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Design of Narrow Band FIR Filters

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Abstract: A new concept for the design of finite impulse response (FIR) filters is introduced here. This analytical approach has led to three fundamental and new results. First, a differential equation for the approximating polynomial of the filter is developed. Second, the linear differential equation is solved by iteration, yielding an algorithm for recursive evaluation of the impulse response of a filter. Finally, the degree equation of the filter is introduced. We present the design of 4 types of narrow band FIR filters: the maximally at notch FIR filter, the equiripple notch FIR filter, the equiripple DC-notch FIR filter and the equiripple comb FIR filter. Equiripple filters are optimal in the Chebyshev sense. The design procedure starts with frequency specifications of the filter and ends with a recursive evaluation of the impulse response coefficients refraining from other numerical procedures.

Key words: FIR notch filter, FIR comb filter, degree equation of FIR filter, recursive algorithms

INTRODUCTION

The design of digital FIR filters may seem to be a closed chapter in the history of digital signal processing. Several textbooks devoted to DSP, Mitra (1998) and Orfanidis (1996) have suggested that a few numerical methods including the Fast Fourier Transform and the Remez exchange algorithm implemented by the famous Parks-McClellan code (McClellan *et al*., 1973), are all that is needed for FIR filter design. Until recently, however, only a limited number of closed form solutions have been available (Zahradnik and Vlček, 2005, 2007). This study deals with the analytical design of FIR filters including the class of optimal filters, which cannot be obtained at all with the Remez algorithm. Our analytical approach in the design of FIR filters has led to three fundamental and new results:

- A differential equation for the approximating function and its linear version.
- A degree equation of the filter.
- A recursive algorithm for the impulse re-sponse coefficients.

The development of the differential equation is novel, and its concept is essential. It consequently plays a fundamental role in replacing spectral transformation by an algebraic evaluation of the impulse response coefficients. The filter degree is evaluated through closed form formulae. This is a novel and non-standard result, as an empirical estimation for the filter degree is frequently used (Rabiner, 1973; Rabiner and Gold, 1975). The solution of the differential equation provides the recurrence algorithm for the impulse response coefficients, refraining from the Fast Fourier Transform. This is important, because the Fast Fourier Transform cannot be used on several analytical filter design methods. Some analytical design methods are worth noting (Chen and Parks, 1986), but it should be emphasized that various numerical methods have always been used. In following filter design starts with frequency specifications and ends with a recursive evaluation of the impulse response coefficients without any other numerical recourse. In addition, the analytical approach pre-sented here is extremely robust and fast. These properties are advantageous in adaptive filtering.

EQUIRIPPLE APPROXIMATION

In this study we present the fundamental principles which govern FIR filter design procedures presented in the study.

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Chebyshev polynomials and Zolotarev polynomials and their relatives form the mathematical core of practical applications including filter design and data interpolation. Chebyshev polynomials of the first kind are defined as a solution of an approximation problem (Achieser, 1928). The objective of this problem is to find, within all polynomials of the n–th order and fixed highest coefficient b(n), such a polynomial

$$T_n(w) = \sum_{k=0}^{n} b(k)\, w^k \tag{1}$$

that minimizes the quantity

$$\max_w |T_n(w)|, \quad w \in [-1, 1] \tag{2}$$

The polynomials fulfill the differential equation (Abramowitz and Stegun, 1972):

$$(1-w^2)\left(\frac{dT_n(w)}{dw}\right)^2 = n^2\left(1-T_n^2(w)\right) \tag{3}$$

which reflects the equiripple behaviour of $T_n(w)$ within the interval w ∈ [-1, 1]. By separating the variables

$$\frac{dT_n}{\sqrt{1-T_n^2}} = \frac{dw}{\sqrt{1-w^2}} \tag{4}$$

we can arrive at the parametric solution

$$T_n(\varphi) = \cos n\varphi \tag{5}$$

$$w = \cos\varphi \tag{6}$$

This can alternatively be expressed as

$$T_n(w) = \cos(n \arccos(w)),\ w \in [-1, 1] \tag{7}$$

The coefficients b(k) from Eq. 1 are usually given in closed form. Using a standard identity for the cosine of multiple angle Eq. 5 we get

$$T_n(\cos\varphi) = \cos n\varphi = 2^{n-1}\cos^n\varphi + n\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k}{k}\binom{n-k-1}{k-1} 2^{n-2k-1}\cos^{n-2k}\varphi \tag{8}$$

If there were no De Moivre's theorem, we would need to develop an alternative way of finding algebraic form Eq. 1. For this purpose, we use the second order linear differential equation for Chebyshev polynomials $T_n(w)$. The derivative of Eq. 4 and simple algebraic manipulation gives the well known equation

$$(1-w^2)\frac{d^2T_n}{fw^2} - w\frac{dT_n}{dw} + n^2T_n = 0 \tag{9}$$

Inserting identity Eq. 1 into 9 we obtain a recursive evaluation of the coefficients b(k) and the explicit algebraic representation of Eq. 5 and 6 consequently. This alternative and robust way for evaluating the algebraic form of the Chebyshev polynomials avoids using De Moivre's theorem (Table 1).

Table 1: Recursive evaluation of the coefficients b(k)

Given	n
Initialisation	$b(n) = 2^{n-1}$
	$b(n-1) = 0$
Recursive body	
(for	$k = n-2$ to 0
	$b(k) = 1\frac{(k+2)(k+1)}{n^2-k^2}\, b(k+2)$
(end	

- The algorithm produces the coefficients b(k) for Chebyshev polynomials $T_n(x)$, as expected.
- The second order differential equation converts the parametric representation to an explicit form.
- In contrast to the explicit formula

$$T_n(x) = \frac{n}{2}\sum_{m=0}^{n}\frac{(-1)^m(n-m-1)!}{m!(n-2m)!}(2x)^{n-2m}$$

the algorithm computes b(k) for quite high order polynomials (n ≈ 100).

- The maximum of the coefficients appears at approximately

$$\frac{\sqrt{2}}{2}\times n$$

BASIC TERMS

Here we assume the impulse response h(k), with odd length N = 2n + 1 and with even symmetry

$$a(0) = h(n), \quad a(k) = 2h(n+k) = 2h(n-k),\ k = 1 \ldots n \tag{10}$$

The vector α(k) is more useful for further manipulations than the corresponding impulse response h (k). For brevity we call α(k) the a-vector of the filter. Here and in the following we will use the transformed variable w (Vlček and Unbehauen, 1989):

$$w = \frac{1}{2}(z + z^{-1}) \qquad (11)$$

which transforms the z-plane onto a two-leaved w-plane so that the unit circle itself $|z| = 1$ is mapped into the real interval $-1 \le w = \cos \omega T \le 1$ along which both leaves are interconnected. The transfer function H(z) of an FIR filter of the order N-1 is

$$\begin{aligned} H(z) &= \sum_{k=0}^{2n} h(k) z^{-k} \\ &= z^{-n}\left[h(n) = 2\sum_{k=1}^{n} h(n \pm k)\frac{1}{2}(z^{k} + z^{-k})\right] \qquad (12) \\ &= z^{-n}\sum_{k=0}^{n} a(k) T_k(w) = z^{-n} Q(w) \end{aligned}$$

where, $T_k(w)$ is the Chebyshev polynomial of the first kind. The function

$$Q(w) = \sum_{k=0}^{2n} a(k) T_k(w) \qquad (13)$$

represents a polynomial in the variable w which on the unit circle $z = e^{j\omega T}$ reduces to the real valued zero phase transfer function Q(w) of the real argument

$$w = \cos(\omega T) \qquad (14)$$

The zero phase transfer function is formed by the approximating polynomial. The approximating polynomial has a particular form for a particular type of approximation. The frequency response $H(e^{j\omega T})$ of the filter can be expressed by the zero phase transfer function

$$\begin{aligned} H(e^{j\omega T}) &= e^{-jn\omega T}\, Q(\cos \omega T) \\ &= z^{-k}\, Q(w)\big|_{z=ej\omega T} \end{aligned} \qquad (15)$$

