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Saturation-Induced Limit Cycles in Observer-Based State Feedback Control

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Abstract: To avoid significant performance degradation due to control signal limitation usually termed as controller windup, feeding the actual limited plant input signal into the controller's state observer is stipulated by state-of-the-art anti-windup schemes. Based on piecewise affine system description, this study presents necessary and suffcient conditions for such an observer-based controller structure to give rise to sustained non-linear oscillations. An effective numerical procedure is devised to evaluate period and the trajectories of the resulting limit cycles. To demonstrate that existence of limit cycles is related to the structure of anti-windup, it is shown that global closed-loop exponential stability can, on the contrary, be guaranteed for the case when the computed (unlimited) control signal is fed to the observer.

Key words: Input saturation, hybrid systems, feedback, stability, limit cycles

INTRODUCTION

Since, all controlled plants are somehow subjected to signal limitations, consequences due to constrains are always of great concern to engineers. Controller performance degradation, generally termed as windup, can be as severe as loss of stability or sustained oscillations even in a nominally well-damped loop (Hippe and Wurmthaler, 1999). In their probably most spectacular form, oscillations due to control signal limitations are known in avionics as Pilot Induced Oscillations and have been cited as the cause of aircraft accidents with high performance fighters (Rundquist, 1997; Dornheim, 1992) as well as jetliners (Dornheim and Hughes, 1995).

Controller structures explicitly addressing the issue of control signal limitation are known as antiwindup controllers and have been intensively studied before, see e.g., Hippe and Wurmthaler (1999), Röonnböck (1993), Kothare *et al*. (1994) and Öhr *et al*. (1997). Since, any linear dynamic feedback controller can be interpreted as an observer-based state feedback, it will suffice here to consider only controllers with an explicit state observer (Walgama and Sternby, 1990).

According to Hippe and Wurmthaler (1999), a major insight achieved in theory of anti-windup controllers and shared by most of the research community is that "the windup phenomenon is an observation error, occurring if the observer is driven by the control output of an unconstrained compensator, whereas the plant receives a value limited by the saturating actuator". Thus, all state-of-the-art anti-windup controllers use the actual (limited) plant input for state reconstruction. However, it is also well understood that the lattermeasure alone does not guarantee neither system stability nor absence of limit cycles once the control signal is saturated. Hippe and Wurmthaler (1999) attribute the remaining negative effects of input saturation to the plant itself while the analysis presented further on in this paper proves that the state feedback gain matrix still plays an important role in evoking limit cycles in anti-windup controllers. Observer design is though immaterial to existence of limit cycles when the saturated signal is used for state estimation.

Considering a closed-loop anti-windup system over the whole state space, i.e., both with and without control signal saturation, calls upon non-linear analysis. For instance, presence of friction-induced limit cycles Preprint submitted to 5 February 2008 in observer-based controllers for mechanical systems is shown by Putra and Nijmeijer (2003). While dealing with an H_2-controlled active engine vibration isolation system, Olsson (2002) pointed out that using the applied control signal for state estimation could evoke sustained oscillations. This was further explored using approximate describing function analysis of a simple one kinematic degree of freedom system (Olsson, 2003).

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The effect of actuator output magnitude limitations in an otherwise linear observer-based feedback control system can be exactly modeled and analyzed by means of existing tools for piecewise affine dynamic systems and autonomously switched dynamic systems (Branicky, 1998). In an anti-windup controller, switching occurs between the plant open-loop dynamics under controller signal saturation and the nominal closed-loop dynamics within the actuator's linear range. Affine system structure is due to the constant input signal to the plant under saturation. Yet, no analysis of anti-windup controllers based on piecewise affine dynamic system description can be readily found in the literature.

Direct application or specialization of the results developed in the field of hybrid systems to anti-windup controllers is problematic for 2 reasons.

- The phenomena of switching (relay feedback) and affine dynamics due to signal saturation are usually treated as separate instances of hybrid systems, Gonçalves *et al.* (2003) and Johansson (1997), while they occur simultaneously in a mathematical description of an anti-windup system. More specifically, existence and symmetry of limit cycles corresponding to relay feedback systems are studied e.g., in Di Bernardo *et al.* (2001) but not for piecewise affine dynamics.
- As it will be demonstrated in the sequel, a certain necessary eigenvalue condition on a product of 2 matrix exponentials plays a key role in existence of stable limit cycles in systems controlled by anti-windup controllers. This condition is, for simplicity reason, excluded from consideration when limit cycles in hybrid systems are studied. Especially, the setup for analysis of limit cycles in Rubensson (2003) would suite the purpose of investigating anti-windup controllers if it could accommodate for the mentioned eigenvalue condition.

