Asian Journal of Information Technology 5 (1) : 20-23, 2006

On Matching Polynomials of a Simple Hexagonal Lattice

S.A. Wahid
Department of Mathematics and Computer Sciences, The Centre for Graph Polynomials,
The University of the West Indies, St.Augustine, Trinidad and Tobago

Abstract: A recurrence relation is derived for the matching polynomial of a 2 x n hexagonal lattice.Explicit formulae are then obtained for the first ten and the final four coefficients.

Key words: Polynomial, hexagonal lattice, coefficients

INTRODUCTION

The graphs considered here are all finite and without loops or multiple edges. By a matching in a graph G, we mean a spanning subgraph of G whose components are nodes and edges. We define a defect-d matching to be a matching with d component nodes, see Berge[1,2] and Little[3].We denote the number of defect – d matchings in G by $N_d(G)$.A perfect matching is a defect-0 matching. Let G be a graph with p nodes. We associate the weights $w_1$and w_2 with each node and edge respectively. With each matching in G, we associate the monomial $w_1^{2k}w_2^k$ i.e., the product of the weights of the components.Then the matching polynomial of G is $M(G;w_1,w_2)= \sum_k \alpha_k w_1^{p-2k} w_2^k$

where α_k is the number of matchings in G with k edges and the summation is taken over all values of k.The basic properties of matching polynomials can be found in Farrell[4].A momomer – dimer covering of a molecule is equivalent to a matching in a graph.

The importance of monomer-dimer coverings in statistical mechanics have been examined by Heilmann and Lieb[5].

The acyclic polynomial which is obtained from the matching polynomial by putting w_1 = x and w_2 = -1 has been been shown to be a useful tool in mathematical chemistry, see Gutman[6,7] and Aihara[8].

We denote by A_n the lattice formed by concatenating two layers of hexagons.

MATERIALS AND METHODS

The following results given in[4] are the essential methods used to reduce a graph. By applying this reduction process we can set up recurrences for matching polynomials of graphs.They are useful in the material which follows.

The fundamental reduction process: Let G be a graph having an edge e. Let $G^{/}$ be the graph obtained from G by deleting e and $G^{//}$, the graph obtained from G by removing the nodes at the ends of e. Then

$$M(G;w_1,w_2)= M(G^{/};w_1,w_2) + w_2\, M(G^{//};w_1,w_2).$$

The component theorem: If G consists of two components R and S, then $M(G;w_1,w_2) = M(R;w_1,w_2)\, M(S;w_1,w_2)$.

We use generating functions to solve the system of recurrences that are obtained by using (a) and (b).The following result (c) is used to find the first three coefficients.

(c)Let G have p nodes and q edges.Then in $M(G;w_1,w_2)$,

(i)$a_0 = 1$, $a_1 = q$ and

(ii)$a_2 = \binom{q}{2} - \beta$, where β is the number of paths of length 2 in G.

It is easy to find β by choosing two edges at each node in all different ways i.e.,

$$\beta= \sum_{i=1}^{p} \binom{v_i}{2},$$ where v_i is the valency of node i in G.

The successive coefficients are found by first obtaining a recurrence for the hexagonal lattice A_n .Once an expression for a coefficient is found, we use these formulae to generate the value of the successive coefficient.

RESULTS

The matching polynomial of A_n written as a recurrence: We apply the reduction process as stated in result (a) initially to the graph A_n and all reduced graphs that are shown in Fig. 1.

In so doing the following recurrences are obtained that are valid for $n \geq 1$.

Lemma 1

(I) $A_n = B_{n-1} + w_2 C_{n-1}$.

(ii) $B_n = w_1 D_n + w_2 E_n$.

(iii) $C_n = w_1 F_n + w_1 w_2 G_{n-1} + w_2^2 w_1 D_{n-1} + w_2^2 J_{n-1} + w_2^3 C_{n-1} + w_2^3 G_{n-1}$.

(iv) $D_n = w_1 C_n + w_2 w_1 K_n + w_2^2 w_1 A_n + w_2^3 D_{n-1}$.