ANALYTICAL DESIGN OF MAXIMALLY FLAT NOTCH FIR FILTERS

Approximation: Generating function of a maximally at notch FIR filter is the approximating polynomial $A_{p,q}(w)$ (Vlček and Jireš, 1994; Zahradnik and Vlček, 2004):

$$A_{p,q}(w) = C\,(1-w)^{p}\,(1+w)^{q} \qquad (16)$$

The approximating polynomial Ap,q(w) fulfils the differential equation

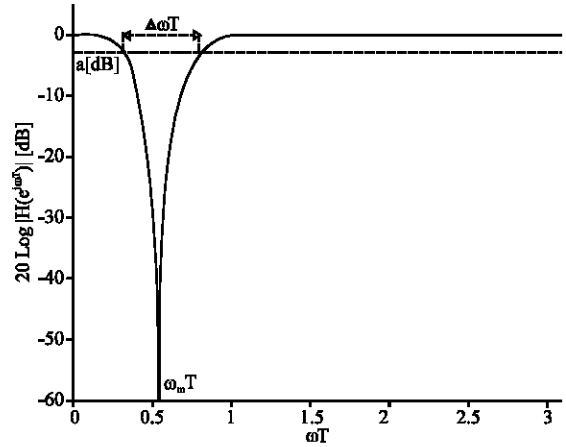

Fig. 1: Amplitude frequency response 20 log $|H(e^{j\omega T})|$ based on the approximating polynomial Q (w) = 1 -A3,37 (w). The parameters are $\omega_m T = 0.1766\pi$ and $\Delta\omega T = 0.1555\pi$ for a = 20 log(2/2) = -3.0103 dB

$$\left(1 - w^{2}\right)\frac{dA_{p,q}}{dw} + [p - q + (p+q)w]\, A_{p,q} = 0 \qquad (17)$$

Differential Eq. 17 is indispensable for deriving the algorithm for analytical evaluation of the impulse response. Normalization of the approximating polynomial $A_{p,q}(w)$ results in

$$A_{p,q}(w) = \left[\frac{p+q}{2p}(1-w)\right]^{p}\left[\frac{p+q}{2p}(1-w)\right]^{q} \qquad (18)$$

The polynomial $Q(w) = 1 - A_{p,q}(w)$ represents the zero phase transfer function of the maximally at notch FIR filter. For illustration, the amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] corresponding to the zero phase transfer function $Q(w) = 1 - A_{3,37}(w)$ is shown in Fig. 1. The notch frequency $\omega_m T$ of the filter is derived from the minimum value of the zero phase transfer function Q (w)

$$w_m = \cos \omega_m T = \frac{q-p}{q+p} \qquad (19)$$

The notch frequency $\omega_m T$ (19) of the filter is given by the integer values p and q exclusively. It is obvious, that for the specified filter length N = 2(p + q) + 1, exactly p + q-1 discrete notch frequencies $\omega_m T$ are available. The goal in the design of a maximally at notch FIR filter is to obtain as precisely as possible the two integers p and q in order to satisfy the filter specification (notch frequency $\omega_m T$, width of the notch band $\Delta\omega T$ and attenuation in the passbands a [dB]). The width of the notchband is:

$$\Delta\omega T = \pi - 2\arccos\sqrt{1-\left(1-10^{0.05\alpha[dB]}\right)^{2/n}} \qquad (20)$$

and degree n is given by

$$n \ge \frac{\log\left(1-10^{0.05\alpha[dB]}\right)}{\log\ \cos\dfrac{\Delta\omega T}{2}} \qquad (21)$$

We call Eq. 21 the degree equation of the maximally at notch FIR filter. The integer values p and q are

$$p = \left[n\ \sin^2\left(\frac{\omega_m T}{2}\right)\right],\ q = \left[n\ \cos^2\left(\frac{\omega_m T}{2}\right)\right] \qquad (22)$$

where the square brackets stand for rounding. The approximating polynomial $A_{p,q}$ (w) of the degree n = p + q can be expressed using Chebyshev polynomials of the first kind

$$A_{p,q}(w) = \sum_{k=}^{n} \alpha(k)\ T_k(w) \qquad (23)$$

Based on differential Eq. 17 we have deduced a fast and robust recursive algorithm (Table 2) for evaluating the impulse response h (k) of the length N = 2 (p + q) + 1 coefficients.

Design procedure: The design procedure reads as follows:

- Specify the notch frequency $\omega_m T$, the maximal width of the notchband $\Delta\omega T$ and the maximal attenuation in the passbands a [dB], as demonstrated in Fig. 1.
- Calculate the minimum degree n (21).
- Calculate the integer values p and q (22).
- Evaluate the impulse response h (k) analytically (Table 2).
- Check the notch frequency using (19).
- If required, tune the notch frequency to the proper value.

It is worth noting that many coefficients of the impulse response h (k) of the maximally at notch FIR filter exhibit negligible values. Consequently, the impulse response of the maximally at notch FIR filter can be greatly abbreviated by rectangular windowing without significant deterioration of the frequency properties of the filter, as emphasized in Vlček and JireJireš (1994).

Example 1: Design the maximally at notch FIR filter specified by $\omega_m T = 0.35\pi$ and $\Delta\omega T = 0.15\pi$ for

a = -3.0103 dB

Table 2: Recursive algorithm for the impulse response coefficients

Given	p, q (Integer values)
Initialisation	n = p + q
	a (n + 1) = 0
	$a(n) = (-1)^p\ 2^{(-p-q+1)} \left(\frac{p+q}{2p}\right)^p \left(\frac{p+q}{2p}\right)^q$
Recursive body (for k = n + 1 to 3)	$a(k-2) = -\frac{(n+k)a(k) + 2(2p-n)a(k-1)}{n+2-k}$
(end loop on k)	$a(0) = -\frac{(n+2)a(2) + 2(2p-n)\ a(1)}{2n}$
Impulse response	h (n) = 1-a (0)
(for k = 1 to n)	h (n±k) = -a (k)/2
(end loop on k)	

Table 3: Impulse response h(k) of the notch filter from example 1

k	k	h (k)	k	k	h (k)
14	74	-0.000002	30	58	0.012289
15	73	-0.000003	31	57	0.002278
16	72	0.000000	32	56	-0.019427
17	71	0.000018	33	55	-0.027483
18	70	0.000037	34	54	-0.003357
19	69	0.000010	35	53	0.042804
20	68	-0.000111	36	52	0.048063
21	67	-0.000245	37	51	-0.009353
22	66	-0.000101	38	50	-0.075616
23	65	0.000537	39	49	-0.065324
24	64	0.001173	40	48	0.029196
25	63	0.000480	41	47	0.106554
26	62	-0.002149	42	46	0.068113
27	61	-0.004302	43	45	-0.053105
28	60	-0.001388	44		0.880514
29	59	0.007135			