One way to exclude limit cycles is to prove global asymptotic stability, implying convergence to the origin. For piecewise affine systems (Johansson, 1999) developed tools based on piecewise quadratic Lyapunov functions and an extension to this research is presented in Nakada and Takaba (2003). A more versatile approach, covering e.g., switching systems with unstable dynamics, is given by Gonçalves *et al.* (2003), presenting an analysis methodology based on considerations of the behaviour at switching surfaces associated with piecewise linear systems.

In this study, necessary and su±cient conditions for limit cycles in the generic structure of observer-based anti-windup with the state observer fed by the limited control signal are given. An effective numerical method for deciding on limit cycle existence and evaluating its period is also provided. Somewhat unexpectedly, no limit cycles are found in an alternative design where the computed (unlimited) control signal is used for state estimation and no explicit anti-windup measures are taken. The method of piecewise quadratic Lyapunov functions suffices for showing global closed-loop stability for openloop stable plant. Possibility of limit cycles does not necessarily imply that anti-windup controllers have otherwise inferior performance compared to controllers without explicit anti-windup.

STATEMENT OF THE PROBLEM

Observer-based state feedback control in the presence of actuator output magnitude limitations, i.e. input saturation, is considered. Specifically, the limitations are assumed to be described by a standard saturation function according to (1) where $\alpha > 0$.

$$f(x) = \begin{cases} -\alpha, & x \leq -\alpha \\ x, & |x| < \alpha \\ -\alpha, & x \geq \alpha \end{cases} \tag{1}$$

Two usual ways of implementing observer-based state feedback controllers are schematically shown in Fig. 1. One implementation uses the computed control signal for observation while the other uses the applied, possibly saturated, control signal. If the saturated control signal could not be measured, a model f_m (x) of the saturation function could be used in the controller to estimate the actual control signal.

Clearly, in case of an implementation according to Fig. 1a there will be state estimation errors when the input signal is saturated since the observer is unaware of the actual control input applied to the plant. This is usually pointed out as the reason to controller windup, Hippe and Wurmthaler (1999). The state estimation error due to the input signal mismatch could therefore be eliminated by feeding the observer with the possibly saturated control signal according to Fig. 1b which is the solution implemented in state-of-the-art anti-windup controllers (e.g., Åström and Wittenmark (1997)).

Let $x(t) \in R^n$ denote the internal states of the plant, $\hat{x}(t) \in R^n$ be the state estimates of the observer and introduce the state estimation error $\epsilon = x - \hat{x}$. In terms of the augmented state vector $\tilde{z} = [\,x^T\ \epsilon^T]^T \in R^{2n}$, the evolution of the closed loop system is given by:

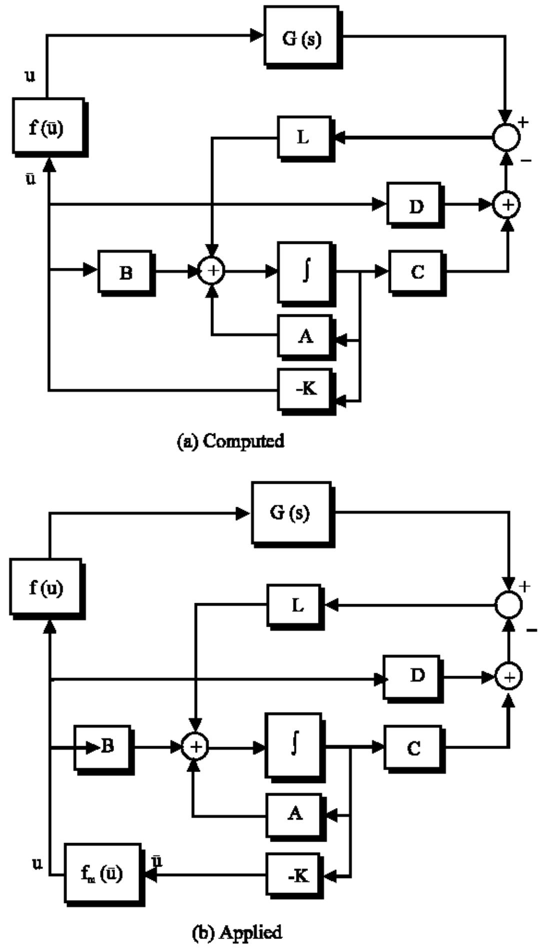

Fig. 1: Implementations of observer-based state feedback controllers using (a) the computed control signal for state estimation and (b) using fundamental anti-windup technique where the observer is fed by the control signal applied to the plant

