ-M/K-module with basis Π(B) and Π(B′) such that | Π(B)| = | B | and | Π(B′)| = |B′|, where Π:P→P/KΓP is the canonical Γ- epimorphism. Hence | B | = | B′|. Thus the theorem proved.

Theorem: Let M be a Γ-ring with the invariant rank property and let P and Q be free left ΓM-modules. Then P and Q are isomorphic if and only if rankP = rankQ.

Proof: Let φ: P→Q be a ΓM-isomorphism. If B is a basis of P, then we verify that φ(B) is a basis of Q. Since φ is one-one and onto, then | B| = |φ(B)|. Hence rankP = rankQ.

Conversely, let rankP = rankQ. Let B be a basis of P and B′ be a basis of Q.

Since rankP = rankQ, so | B | = | B′|. Thus, there is a one-one and onto mapping Ψ: B→B′. By Theorem 2.8, there is a ΓM-homomorphism φ: P→Q such that $\varphi|_B = \Psi$. Clearly φ is also one-one and onto. Hence P≅Q. Thus the theorem is proved.

Theorem: Let P be a left free Γ-module over a Γ-ring M and let Q be a sub ΓM-module of P. Then rankP = rankQ+rankP/Q.

Proof: Let B′ is a basis of Q. Then it can be extended to form a basis B of P. Let $B = \{x_i \mid i\in\Lambda\}$ be such that $B' = \{x_j \mid j\in\Lambda'\}$ and Λ′⊆Λ. Therefore |B| = |Λ| = |Λ′|+|Λ\Λ′| = rankQ+|Λ\Λ′|. We now show that rank P/Q= |Λ\Λ′| and this will prove the theorem. Let $B'' = \{ x_i \mid i\in\Lambda\backslash\Lambda'\}$ and $\overline{B} = \{x_i+Q \mid i\in\Lambda\backslash\Lambda'\}$. Then for each i∈Λ\Λ′, $x_i+Q\neq Q$, since otherwise $x_i\in Q$ implies that x_i is a linear Γ- combination of elements of Q, a contradiction. Similarly for i,j∈Λ\Λ′, i ≠j, we have $x_i+Q\neq x_j+Q$. Hence |B| = |B′′| = |Λ\Λ′|. Now we show that $\overline{B}$ is a basis of P/Q. First we show that $\overline{B}$ is a linearly Γ-independent set. Let $x_{i1}+Q, x_{i2}+Q, \ldots, x_{it}+Q$ be distinct elements of $\overline{B}$ such that $\sum_{r=1}^{t} m_l\gamma(x_{ir}+Q) = Q$, where $m_l\in M$, so $\sum_{r=1}^{t} m_l x_{i_r}+Q = Q$. then $\sum_{r=1}^{t} m_l\gamma x_{i_r}Q$. If $\sum_{r=1}^{t} m_l\gamma x_{i_r}\neq 0$, then it is a linear Γ-combination of element of B′ and this contradicts the linear Γ- independence of B. Therefore $\sum_{r=1}^{t} m_l\gamma x_{i_r}= 0$ and it implies that $m_r = 0$ for all r = 1,2, ... , t as $\{x_{i_1}, x_{i_2}, \ldots, x_{i_T}\}\subseteq B$. Hence every finite subset of $\overline{B}$ is linearly Γ- independent and so $\overline{B}$ is linearly Γ- independent.

Now, if x+Q ∈P/Q, then as x∈P so $x = \sum_{i\in\Lambda} m_i\gamma x_i$, where $m_i \in M$ and $m_i\neq 0$ for finitely many indices i∈Λ. Thus $x+Q = \sum_{i\in\Lambda} m_i\gamma x_i+Q = \sum_{i\in\Lambda} m_i\gamma(x_i+Q) = \sum_{i\in\Lambda'\backslash\Lambda} m_i\gamma(x_i+Q)$, as $x_i\in Q$ for all i∈Λ′. Hence $\overline{B}$ generates P/Q. Therefore $\overline{B}$ is a basis of P/Q. Hence rank P/Q=|Λ\Λ′|. Thus rank P = rankQ+rank P/Q. Hence the theorem is proved.

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