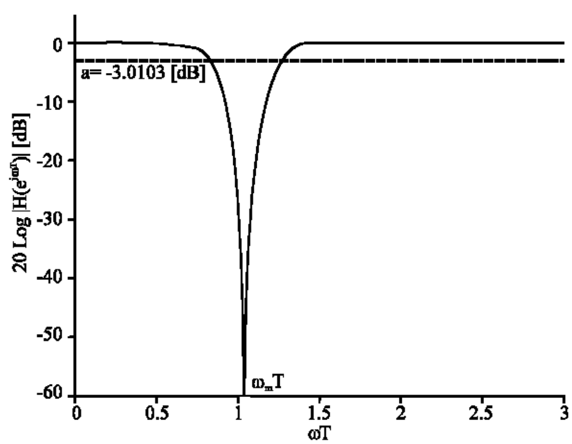

Fig. 2: Amplitude frequency response 20 log |H(e jωT) |[dB] of the maximally at FIR notch filter from example 1

Using the proposed design procedure we get n = [43.8256] → 44 (21), p = [11.9644] → 12 and q = [31.8610] → 32 (22). The actual filter parameters are $\omega_m T$ = 0.3498π and ΔωT = 0.1496π. The impulse response coefficients h(k) of the length N = 89 (Table 3) are evaluated by the recursive algorithm (Table 2). The amplitude frequency response 20 log $|H(e^{j\omega T})|$[dB] of the maximally at notch FIR filter is shown in Fig. 2.

ANALYTICAL DESIGN OF EQUIRIPPLE NOTCH FIR FILTERS

Approximation: The approximating polynomial of the equiripple notch FIR filter is the Zolotarev polynomial (Zahradnik and Vlček, 2004):

$$Z_{p,q}(u|k) = \frac{(-1)^p}{2} \left[\left(\frac{H(u-\frac{p}{n}K(\kappa))}{H(u-\frac{p}{n}K(\kappa))} \right)^n + \left(\frac{H(u+\frac{p}{n}K(\kappa))}{H(u+\frac{p}{n}K(\kappa))} \right)^n \right] \quad (24)$$

It approximates zero value in 2 disjoint intervals. H (u ± p/n) K (κ)) is the Jacobi Eta function, K (κ) is the quarter-period given by the complete elliptic integral of the first kind of the Jacobi elliptic modulus κ. The degree of the Zolotarev polynomial is n = p + q. The indices p and q emphasize that p counts the number of zeros right from the maximum wm and q corresponds to the number of zeros left from the maximum wm. The extremal values of the Zolotarev polynomial alternate between -1 and +1 (q + 1)-times in the interval (-1, w_s) and (p + 1)-times in the interval (w_p, 1). Assuming the conformal transformation (Achieser, 1928; Levy, 1970) between the u domain and the w domain

$$w = \frac{sn^2(u)\, cn^2\left(\frac{p}{n}K(\kappa)|\kappa\right) + cn^2(u)\, sn^2\left(\frac{p}{n}K(\kappa)|\kappa\right)}{sn^2(u) - sn^2\left(\frac{p}{n}K(\kappa)|\kappa\right)} \quad (25)$$

we denote $Z_{p,q}(w) = Z_{p,q}(u|\kappa)$ the Zolotarev polynomial in the w-domain. It was derived in Vlček and Unbehauen (1999) that the Zolotarev polynomial $Z_{p,q}(w)$ satisfies the differential equation

$$P(w)\left[(1-w^2)\frac{d^2 Z_{p,q}(w)}{dw^2} - w\frac{dZ_{p,q}(w)}{dw}\right] - Q(w)(1-w^2)\frac{dZ_{p,q}(w)}{dw} + R(w)Z_{p,q}(w) = 0 \quad (26)$$

$$P(w) = (w - w_p)(w - w_s)(w - w_m) \quad (27)$$

$$Q(w) = (w - w_p)(w - w_s) - (w - w_m)\left(w - \frac{w_p + w_s}{2}\right) \quad (28)$$

$$R(w) = n^2 (w - w_m)^2 \quad (29)$$

The band edges wp and ws correspond to

$$w_p = 2sn^2\left(\frac{q}{n}K(\kappa)|\kappa\right) - 1, \qquad w_s = 1 - 2sn^2\left(\frac{p}{n}K(\kappa)|\kappa\right) \quad (30)$$

The position of the maximum value $y_m = A_{p,q}(w_m)$ is:

$$w_m = w_s + 2\frac{sn\left(\frac{p}{n}K(\kappa)|\kappa\right) cn\left(\frac{q}{n}K(\kappa)|\kappa\right)}{dn\left(\frac{q}{n}K(\kappa)|\kappa\right)} Z\left(\frac{p}{n}K(\kappa)|\kappa\right) \quad (31)$$

Where,

$$Z\left(\frac{p}{n}K(\kappa)|\kappa\right)$$

represents the Jacobi zeta function. The integer values p, q, n = p + q and the real valued elliptic modulus κ are related by the partition equation

$$\frac{p}{n}K(\kappa) + \frac{q}{n}K(\kappa) = F(\varphi_s|\kappa) + F(\varphi_p|\kappa) = K(\kappa) \quad (32)$$

Function F (ϕ |κ) is the incomplete elliptic integral of the first kind of Jacobi elliptic modulus κ. The goal in the approximation of the equiripple notch FIR filter is to obtain the 3 parameters p, q and κ in order to satisfy the specified notch frequency $\omega_m T$, the width of the notchband ΔωT and the attenuation in the passbands a [dB] (Fig. 3) as precisely as possible. The degree of the Zolotarev polynomial is expressed by the degree equation

$$n \geq \frac{\ln(y_m + \sqrt{y_m^2 - 1})}{2\sigma_m Z(\frac{p}{n}K(\kappa)|\kappa) - 2\Pi(\sigma_m, (\frac{p}{n}K(\kappa)|\kappa)} \quad (33)$$

where the auxiliary parameter σ_m is

$$\sigma_m = F\left(\arcsin\left(\frac{1}{\kappa\,\mathrm{sn}\,(\frac{p}{n}K(\kappa)|\kappa)}\sqrt{\frac{w_m - w_s}{w_m + 1}}\right)\Big|\kappa\right) \quad (34)$$

The maximum value ym of the Zolotarev polynomial reads

$$y_m = \cosh 2n(\sigma_m Z(\frac{p}{n}K(\kappa)|\kappa) \quad (35)$$
$$\Pi(\sigma_m, \frac{p}{n}K(\kappa)|\kappa)$$

where, $\Pi(\sigma_m, p/n\,K(\kappa)|\kappa)$ is the incomplete elliptic integral of the third kind. The maximizer ym is related to the attenuation in the passbands α [dB]

$$\alpha[dB] = 20\ \log\left(1 - \frac{2}{y_m + 1}\right) \quad (36)$$

The zero phase transfer function of the equiripple notch FIR filter is

$$Q(w) = 1 - \frac{Z_{p,q}(w) + 1}{y_m + 1} \quad (37)$$

Based on the differential Eq. 26 we have developed a fast and robust algorithm for evaluating the a-vector of the Zolotarev polynomial

$$Z_{p,q}(w) = \sum_{k=0}^{n} \alpha(k)\ T_k(w) \quad (38)$$

and of the impulse response h (k), Table 4 and 5.