$$\dot{\tilde{z}} = F_0\, \tilde{z} + b^{(1)}\, \tilde{f}\,(Kx - K \in) \qquad (2)$$

$$F_0 = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix};\; b^{(1)} = \begin{bmatrix} B \\ B \end{bmatrix}$$

when the computed control signal is used for state estimation. In (2), $\tilde{f}(x)$ is a static, odd, "dead zone"-type non-linear function defined as:

$$\tilde{f}(x) = \begin{cases} x + \alpha, & x \le -\alpha \\ 0, & |x| < \alpha \\ x - \alpha, & x \ge \alpha \end{cases} \qquad (3)$$

When using the applied control signal for estimation, the closed-loop dynamics are governed by:

$$\dot{\tilde{z}} = F^{(2)}\, \tilde{z} - b^{(2)}\, f\,(Kx - K \in) \qquad (4)$$

$$F^{(2)} = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix};\; b^{(2)} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

By scaling the state vector as

$$z(t) = \tilde{z}(t) / \alpha \qquad (5)$$

The dependence on the saturation limit α could be eliminated without loss of generality in both (2) and (4) so that $\alpha = 1$.

Remark 1: Since, the value of the saturation limit α could be excluded from the system descriptions using state transformation, the characteristics of a specific solution could not depend on this limit. However, the state space trajectory corresponding to any given solution will of course depend on α through (5).

Remark 2: This study focuses on the case where the observer is not augmented with any additional dynamics. However, the analysis could easily be generalised to deal with disturbance modeling or dynamic weighting functions from e.g., H_2 or H_∞ design (Olsson, 2002).

Due to the non-linear characteristics of the 2 systems, it is natural to partition the state space into regions corresponding to no saturation, positive saturation and negative saturation, respectively. These 3 regions could be expressed as:

$$\begin{aligned} \mathcal{Z}_0 &= \{ z \,|\, |\tilde{K} z| < 1 \} \\ \mathcal{Z}_1 &= \{ z \,|\, \tilde{K} z \ge 1 \} \\ \mathcal{Z}_2 &= \{ z \,|\, \tilde{K} z \le -1 \} \end{aligned} \qquad (6)$$

where, $\tilde{K}$ = [K - K]. Introducing

$$F^{(1)} = \begin{bmatrix} A & 0 \\ BK & A - LC - BK \end{bmatrix} \qquad (7)$$

both systems are suitably represented using a piecewise affine system model of the form

$$\dot{z}(t) = F_0\, z(t),\; z \in \mathcal{Z}_0 \qquad (8)$$

$$\dot{z}(t) = F^{(p)}\, z(t) - b^{(p)},\; z \in \mathcal{Z}_1 \qquad (9)$$

$$\dot{z}(t) = F^{(p)}\, z(t) + b^{(p)},\; z \in \mathcal{Z}_2 \qquad (10)$$

where, p = 1 for the implementation using the computed control signal for state estimation according to (2) and p = 2 for the implementation given by (4), using the applied control signal for state estimation.

For notational convenience, the superscript p is dropped in the sequel whenever, an expression holds for both systems.

CLOSED ORBITS AND LIMIT CYCLES

A periodic solution Z to system (8)-(10) of least period T > 0, i.e., such that $Z(t_0) = Z(t_0 + T)$ and $Z(t_0 + \tau) \neq = Z(t_0)$ for $0 < \tau < T$, is a closed orbit in R 2n , see e.g., Nayfeh and Balachandran (1995). Figure 2 shows a 2-dimensional visualisation of such a closed trajectory passing through the 3 regions defined by (6). The dashed lines correspond to the boundaries between the regions $\mathcal{Z}_0$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$, represented by the hyperplanes P^+, given by $\tilde{K}z = +1$ and P^-, given by $\tilde{K}z = -1$. The notation τ_j is used to represent the duty times, i.e., the time elapsed between 2 subsequent crossings of the region boundaries.

Even though existence of closed orbits corresponding to periodic solutions is of interest, the existence of limit cycles has been of main concern for this research. According to the definition by Poincaré (1892-1899), a limit cycle is an isolated periodic solution corresponding to an isolated closed orbit in state space and every trajectory initiated near the limit cycle approaches it either as $t \to \infty$ or as $t \to -\infty$.