(v) $E_n = w_1 F_n + (w_2^2 + w_2 w_1^2) A_n + w_2^2 w_1 M_{n-1}$.

(vi) $F_n = w_1 P_n + 2 w_2 w_1 A_n + w_2^2 M_{n-1} + w_2^2 Q_{n-1}$.

(vii) $G_n = w_1 J_n + (w_2 w_1^2 + w_2^2) K_n + w_2^2 w_1^2 A_n + w_2^3 w_1 D_{n-1}$.

(viii) $J_n = w_1 C_n + w_2 R_n$.

(ix) $K_n = w_1^2 A_n + w_2 w_1 M_{n-1} + w_2 w_1 D_{n-1} + w_1 w_2^2 R_{n-1} + w_2^3 K_{n-1}$.

(x) $M_n = (w_1^2 + w_2) R_n + w_2 w_1 K_n + w_2 w_1 P_n + w_2^2 w_1 A_n + w_2^3 M_{n-1}$.

(xi) $P_n = w_1^2 A_n + w_2 w_1 M_{n-1} + w_2 w_1 J_{n-1} + w_2^2 w_1 F_{n-1} + w_2^3 P_{n-1}$.

(xii) $Q_n = w_1 C_n + w_2 F_n$.

(xiii) $R_n = w_1 K_n + (w_2 w_1^2 + w_2^2) J_{n-1} + w_2^2 w_1^2 F_{n-1} + w_2^3 w_1 P_{n-1}$.

We then use standard techniques of generating functions to get the following recurrence in A_n.

Theorem 1

$$A_n = (w_1^6 + 8w_2 w_1^4 + 17 w_2^2 w_1^2 + 8 w_2^3) A_{n-1} - (3 w_2^2 w_1^8 + 28 w_2^3 w_1^6 + 84 w_2^4 w_1^4 + 88 w_2^5 w_1^2 + 28 w_2^6) A_{n-2} + (3 w_2^4 w_1^{10} + 32 w_2^5 w_1^8 + 126 w_2^6 w_1^6 + 228 w_2^7 w_1^4 + 191 w_2^8 w_1^2 + 56 w_2^9) A_{n-3} - (w_2^6 w_1^{12} + 12 w_2^7 w_1^{10} + 61 w_2^8 w_1^8 + 172 w_2^9 w_1^6 + 281 w_2^{10} w_1^4 + 224 w_2^{11} w_1^2 + 70 w_2^{12}) A_{n-4} + (3 w_2^{10} w_1^{10} + 28 w_2^{11} w_1^8 + 102 w_2^{12} w_1^6 + 176 w_2^{13} w_1^4 + 151 w_2^{14} w_1^2 + 56 w_2^{15}) A_{n-5} - (3 w_2^{14} w_1^8 + 20 w_2^{15} w_1^6 + 52 w_2^{16} w_1^4 + 56 w_2^{17} w_1^2 + 28 w_2^{18}) A_{n-6} + (w_2^{18} w_1^6 + 4 w_2^{19} w_1^4 + 9 w_2^{20} w_1^2 + 8 w_2^{21}) A_{n-7} - w_2^{24} A_{n-8}. (n \geq 9).$$

A Table of coefficients of $M(A_n; w_1, w_2)$ for n = 1 to 9 is given below. We note that A_1 is the path on four nodes.

Theorem 1 is now used to give the number of k-matchings ie. the coefficients of matching polynomials.

Number of k-matchings in A_n: As mentioned before, the number of defect – k matchings in G is written as $N_k(G)$ and the number of k-matchings is $\alpha_k(G)$. We state the formulae for $\alpha_k(A_n)$ for the first three values of k.