Design procedure: The design procedure reads as follows:

- Specify the notch frequency $\omega_m T$, the width of the notchband $\Delta\omega T$ and the maximal attenuation in the passband α [dB] demonstrated in Fig. 3.
- Calculate the band edges

$$\omega_p T = \omega_m T - \frac{\Delta\omega T}{2},\ \omega_s T = \omega_m T + \frac{\Delta\omega T}{2} \quad (39)$$

- Evaluate the Jacobi elliptic modulus κ

$$\kappa = \sqrt{1 - \frac{1}{\tan^2(\varphi_s)\tan^2(\varphi_p)}} \quad (40)$$

Table 4: Impulse response h(k) of the equiripple FIR notch filter

-------k--------		h (k)	---------k---------		h (k)
0	76	-0.029617	20	56	0.023583
1	75	0.009026	21	55	-0.016098
2	74	-0.006910	22	54	0.004079
3	73	0.002540	23	53	0.009627
4	72	0.003215	24	52	-0.021585
5	71	-0.008960	25	51	0.028644
6	70	0.013104	26	50	-0.028782
7	69	-0.014293	27	49	0.021678
8	68	0.011819	28	48	-0.008872
9	67	-0.005907	29	47	-0.006575
10	66	-0.002247	30	46	0.020828
11	65	0.010702	31	45	-0.030245
12	64	-0.017226	32	44	0.032332
13	63	0.019891	33	43	-0.026411
14	62	-0.017624	34	42	0.013833
15	61	0.010583	35	41	0.002336
16	60	-0.000216	36	40	-0.018080
17	59	-0.011026	37	39	0.029453
18	58	0.020271	38		0.916626
19	57	-0.024966			

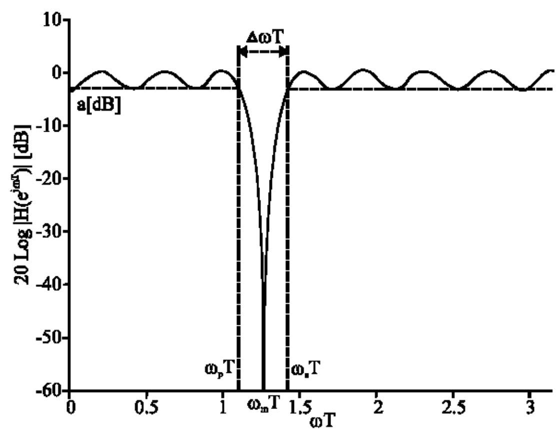

Fig. 3: Amplitude frequency response 20 log $|H(e^{j\omega T})|$[dB] based on the Zolotarev polynomial $Z_{6,9}(w)$. The parameters are $\omega_p T = 0.3506\pi$, $\omega_m T = 0.4006\pi$, $\omega_s T = 0.4507\pi$, $\Delta\omega T = 0.1001\pi$ and a = -.3.2634 dB

for the auxiliary parameters φ_s and φ_p

$$\varphi_s = \frac{\omega_s T}{2},\ \varphi_p = \frac{\pi - \omega_p T}{2} \quad (41)$$

- Calculate the rational values p/n and q/n (32).
- Determine the required maximum value y_m (35).
- Calculate the minimum degree n, using the degree Eq. 33.
- Calculate the integer values p and q defining the Zolotarev polynomial $Z_{p,q}$ (w).

$$P = \left[n\frac{F(\varphi_s|\kappa)}{K(\kappa)}\right],\ q = \left[n\frac{F(\varphi_p|\kappa)}{K(\kappa)}\right] \quad (42)$$

Table 5: Evaluation of the impulse response of an equiripple FIR notch lter

Given	p, q (integers), κ (real)
Initialisation	n = p + q
	$w_p = 2sn^2\left(\frac{q}{n}K(\kappa)\middle|\kappa\right) - 1,\ w_s = 1 - 2sn^2\left(\frac{p}{n}K(\kappa)\middle|\kappa\right),\ w_a = \frac{w_p + w_s}{2}$
	$w_m = w_s + 2\,\frac{sn\left(\frac{p}{n}K(\kappa)\middle|\kappa\right)\,cn\left(\frac{p}{n}K(\kappa)\middle|\kappa\right)}{dn\left(\frac{p}{n}K(\kappa)\middle|\kappa\right)}\,Z\left(\frac{p}{n}K(\kappa)\middle|\kappa\right)$
	$\alpha(n) = 1,\ \alpha(n+1) = \alpha(n+2) = \alpha(n+3) = \alpha(n+4) = \alpha(n+5) = 0$
body	
(For	m = n + 2 to 3
	$8c(1) = n^2 - (m+3)^2$
	$4c(2) = (2m+5)(m+2)(w_m - w_a) + 3w_m[n^2 - (m+2)^2]$
	$2c(3) = \frac{3}{4}[n^2 - (m+1)^2] + 3w_m[n^2 w_m - (m+1)^2 w_a] - (m+1)(m+2)(w_p w_s - w_m w_a)$
	$c(4) = \frac{3}{2}(n^2 - m^2) + m^2(w_m - w_a) + w_m(n^2 w_m^2 - m^2 w_p w_s)$
	$2c(5) = \frac{3}{4}[n^2 - (m-1)] + 3w_m[n^2 w_m - (m-1)^2 w_a] - (m-1)(m-2)(w_p w_s - w_m w_a)$
	$4c(6) = (2m-5)(m-2)(w_m - w_a) + 3w_m[n^2 - (m-2)^2]$
	$8c(7) = n^2 - (m-3)^2$
	$\alpha(m-3) = \frac{1}{c(7)}\sum_{\mu=1}^{6} c(\mu)\alpha(m+4-\mu)$
(end loop on m)	
Normalisation	$s(n) = \frac{\alpha(0)}{2} + \sum_{m=1}^{n}\alpha(m)$
	$a(0) = (-1)^p\,\frac{\alpha(0)}{2_s(n)}$
(for m = 1 to n)	$a(m) = (-1)^p\,\frac{\alpha(m)}{s(n)}$
(for loop on m)	
Impulse response	$h(n) = \frac{y_m - \alpha(0)}{y_m + 1}$
(for k = 1 to n)	$h(n \pm k) = -\frac{\alpha(k)}{2(y_m + 1)}$
(end loop on k)	

where the square brackets stand for the rounding.

- Calculate the actual attenuation in the passbands a [dB] (36) for the corresponding maximal value y_m (35).
- Calculate the actual width of the passband

$$\Delta\omega T = \arccos(w_p) - \arccos(w_s) \qquad (43)$$

- For p, q and κ evaluate the impulse response h(k) analytically (Table 5).
- Check the notch frequency using (31).
- If required, tune the notch frequency to the proper value.

Example 2: Design the equiripple notch FIR filter specified by the notch frequency $\omega_m T = 0.84\pi$ and by the width of the stopband $\Delta\omega T = 0.0610\pi$ for the maximal attenuation in the passband a = -0.95 dB. Using the proposed design procedure we get $\omega p = 0.8095\pi$, $\omega s = 0.8705\pi$ (39), $\varphi_s = 0.2992$, $\varphi_p = 1.3674$ (41), κ = 0.743599 (40), n = [37.2896] → 38 (33), p = [31.9713] → 32 and q = [6.0287] → 6 (42). For the calculatedvalues p, q, κ the actual filter

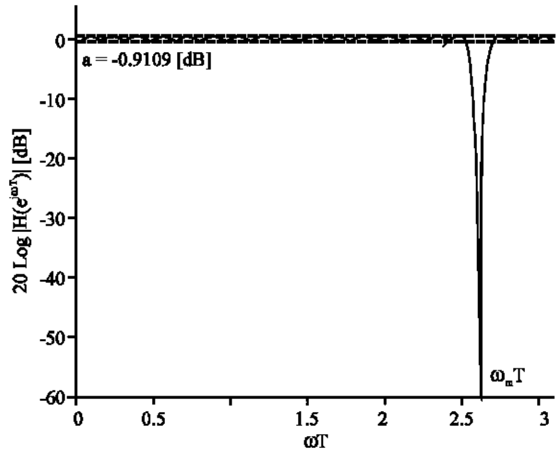

Fig. 4: Amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] of the equiripple FIR notch filter from example 2

parameters are $\omega_m T = 0.8408\pi$ (31), $\Delta\omega T = 0.0607\pi$ (43) and a = -0.9109 dB (36). The filter length is N = 77 coefficients. The impulse response coefficients h(k) of the filter were evaluated recursively (Table 5) and they are summarized in Table 4. The amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 4.