As demonstrated in Di Bernardo *et al.* (2001), unforced linear dynamic systems with symmetric non-linearities might have asymmetric orbits. For the special case of relay feedback, it has been shown that systems with real stable poles and no zeros can only have symmetric orbits. Relay feedback implies though that the system dynamics is the same before and after relay switching. This is, however, not the case with anti-windup controllers where the dynamics alters from the nominal closed-loop to the plant open-loop under control signal saturation.

The following theorem specifically concerns symmetry properties of the system at hand and will be used in the sequel to investigate the existence of closed orbits in anti-windup controllers.

Theorem 3.1: Consider either of the systems (2) or (4) described by (8), (9) and (10). If $z(\tau)$ is a solution to any of these systems, then the symmetrically opposite time dependent vector $-z(\tau)$ must correspond to another solution to this system.

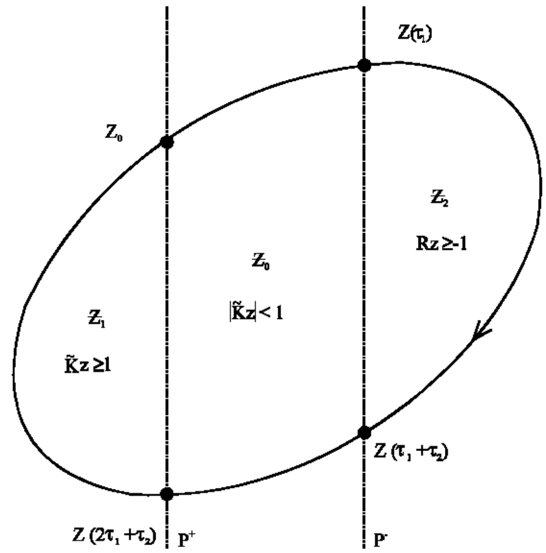

Fig. 2: A 2-dimensional visualisation of a closed orbit cor responding to a periodic solution

Proof: Assume there exist a solution

$$Z(t) \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2$$

to the systems described by (8)-(10). As a consequence of the odd non-linear characteristics of the saturation function the bidirectional mappings:

$$\mathcal{Z}_0 \overset{-1}{\rightarrow} Z_0, \mathcal{Z}_1 \overset{-1}{\rightarrow} Z_2, \mathcal{Z}_2 \overset{-1}{\rightarrow} Z_1$$

yield

$$\begin{aligned} \dot{\tilde{z}}(t) &= F_0\tilde{z}(t), & \tilde{z} &\in \mathcal{Z}_0 \\ \dot{\tilde{z}}(t) &= F\tilde{z}(t) + b & \tilde{z} &\in \mathcal{Z}_2 \\ \dot{\tilde{z}}(t) &= F\tilde{z}(t) - b & \tilde{z} &\in \mathcal{Z}_2 \end{aligned} \quad (11)$$

Where,

$$\tilde{z}(t) = -z(t) \quad (12)$$

Since, the equations given by (11) are the same as (8)-(10), but with $\tilde{z}$ written for z, a solution $\tilde{Z}(t)$ corresponding to the solution Z(t) must exist with $\tilde{Z}(t) = -Z(t)$ $\forall t$, t > 0. Thus, the 2 solutions Z (t) and $\tilde{Z}(t)$ represents 2 unique, identical but symmetrically opposite, trajectories.

Remark 3: As a consequence of Theorem 3.1, a solution Z(t) to the systems described by (8)-(10) with Z (t) = −Z(0) for some t > 0, corresponds to a half of a closed symmetric orbit.

Introduce the following matrices and vectors.

$$\Phi_1(\tau_1) = e^{F_0\tau_1},\ \Phi_2(\tau_2) = e^{F\tau_2}$$

$$M(\tau_1, \tau_2) = \begin{bmatrix} I + \Phi_2(\tau_2)\Phi_1(\tau_1) \\ \tilde{K} \\ \tilde{K}\Phi_1(\tau_1) \end{bmatrix}$$

$$N(\tau_2) = \left[\left(-\int_0^{\tau 2} e^{F(\tau 2 - \tau)} b d\tau \right)^T \; 1 - 1 \right]^T$$

Based on the symmetry argument, necessary and sufficient conditions for existence of closed orbits and limit cycles in the systems under consideration are formulated in the theorem below.