Theorem 2

$$\alpha_0(A_n) = 1, \qquad n \geq 1$$
$$\alpha_1(A_n) = 8n - 5, \qquad n \geq 1$$
$$\text{and } \alpha_2(A_n) = 32n^2 - 58n + 29, \qquad n \geq 2.$$

Table 1: Coefficients of matchimg polynomials

n	Coefficients of $M(A_n; w_1, w_2)$ written in increasing powers of w_2
1	1,3,1
2	1,11,41,61,31,3
3	1,19,143,547,1132,1244,661,135,6
4	1,27,309,1961,7579,18441,28218,26354,14086,3817,414,10
5	1,35,539,4815,27694,107618,288453,534829,677921,571751,306251,96534,15742,1038,15
6	1,43,833,9621,73893,398535,1553872,4442120,9345547,14398537,16030128,12605006,6759506,485317,52357,2276,21
7	1,51,1191,16891,162688,1127448,5810931,22707547,67973178,156433962,276262234,371712868,376265551,281123407,150857707,55938623,13553613,1971415,149623,4529,28
8	1,59,1613,27137,314687,2669877,17169350,85552126,334853938,1037589342,2553803425,4990433843,7711656287,9352803755,8801483951,6324560617,3395386889,1322124110,358471282,63850156,6842838,381543,8368,36
9	1,67,2099,40871,554594,5573374,43033569,261410761,1268965511,4972794985,15828788075,41046957489,86738080903,149025190015,207199626684,231437473113,20557734315.143255713873,76929903871,31099715551,9174548032,1893084337,257272466,21031001,889793,14577,45

Proof: A_n has 6n - 2 nodes. There is only one matching with zero edges i.e the empty graph with 6n - 2 nodes. Therefore(i) follows. Now A_n has n -1 cells. The first cell has 11 edges and the remaining (n-2) cells each have 8 edges. Thus $\alpha_1 = 11 + 8(n-2) = 8n - 5$.

There are 2n + 4 nodes of degree 2 and 4n-6 nodes of degree 3 in A_n. By using result (c) above, the expression follows easily for $\alpha_2(A_n)$.

By equating the coefficient of w_2^k in Theorem 2, we obtain the following result.

Theorem 3: A_n has a k-matching if and only if $0 < k \leq (3n-1)$ In this case,

$$a_k(A_n) = a_k(A_{n-1}) + 8a_{k-1}(A_{n-1}) + 17a_{k-2}(A_{n-1}) + 8a_{k-3}(A_{n-1}) - 3a_{k-2}(A_{n-2}) - 28a_{k-3}(A_{n-2}) - 84a_{k-4}(A_{n-2}) - 88a_{k-5}(A_{n-2}) - 28a_{k-6}(A_{n-2}) + 3a_{k-4}(A_{n-3}) + 32a_{k-5}(A_{n-3}) + 126a_{k-6}(A_{n-3}) + 228a_{k-7}(A_{n-3}) + 191a_{k-8}(A_{n-3}) + 56a_{k-9}(A_{n-3}) - a_{k-6}(A_{n-4}) - 12a_{k-7}(A_{n-4}) - 61a_{k-8}(A_{n-4}) - 172a_{k-9}(A_{n-4}) - 281a_{k-10}(A_{n-4}) - 224a_{k-11}(A_{n-4}) - 70a_{k-12}(A_{n-4}) + 3a_{k-10}(A_{n-5}) + 28a_{k-11}(A_{n-5}) + 102a_{k-12}(A_{n-5}) + 176a_{k-13}(A_{n-5}) + 151a_{k-14}(A_{n-5}) + 56a_{k-15}(A_{n-5}) - 3a_{k-14}(A_{n-6}) - 20a_{k-15}(A_{n-6}) - 52a_{k-16}(A_{n-6}) - 56a_{k-17}(A_{n-6}) - 28a_{k-18}(A_{n-6}) + a_{k-18}(A_{n-7}) + 4a_{k-19}(A_{n-7}) + 9a_{k-20}(A_{n-7}) + 8a_{k-21}(A_{n-7}) - a_{k-24}(A_{n-8})(n ^3 9).$$

By putting k = 3,4,5,6,7 and 8 respectively in Theorem 3, we use generating functions to obtain the following result.