ANALYTICAL PROCEDURE FOR TUNING FIR FILTERS

Precise tuning (Zahradnik and Vlček, 2006) of frequency properties is an useful operation in the design of digital filters. Instead of designing the filter from scratch, the impulse response of the available filter can be reused. Adaptive filtering is one of the applications. Tuning is also useful in the analytical design of digital FIR filters where the available critical frequencies are quantized. Hence analytical design combined with tuning the filter represents a powerful design tool. The proposed fast versatile tuning procedure adjusts a single frequency of the frequency re- sponse of the FIR filter to the specified value while preserving the nature of the filter, e.g., maximally at, equiripple etc. The tuning procedure is based on expanding the Chebyshev polynomial of the transformed argument into the sum of the Chebyshev polynomials, resulting in the transformation matrix. The impulse response of the final filter is obtained from the impulse response of the original filter by applying the transformation matrix. The purpose of tuning is to map the critical frequency $\omega_m T$ of the frequency response of the filter to the desired value $\omega_0 T$. The mapping $\omega_m T \leftrightarrow \omega_0 T$ in the frequency domain is equivalent to the mapping $w_m \leftrightarrow w_0$ in the w-domain. Due to (14) the shift in the 2 domains occurs in opposite directions. We propose the transformed zero phase transfer functions in the form

$$Q_t(w) = Q(\lambda w + \lambda'),\ \lambda = \frac{w_m - 1}{w_0 - 1},\ \text{if}\ \omega_m T < \omega_0 T \tag{44}$$

And

$$Q_t(w) = Q(\lambda w + \lambda'),\ \lambda = \frac{w_m + 1}{w_0 + 1},\ \text{if}\ \omega_0 T < \omega_m T$$

The real number λ is confined to $0 < \lambda \le 1$. The tuning procedure provides the impulse response coefficients for an FIR filter with the following properties:

- The frequency $\omega_m T$ is adjusted to the specified value $\omega_0 T$.
- The maximal attenuation(s) in the passband(s) and the minimal attenuation(s) of the stopband(s) of the filter are preserved.
- The nature (maximally at, equiripple etc.) of the filter is preserved.
- The bands of the filter are broadened.

The transformed zero phase transfer functions

$$\begin{aligned} Q_t(w) &= \sum_{k=0}^{n} a(k) T_k(\lambda w + \lambda') \\ &= \sum_{k=0}^{n} a(k) \sum_{m=0}^{k} \alpha_k(m) T_m(w) \end{aligned} \tag{45}$$

can be rewritten in matrix form

$$Q_t(w) = [a(0)\ a(1) \ldots a(n)] \times \begin{bmatrix} \alpha_0(0) & 0 & 0 & 0 & \ldots 0 \\ \alpha_1(0) & \alpha_1(1) & 0 & 0 & \ldots 0 \\ \alpha_2(0) & \alpha_2(1) & \alpha_2(2) & 0 & \ldots 0 \\ \alpha_3(0) & \alpha_3(1) & \alpha_3(2) & \alpha_3(3) & \ldots 0 \\ \vdots & & & & \vdots \\ \alpha_n(0) & \alpha_n(1) & \alpha_n(2) & \alpha_n(3) & \alpha_n(n) \end{bmatrix} \times \begin{bmatrix} T_0(w) \\ T_1(w) \\ T_2(w) \\ T_3(w) \\ \vdots \\ T_n(w) \end{bmatrix} \tag{46}$$

We call the low triangular matrix A the transformation matrix. The a_t-vector of the transformed filter is given by the product of the a-vector of the original filter and the transformation matrix A

$$a_t = a\,A \tag{47}$$

Table 6: Evaluation of the coefficients α_k (m) of transformation matrix A+ for filter tuning

Given	k (integer value), $0 < \lambda \le 1$ (real value)
Initialization	$\lambda' = 1 - \lambda$
	$\alpha^k (k+1)\ \alpha k (k+2) = \alpha_k (k+3) = 0$
	$\alpha_k (k) = \lambda^k$
body	
(for μ = -3 ... k - 4)	
	α_k (k - μ - 4)
	{
	$-2\left[(\mu+3)(2k-\mu-3)-\frac{\lambda'}{\lambda}(k-\mu-3)(2k-2\mu-7)\right]\alpha_k(k-\mu-3)$
	$+2\frac{\lambda'}{\lambda}2(k-\mu-2)\ \alpha_k(k-\mu-2)$
	$+2\left[(\mu+1)(2k-\mu-1)-\frac{\lambda'}{\lambda}(k-\mu-1)(2k-2\mu-1)\right]\alpha_k(k-\mu-1)$
	$+\mu(2k-\mu)\ \alpha_k(k-\mu)$
	}\(μ + 4) (2k - μ - 4)
(end loop on μ)	

There are 2 transformation matrices A_+ and A_- corresponding to the transformed zero phase transfer functions (44). Fast evaluation of the coefficients α_k (m) of the transformation matrices is essential in adaptive filtering. Using the differential Eq. 9 for the Chebyshev polynomial of the first kind T_k (x) we have derived the differential equation

$$(1-w^2+2\frac{\lambda'}{\lambda}(1-w))\frac{d^2F+(w)}{dw^2} -(w+\frac{\lambda'}{\lambda})\frac{dF+(w)}{dw}+k^2F+(w)=0 \quad (48)$$

for the polynomial

$$F_+(w) = T_k(\lambda w + \lambda') \quad (49)$$

and the differential equation

$$(1-w^2+2\frac{\lambda'}{\lambda}(1+w))\frac{d^2F-(w)}{dw^2} -(w-\frac{\lambda'}{\lambda})\frac{dF-(w)}{dw}+k^2F-(w)=0 \quad (50)$$

for the polynomial

$$F\_(w) = T_k(\lambda w - \lambda') \quad (51)$$

where the real values λ and λ' are related by $\lambda + \lambda' = 1$. Based on differential Eq. 48 and 50 we have developed a fast and robust procedure for evaluating the coefficients α_k (m) of the transformation matrices A_+ and A_-. The fast algorithm for evaluating the coefficients of the transformation matrix A_+ is summarized in Table 6. The evaluation of the transformation matrix A_- is by analogy. The 2 matrices differ by the signs of the "odd" coefficients α_k (k-μ-3) and α_k (k-μ-1) only.