Theorem 3.2: A closed orbit corresponding to a solution to (8)-(10) exists iff $\exists\ \tau_1 > 0$ and $\tau_2 > 0$ such that

$$N(\tau_2) \in \mathrm{Im}\,(M(\tau_1, \tau_2)) \tag{13}$$

Moreover, this closed orbit corresponds to a limit cycle solution iff $|\lambda_i| < 1$, $\forall_i \neq 1$, where λ_i are the eigenvalues of the matrix

$$D_{T/2} = \Phi_2(\tau_2)\,\Phi_1(\tau_1) \tag{14}$$

with increasing index for non-increasing absolute values, i.e., $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{2n}|$.

Proof: Assume a solution to (8)-(10) initiated at Z_0 belonging to the hyperplane P^+ given by $\tilde{K}z = 1$, evolving through $\mathcal{Z}_0$, entering $\mathcal{Z}_2$ at $t = \tau_1$ and returning to the other hyperplane P^- at $Z(\tau_1 + \tau_2)$ with $Z(\tau_1 + \tau_2) = -Z_0$. Consequently, at $t = \tau_1$ and $t = \tau_1 + \tau_2$, this solution is given by

$$Z(\tau_1) = e^{F_0 \tau 1}\, Z_0$$

$$Z(\tau_1 + \tau_2)\ e^{F\tau 2}\ Z(\tau_1) + \int_0^{\tau 2} e^{F(\tau 2 - \tau)} b d r$$

Substituting the expression for $Z(\tau_1)$ in the expression for $Z(\tau_1 + \tau_2)$ and using $Z(\tau_1 + \tau_2) = -Z_0$ gives

$$(I + e^{F\tau 2}\, e^{F_0 \tau 1}) Z_0 = -\int_0^{\tau 2} eF(\tau_2 - \tau) b d\tau \tag{15}$$

Z_0 belonging to the hyperplane P^+ and $Z(t)$ entering $\mathcal{Z}_2$ (i.e., intersecting $P-$) at $t = \tau_1$ implies

$$\tilde{K}\ Z_0 = 1, \tilde{K}\ e^{F_0 \tau 1}\ Z_0 = -1 \tag{16}$$

From Theorem 3.1, $Z(t)$; $t_\epsilon\ [0, 2(\tau_1 + \tau_2)]$, constitutes a closed orbit, i.e., from (15) and (16), a closed orbit exists iff $\exists\ \tau_1 > 0$, $\tau_2 > 0$ and $Z_0 \in R^{2n}$ satisfying

$$M(\tau_1, \tau_2)\, Z_0 = N(\tau_2) \tag{17}$$

which yields (13).

The eigenvalue condition for limit cycle solutions follows from Floquet theory (Nayfeh and Balachandran, 1995) where the eigenvalues of the monodromy matrix, called the Floquet or characteristic multipliers, determine the stability properties of the corresponding periodic solution. A periodic solution requires one of the Floquet multipliers to be located on the unit circle in the complex plane and is asymptotically stable, i.e. con- stitutes a limit cycle, if there are no multipliers outside the unit circle. For a periodic solution corresponding to (13), the monodromy matrix is given by:

$$D_T = (e^{F\tau 2}\, e^{F_0 \tau 1})^2$$

As a consequence of symmetry, DT could be written $D_T = (D_{T/2})^2$, with $D_{T/2}$ according to (14). Since, the eigenvalues of DT/2 are the square root of the eigenvalues of D_T, the requirements on the Floquet multipliers for stability of a periodic solution, apply also to the eigenvalues of $D_{T/2}$.

Remark 4: Notice that the sought periodic solution is initiated at the hyperplane P^+ without loss of generality. Furthermore, for a non-singular matrix F, the integral in the left hand side of (13) evaluates to $F^{-1}(e^{F\tau 2} - I)b$.

Corollary 1: A limit cycle requires $\lambda_1 = -1$ where λ_1 is the eigenvalue of the matrix $D_{T/2}$ given by (14) with largest absolute value.