Theorem 4:
(i)$\alpha_3(A_n)$-175 + (1154/3)n -304n^2 + (256/3)n^3. ($n \geq 2$).
(ii)$\alpha_4(A_n)$ =1079 – (7501/3)n + (6934/3) n^2 – (3008/3) n^3 + (512/3) n^4 ($n \geq 3$).
(iii)$\alpha_5(A_n)$ = -6755 + (242669/15)n - (49022/3) n^2 + 8656 n^3 - (7168/3) n^4 + (4096/15)n^5. ($n \geq 3$).
(iv)$\alpha_6(A_n)$ = 42798 – (314279/3)n + (5038516/45) n^2 – 66620 n^3 + (208448/9) n^4 -(13312/3) n^5 + (16384/45)n^6 . ($n \geq 4$)
(v)$\alpha_7(A_n)$ = -273818 + (71436107/105)n - (34011836/45) n^2 + (21830804/45) n^3 – (1741984/9) n^4 + (2150912/45) n^5 – (303104/45)n^6 + (131072/315)n^7. ($n \geq 4$).
(vi)$\alpha_8(A_n)$ = 1766126 - (186256835/42)n + (3190961899/630) n^2 - (154254958/45) n^3 + (67447294/45) n^4 - (3888512/9) n^5 + (3595264/45)n^6 - (2718744/315)n^7 + (131072/315)n^8. ($n \geq 5$)

Defect –d-matchings in A_n: The following result is obtained from Theorem 1 by equating coefficients of the terms in w_1^d.

Theorem 5: A_n has a defect – d matching for $0 \leq d \leq (3n-1)$. In this case,

$N_d(A_n) = N_{d-6}(A_{n-1}) + 8 N_{d-4}(A_{n-1}) + 17 N_{d-2}(A_{n-1}) + 8 N_d(A_{n-1}) - 3 N_{d-8}(A_{n-2}) -$
$28N_{d-6}(A_{n-2}) - 84N_{d-4}(A_{n-2}) - 88N_{d-2}(A_{n-2}) - 28N_d(A_{n-2}) + 3N_{d-10}(A_{n-3}) +$
$32N_{d-8}(A_{n-3}) + 126N_{d-6}(A_{n-3}) + 228N_{d-4}(A_{n-3}) + 191N_{d-2}(A_{n-3}) + 56N_d(A_{n-3}) -$
$N_{d-12}(A_{n-4}) + 12N_{d-10}(A_{n-4}) - 61N_{d-8}(A_{n-4}) - 172N_{d-6}(A_{n-4}) - 281N_{d-4}(A_{n-4}) -$
$224N_{d-2}(A_{n-4}) - 70N_d(A_{n-4}) + 3N_{d-10}(A_{n-5}) + 28N_{d-8}(A_{n-5}) + 102N_{d-6}(A_{n-5}) +$
$176N_{d-4}(A_{n-5}) + 151N_{d-2}(A_{n-5}) + 56N_d(A_{n-5}) - 3N_{d-8}(A_{n-6}) - 20N_{d-6}(A_{n-6}) -$
$52N_{d-4}(A_{n-6}) - 56N_{d-2}(A_{n-6}) - 28N_d(A_{n-6}) + N_{d-6}(A_{n-7}) + 4N_{d-4}(A_{n-7}) +$
$9N_{d-2}(A_{n-7}) + 8N_d(A_{n-7}) - N_d(A_{n-8})$. ($n \geq 9$)

Lemma 2
$N_0(A_n) = 8N_0(A_{n-1}) - 28 N_0(A_{n-2}) + 56N_0(A_{n-3}) - 70N_0(A_{n-4}) + 56N_0(A_{n-5}) - 28N_0(A_{n-6}) + 8N_0(A_{n-7}) - N_0(A_{n-8})$.

We then use the method of generating functions to get the following result.