Table 7: Impulse responses h(k) and h_t (k) from example 3

------k-----		h (k)	h_t (k)	-----k------		h (k)	h_t (k)
0	72	0.016832	0.011622	19	53	0.022942	0.028084
1	71	0.004953	-0.005198	20	52	0.028636	0.024800
2	70	-0.002260	-0.009660	21	51	0.009196	0.000144
3	69	-0.009076	-0.010249	22	50	-0.019871	-0.026360
4	68	-0.008700	-0.003681	23	49	-0.033269	-0.032071
5	67	0.000117	0.006768	24	48	-0.018035	-0.010712
6	66	0.010632	0.013012	25	47	0.014008	0.021071
7	65	0.013100	0.008921	26	46	0.035383	0.036681
8	64	0.003678	-0.003396	27	45	0.026667	0.021933
9	63	-0.010795	-0.013938	28	44	-0.005787	-0.012097
10	62	-0.017533	-0.012654	29	43	-0.034396	-0.037428
11	61	-0.009034	0.001287	30	42	-0.033926	-0.032360
12	60	0.009012	0.017271	31	41	-0.003929	-0.000239
13	59	0.021195	0.021210	32	40	0.030153	0.032628
14	58	0.015517	0.007859	33	39	0.038727	0.038693
15	57	-0.004980	-0.013249	34	38	0.013946	0.012424
16	56	-0.023239	-0.024485	35	37	-0.023124	-0.024678
17	55	-0.022375	-0.014929	36		0.933816	0.932507
18	54	-0.001244	0.009158				

Example 3: Design the equiripple notch FIR filter specified by the notch frequency $\omega_0T = 0.3\pi$ and the width of the notchband $\Delta\omega T = 0.075\pi$ for maximum attenuation in the passbands a = -0.5 dB. Using the analytical design procedure we get $\kappa = 0.665619$, n = 36, p = 11 and q = 25. The designed filter of length N = 73 coefficients with "quantized" notch frequency $\omega_mT = 0.3064\pi$ and $\Delta\omega T = 0.075\pi$ for aact = -0.4584 dB is tuned using the proposed tuning procedure in order to get the specified notch frequency $\omega_0T = 0.3\pi$. Because $\omega_0T<\omega_mT$ we evaluate the transformation matrix A_- for $\lambda = 0.9898$ (44).

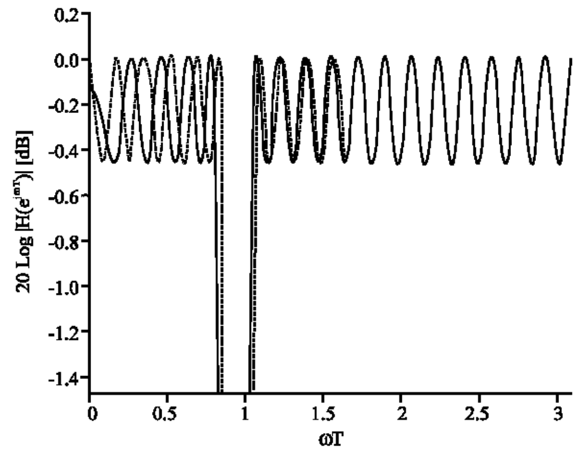

Fig. 5: Passbands of the "quantized" filter (thin line) and of the tuned filter from example 3

The parameters of the tuned filter are $\omega_0 T = 0.3\pi$ and $\Delta_\omega T = 0.0779\pi$ for a = -0.4584 dB. A detailed view of the passbands of the "quantized" filter and of the tuned filter is shown in Fig. 5 and Table 7.

ANALYTICAL DESIGN OF EQUIRIPPLE DC-NOTCH FIR FILTERS

Approximation: The approximating polynomial of the DC-notch FIR filter is the polynomial F(w) (Zahradnik and Vlček, 2007):

$$F(w) = T_k(\lambda w + \lambda' - 1) = \sum_{m=0}^{n} B(m) w_m = \sum_{m=0}^{n} A(m) T_m(w) \quad (52)$$

The approximating polynomial F(w) fulfils the differential equation:

$$(1 - w^2 + \frac{1-\lambda}{\lambda}) \frac{d^2F(w)}{dw^2} - (w - \frac{1-\lambda}{\lambda}) \frac{dF(w)}{dw} + n^2 F(w) = 0 \quad (53)$$

Differential Eq. 53 is again used for deriving the algorithm for recursive evaluation of the impulse response of the filter. The zero phase transfer function of the DC-notch FIR filter is:

$$Q(w) = 1 - \frac{F(w)+1}{T_n(2\lambda - 1) + 1} = 1 - \frac{T_n(\lambda w + \lambda - 1) + 1}{T_n(2\lambda - \lambda) + 1} \quad (54)$$

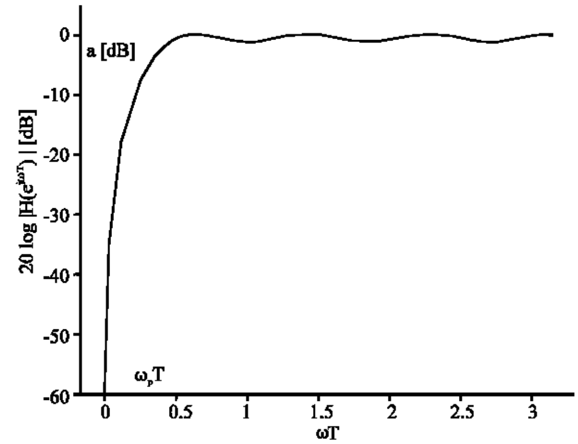

Fig. 6: Amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] of a DC notch filter for n = 7, K = 1.057638, $\omega_p T$ = 0.15π and a = -1.2446 dB

The DC-notch FIR filter is specified by the passband frequency $\omega_p T$ and by the attenuation in the passband a [dB] (Fig. 6). The degree equation reads as follows:

$$n \geq \frac{a\cosh\left(\dfrac{1 + 10^{0.05a[dB]}}{1 - 10^{0.05a[dB]}}\right)}{a\cosh\left(\dfrac{1 + \sin^2 \dfrac{\omega_p T}{2}}{1 - \sin^2 \dfrac{\omega_p T}{2}}\right)} \quad (55)$$

Where,

$$\lambda = \frac{1}{1 - \sin^2 \dfrac{\omega_p T}{2}} \quad (56)$$

Based on differential Eq. 53 we have developed a fast and robust algorithm (Table 8) for evaluating the impulse response of the DC-notch FIR filter.

Design procedure: The design procedure reads as follows:

- Specify the passband frequency $\omega_p T$ and the maximal attenuation in the passband a [dB] demonstrated in Fig. 6.
- Calculate the minimum degree n (55).
- Evaluate the impulse response h(k) analytically (Table 8).

Example 4: Design the DC-notch FIR filter specified by $\omega_p T$ = 0.05π and a = -0.01 dB.

Table 8: Evaluation of the impulse response of a DC notch filter

Given	n (integer value), λ (real value)
Initialization	$A(n) = \lambda^n$, $A(n+1) = A(n+2) = A(n+3) = 0$
body	
(for k = 2 ... n + 1)	
	A (n + 1 - k) = { 2[(K - 1) (2n + 1 - k) - ((1 - λ)\λ) (n + 1 - k) (2n + 1 - 2k)] A (n + 2 - k) + 4((1 - λ)\λ) (n + 2 - k) A (n + 3 - k) - 2[(K - 3) (2n + 3 - k) - ((1 - λ)\λ) (n + 3 - k) (2n + 7 - 2k)] A (n + 4 - k) + (k - 4) (2n + 4 - k) A (n + 5 - k) } \ k (2n - k)
(end loop on k)	
	$h(0) = 1 - \frac{A(0)\backslash 2+1}{T_n(2\lambda-1)+1}$
	$h(\pm k) = -\frac{1}{2}\frac{A(k)+1}{T_n(2\lambda-1)+1}, \; k = 1 \ldots n$

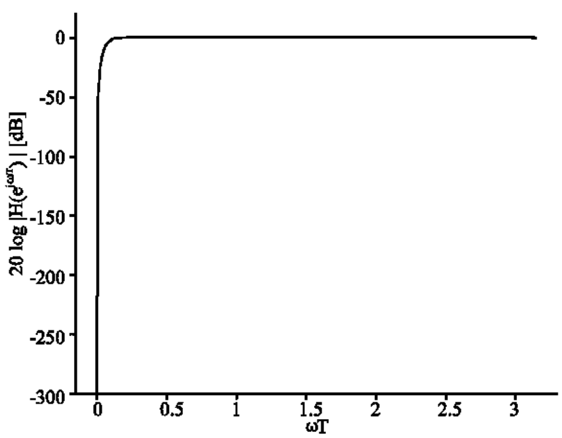

Fig. 7: Amplitude frequency response 20 log $|H(e^{j\omega T})|$[dB] of the DC notch FIR filter from example 4

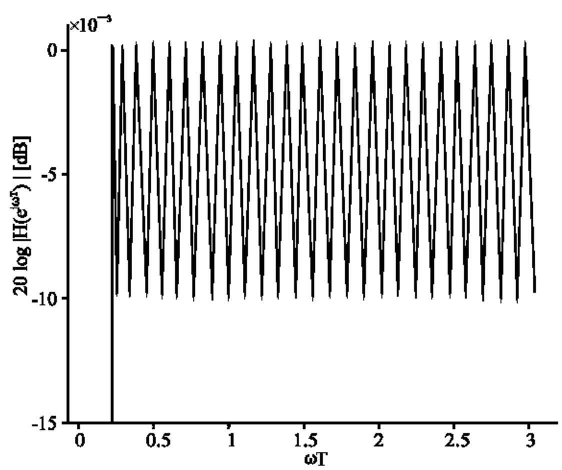

Fig. 8: Passband of the DC notch FIR filter from example 4

Using the proposed design procedure we get $n = 51.8513 \rightarrow 52$ (55) and $\lambda = 1.006194$ (56). The actual filter parameters are $\omega_p T = 0.05\pi$ and $a_{act} = -0.009768$ dB.

Table 9: Impulse response h(k) of the DC notch filter from exmple 4

-------k---------		h(k)	--------k-------		h(k)
0,	104	-0.000387	27,	77	-0.009226
1,	103	-0.000248	28,	76	-0.009866
2,	102	-0.000325	29,	75	-0.010516
3,	101	-0.000416	30,	74	-0.011173
4,	100	-0.000523	31,	73	-0.011834
5,	99	-0.000646	32,	72	-0.012495
6,	98	-0.000787	33,	71	-0.013154
7,	97	-0.000947	34,	70	-0.013807
8,	96	-0.001128	35,	69	-0.014451
9,	95	-0.001330	36,	68	-0.015081
10,	94	-0.001556	37,	67	-0.015696
11,	93	-0.001805	38,	66	-0.016291
12,	92	-0.002079	39,	65	-0.016862
13,	91	-0.002378	40,	64	-0.017407
14,	90	-0.002704	41,	63	-0.017921
15,	89	-0.003056	42,	62	-0.018402
16,	88	-0.003435	43,	61	-0.018848
17,	87	-0.003840	44,	60	-0.019254
18,	86	-0.004273	45,	59	-0.019619
19,	85	-0.004731	46,	58	-0.019941
20,	84	-0.005216	47,	57	-0.020216
21,	83	-0.005725	48,	56	-0.020444
22,	82	-0.006258	49,	55	-0.020623
23,	81	-0.006813	50,	54	-0.020752
24,	80	-0.007390	51,	53	-0.020829
25,	79	-0.007986	52		0.978583
26,	78	-0.008598			

The impulse response h(k) (Table 9) with the length N = 105 is evaluated analytically (Table 8). The amplitude frequency response 20 log $|H(e^{j\omega T})|$[dB] of the DC-notch FIR filter is shown in Fig. 7. Its passband is shown in Fig. 8.

ANALYTICAL DESIGN OF EQUIRIPPLE COMB FIR FILTERS

Approximation: The approximating polynomial F(w) of the equiripple comb FIR filter is given by the compounded Chebyshev polynomial (Zahradnik and Vlček, 2005):

$$F(w) + T_n[\lambda T_r(w)] = \sum_{k=0}^{nr} A(k) T_k(w) \qquad (57)$$

The real parameter $\lambda = 1/\kappa > 1$ affects the ripples in the passbands of the comb FIR filter. The degree r of the inner Chebyshev polynomial determines r narrow bands. The narrow bands of the comb FIR filter positioned at:

$$\omega_{mi} T = \frac{i\pi}{r} \quad i = 0,1,...,r \tag{58}$$

are equally spaced inside the interval [0, π]. The even degree n of the outer Chebyshev polynomial T_n (w) determines n-1 local extremes with the same amplitude between the narrow bands. The approximating polynomial F(w) (57) of the equiripple comb FIR filter fulfils the differential equation:

$$P(w)\left[(1-w^2)\frac{d^2F(w)}{dw^2} - w\frac{dF(w)}{dw}\right] - Q(w)\frac{dF(w)}{dw} + R(w)F(w) = 0 \tag{59}$$

Where,

$$P(w) = U_{r-1}(w)(\kappa^2 - T_r^2(w)) \tag{60}$$

$$Q(w) = r(1-\kappa^2)\, T_r(w) \tag{61}$$

$$R(w) = n^2 r^2 U_{r-1}(w)(1 - T_r^2(w)) \tag{62}$$

and U_r (w) is Chebyshev polynomial of the second kind. The differential Eq. 59 is used for deriving the algorithm for recursive evaluation of the impulse response coefficients. The zero phase transfer function Q (w) of the comb filter is given by the normalization of the approximating polynomial:

$$Q(w) = -\frac{1+F(w)}{C} = = \frac{1 + T_n[\lambda T_r(w)]}{C} = \sum_{k=0}^{nr} a(k)\, T_k(w) \tag{63}$$

The normalizing constant C follows from the approximating polynomial F (w) for w = 1

$$C = 1+F(1) = 1 + T_n[\lambda T_r(1)] = 1+T_n(\lambda) = 1+\cosh[n\, a\cosh(\lambda)] \tag{64}$$

Note that (64) is independent from the degree r of the inner Chebyshev polynomial T_r (w). The goal in the approximation of the equiripple comb FIR filter is to obtain the two parameters n and λ in order to satisfy the specified number of notch bands r, the width of the notch bands ΔωT and the maximal attenuation in the passbands a[dB] (Fig. 9) as precisely aspossible. The degree n of the outer Chebyshev polynomial T_r (w) is

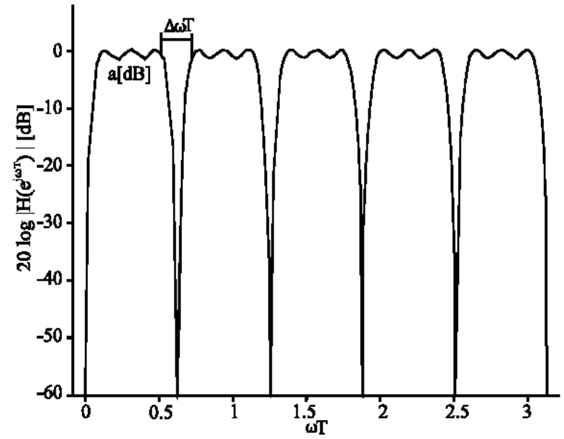

Fig. 9: Amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB]of an equiripple comb FIR filter

$$n \geq \frac{a\cosh(x)}{a\cosh(\lambda)} = \frac{\text{In}\,(x+\sqrt{x^2-1})}{\text{In}\,(\lambda+\sqrt{\lambda^2-1})} \tag{65}$$

where the auxiliary parameters λ and χ are

$$\lambda = \frac{1}{\cos\left(r\frac{\Delta\omega T}{2}\right)},\; x = \frac{1+10^{0.05a[dB]}}{1-10^{0.05a[dB]}} \tag{66}$$

We call (65) the degree equation of the equiripple comb FIR filter. The real value n (65) has to be up-rounded to the next even integer value. This up-rounding preserves the specified number of notch bands and the width of the notchbands. The attenuation in passbands a[dB] is equal or better than the specified value. The impulse response h(k) of the filter consists of 2nr + 1 coefficients, among which there are n + 1 non-zero values. For illustration, the amplitude frequency response 20 log $|H(e^{j\omega})|$[dB] based on the zero phase transfer function:

$$Q(w) = 1 - \frac{1+T_6[1.15T_5(w)]}{1+T_6(1.15)} \tag{67}$$

is shown in Fig. 9. Note that there are true zeros at the notch frequencies. Using differential Eq. 59 we have developed a fast and robust procedure (Table 10) for an algebraic evaluation of the impulse response h (k) of the comb FIR filter.