Theorem 6:

$$N_0(A_n) = \frac{n(n+1)}{2} \quad (n \geq 1).$$

By putting n = 2 in Lemma 2 , we get

$N_2(A_n) = 17N_0(A_{n-1}) + 8N_2(A_{n-1}) - 88N_0(A_{n-2}) - 28N_2(A_{n-2}) + 191N_0(A_{n-3}) + 56N_2(A_{n-3}) - 224N_0(A_{n-4}) - 70N_2(A_{n-4}) + 151N_0(A_{n-5}) + 56N_2(A_{n-5}) - 56N_0(A_{n-6}) - 28N_2(A_{n-6}) + 9N_0(A_{n-7}) + 8N_2(A_{n-7}) - N_2(A_{n-8})$. ($n \geq 9$).

We use Theorem 6 to substitute the values of $N_0(A_{n-k})$ for k =1,2,…,8 to get

$N_2(A_n) = 8N_2(A_{n-1}) - 28N_2(A_{n-2}) + 56N_2(A_{n-3}) - 70N_2(A_{n-4}) + 56N_2(A_{n-5}) - 28N_2(A_{n-6}) + 8N_2(A_{n-7}) - N_2(A_{n-8})$. ($n \geq 9$).

We solve by the method of generating functions to get the following result.

Theorem 7
$N_2(A_n)$ = -216 + (5452/15)n – (89867/360)n^2 + (22459/240)n^3 – (337/18)n^4 + (49/20)n^5 – (53/360)n^6 + (1/240)n^7 . ($n \geq 2$).

This procedure is repeated in turn for the values n = 4 and 6.In so doing we obtain the following result which gives the number of defect-4 and defect-6 matchings.

Theorem 8:
(i)$N_4(A_n)$ = -939 + (25826501/13860)n - (4985567/3150)n^2 + (687729289/907200)n^3 -(1181435/5184)n^4 + (16022679/34560)n^5 - (1009057/172800)n^6 + (99797/151200)n^7 – (979/24192)n^8 + (467/145152)n^9 – (67/518400)n^{10} + (1/316800)n^{11}. ($n \geq 2$).
(ii)$N_6(A_n)$ = 28168 – (4479316873/90090)n + (1821886442977/50450400)n^2 – (3615817719847 /259459200)n^3 + (134323583/44550)n^4 - (39042403529 /119750400)n^5 + (12460517/21772800)n^6 + (426586513 /87091200)n^7 – (23504843/38102400)n^8 + (1698391/21772800)n^9 – (139/43200)n^{10} + (40153/ 95800320)n^{11} – (269/29937600)n^{12} + (2071/3113510400)n^{13} – (17/871782912)n^{14} + (1/1779148800)n^{15}. ($n \geq 2$).

DISCUSSION

This article has paved the way for further research in lattice type graphs.It is possible to get results on other coefficients but the calculations are tedious.It is interesting to note that all thirteen graphs mentioned in the Fig. 2 have the same recurrence for their matching polynomials.It is not simple to get a recurrence for the general lattice with m rows of hexagons with each row having n hexagons.

REFERENCES

1. Berge, C., 1958. Sur Le Couplage d'un Graphe, C.R. Acad. Sci. Paris, 247: 258-259.
2. Berge, C., 1973. Graphs and Hypergraphs North-Holland,Amsterdam.
3. Little, C.H.C., D.D. Grant and D.A. Holton, 1975. On Defect – d Matchings in Graphs, Discrete Math. 13: 41 -54.
4. Farrell, E.J., 1979. Introduction to matching polynomials, J. Combin. Theory (B) 27: 75 -86.
5. Heilmann O.J. and E.H. Lieb, 1972. Theory of Monomer - Dimer Systems, Comm. Math. Phy. 25: 190-232.
6. Gutman, I., 1978. The energy of a graph, bericht der mathematisch-statistiche sektion Im Forschungsentrum Graz, 103: 1-22.
7. Gutman, I., 1979. The Matching Polynomial , Match., 6: 75 – 91.
8. Aihara, J., 1976. A new definition of dewar-type resonance energy, J. Amer. Chem., pp: 2750-2758.