Table 10: Evaluation of the impulse response of a comb FIR filter

Given	n (even integer), r (integer), $\lambda > 1$ (real)
Initialization	$\text{kppa} = \frac{1}{\lambda},\ \alpha(n) = \lambda^n,\ \alpha(n+2) = \alpha(n+4) = \alpha(n+6) = 0$
body	
(for k = 1 ... n/2)	$\alpha(n-2k) = \{\ \alpha(n-2(k-1)) \times [(1-\text{kppa}^2)(n-(2k-1))(n-(2k-2)) + 3(k+1)(n-(k-1))] - \alpha(n-2(k-2)) \times [(1-\text{kppa}^2)(n-(2k-4))(n-(2k-5)) + 3(k-2)(n-(k-2))] + \alpha(n-2(k-3))(k-3)(n-(k-3))\ \} \setminus k(n-k)$
(end loop on k)	$\alpha(0) = \frac{\alpha(0)}{2}$
A (k) of F (w)	
body	
(for k = 0... n\2)	A (nr-2kr) = α (n - 2k)
(end loop on k)	
α (K) of Q (q)	$C = \cosh[n\, a\cosh(\lambda)],\quad a(0) = 1 - \frac{1+A(0)}{C}$
body	
(for k = 1 ... nr)	$a(k) = -\frac{A(k)}{C}$
(end loop on k)	
impulse response h (k)	h (nr) = α (0)
body	
(for k = 1 ... nr)	$h\ (nr \pm k) = \frac{\alpha(k)}{2}$
(end loop on k)	

Design procedure: The design procedure for the equiripple comb FIR filter consists of the following steps:

- Specify the number of notch bands r, the width of the notch bands $\Delta\omega T$ and the maximal attenuation in the passbands a [dB] demonstrated in Fig. 9.
- The degree of the inner Chebychev polynomial is r.
- Determine the auxiliary parameters λ and χ (66).
- Evaluate the real value n (65) and round it to the next even integer value.
- Evaluate the impulse response h (k), recursively (Table 10).
- Evaluate the actual attenuation in the passbands.

$$a_{act}[\text{dB}] = 20\log\left(1 - \frac{2}{1+\cosh[n\, a\cosh(\lambda)]}\right) \quad (68)$$

Example 5: Design an equiripple comb FIR filter with 20 notch bands specified by the width of the notch bands $\Delta\omega T$ = $\pi/50$ and by the maximal attenuation in the passbands a = -1 dB.

The degree of the inner Chebyshev polynomial is r = 20. We get λ = 1.2361, k = 17.3910 (66) and n = 5.2623 → 6 (65). The zero phase transfer function is

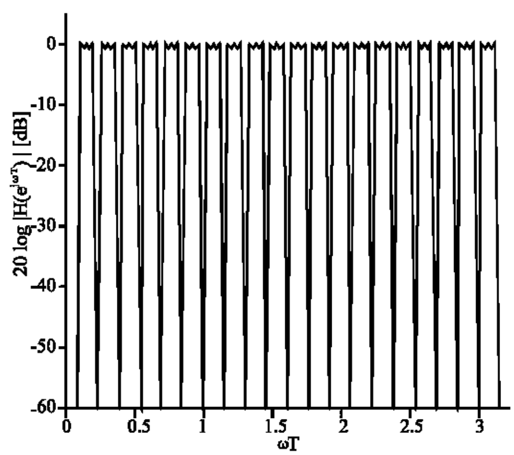

Fig. 10: Amplitude frequency response 20log $|H(e^{j\omega T})|$ [dB] of the equiripple comb FIR filter from example 5

Table 11: Non-zero coefficients of the impulse response h(k) of the comb filter from example 5

k		h (k)
0	240	-0.060281
40	200	-0.124960
80	160	-0.189719
120		0.749920

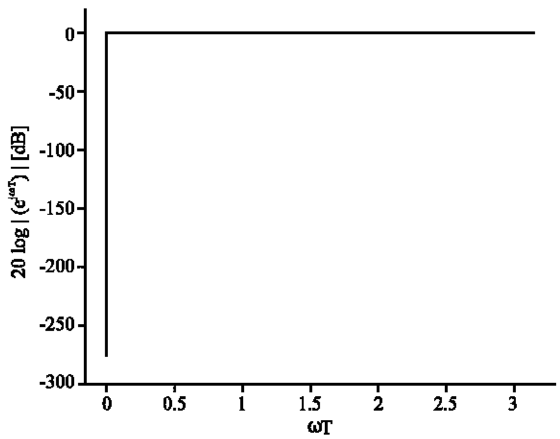

Fig. 11: Amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB]confirming robustness of the DC notch design algorithm

$$Q(w) = 1 - \frac{1 + T_6[1.2361\ T_{20}(w)]}{1 + T_6\ (1.2361)} \tag{69}$$

The impulse response h (k) with a length of 241 coefficients is evaluated recursively (Table 10). It consists of seven non-zero coefficients only. The impulse response is summarized in Table 11. The actual parameters of the comb FIR filter are $\Delta\omega T = \pi/50$ and a = -0.6080 dB. The amplitude frequency response 20 log $|H(e^{j\omega})|$ [dB] of the equiripple comb FIR filter is shown in Fig. 10.

ROBUSTNESS OF THE ANALYTICAL DESIGN

It is worth noting that the analytical design by far outperforms the numerical procedures (McClellan *et al*., 1973). In order to demonstrate the robustness, we wish to design an equiripple DC-notch FIR filter with a quite absurd specification $\omega_p T = 0.00001\pi$ and a = -0.01 [dB]. Using the proposed analytical procedure, we get $\lambda = 1.00000000024674$, $n = 259523.24 \rightarrow 259524$. The length of the filter amounts to N = 519049 coefficients. The zero phase transfer function of the filter is:

$$Q(w) = 1 - \frac{T_n\,(w + 0.24674 \times 10^{-13} \times (w+1)) + 1}{T_n\ (1.00000000049348) + 1} \tag{70}$$

The properties of the designed filter are $\omega_p T = 0.00001\pi$ and a = -0.00999976 [dB]. The amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 11.

CONCLUSION

In this study we have presented robust and fast solutions for the design of narrow band FIR filters. We have developed a new and fundamental approach to the FIR filter design which uses a differential equation to approximate the filter specifications. Then the differential equation is solved recursively. It avoids using Fourier transform in evaluating the impulse response coefficients. Formulae for the degree and for the impulse response of the filters have been also presented. In our future work we will develop a similar robust design procedure for a low-pass FIR filter.

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