Proof: Using a non-linear function f (z; t), a periodic solution Z (t) to (8)-(10) could be written

$$\dot{Z}(t) = f(z, t) \tag{18}$$

Taking the derivative of both sides of (18) yields

$$\ddot{Z} = \frac{\partial f(Z, t)}{\partial Z} \dot{Z} \tag{19}$$

$\dot{Z}(t)$ is thus a solution to the Linear Time Varying (LTV) system (19). Since, Z(t) is a periodic solution, so is $\dot{Z}(t)$. Moreover $Z(0) = -Z(\tau_1 + \tau_2)$ implies that $\dot{Z}(0) = -\dot{Z}(\tau_1 + \tau_2)$. Considering the solution to the LTV system (19) for half a period and writing $\dot{Z}(t) = V(t)$, it follows that:

$$(I + \Phi_2(\tau_2)\,\Phi_1(\tau_1))\,V(0) = (I + D_{T/2})\,V(0) = 0$$

Consequently, for $V(0) \neq 0$, $(I + D_{T/2})$ has to loose rank. This is only possible if $\lambda_i = -1$ for some $i \in [1, ..., 2n]$.

Remark 5: The eigenvalue conditions of Theorem 3.2 and Corollary 1 are crucial to existence of (stable) limit cycles in anti-windup controllers. Unfortunately, the problem of spectral radius characterization of the product of 2 matrix exponentials is apparently yet an unsolved problem in linear algebra (Bernstein, 1992). An analytical example of dimension 2 given in the Appendix reveals how a product of matrix exponentials can have eigenvalues on and outside of the unit circle even for Hurwitz matrices. The eigenvalue conditions imply singularity of the matrix $I + D_{T/2}$, which in its turn effectively prevents direct application of results on limit cycles provided in e.g., Rubensson (2003) and Di Bernardo *et al.* (2001).

The necessary and sufficient conditions of Theorem 3.2 are stated in terms of the block matrices F_0, $F^{(1)}$ and $F^{(2)}$ as given by (2), (4) and (7). To get insight into how the structure of the observer-based system influences the conditions, the latter are treated below in terms of the plant and controller matrices for each one of the cases.

The case of computed control signal: Introduce the following notation:

$$\phi_1(t) = e^{At},\ \phi_2(t) = e^{(A-BK)t}$$
$$\phi_3(t) = e^{(A-BK-LC)t},\ \phi_4(t) = e^{(A-LC)t}$$
$$f_1(\tau_1) = \int_0^{\tau_1} \phi_2(\tau) BK \phi_4(\tau_1 - \tau)\, d\tau$$
$$f_2(\tau_2) = \int_0^{\tau_2} \phi_3(\tau) BK \phi_1(\tau_2 - \tau)\, d\tau$$

Corollary 2: In the case of an implementation according to (2), i.e., using the computed control signal for state estimation, Theorem 3.2 holds for the following values of the involved matrices

$$D_{T/2} = \begin{bmatrix} \phi_1(\tau_2)\,\phi_2(\tau_1) & \phi_1(\tau_2)\, f_1(\tau_1) \\ f_2(\tau_1)\phi_2(\tau_1) & f_2(\tau_2) f_1(\tau_1) + \phi_3(\tau_2)\phi_4(\tau_1) \end{bmatrix}$$

$$M(\tau_1, \tau_2) = \begin{bmatrix} I + \phi_1(\tau_2)\,\phi_2(\tau_1) & \phi_1(\tau_2)\, f_1(\tau_1) \\ f_2(\tau_2)\phi_2(\tau_1) & M_{22}(\tau_1, \tau_2) \\ K & -K \\ K\phi_2(\tau_1) & K(f_1(\tau_1) - \phi_4(\tau_1)) \end{bmatrix}$$

$$N(\tau_2) = \begin{bmatrix} \left(\int_0^{\tau_2} e^{A(\tau_2 - \tau)} B d\tau \right)^T & N_2^T(\tau_2) 1 - 1 \end{bmatrix}^T$$

Where,

$$N_2(\tau_2) = \int_0^{\tau_2} \phi_3(\tau_2 - \tau) BK \left(\int_0^{\tau} \phi_1(\tau - t) dt \right) + \phi_1(\tau_2 - \tau) d\tau B$$

$$M_{22}(\tau_1, \tau_2) = I + f_2(\tau_2) f_1(\tau_1) + \phi_3(\tau_2)\phi_4(\tau_1)$$

Proof: Taking the block matrix structure of the involved matrices into account, the result follows from a direct evaluation of (14) and (13) with F_0, $F^{(1)}$ and $b^{(1)}$ given by (2), (4) and (7).

The case of applied signal: In this study, it is proven that for the case of the observer-based anti-windup controller using the applied (limited) control signal in state estimation, the observer design is immaterial to existence of limit cycles. The necessary and sufficient conditions for existence of closed orbits and limit cycles provided below in Corollary 3 are formulated solely in terms of plant properties and the state feedback gain matrix.

In the case when the applied control signal is used for state estimation, it is convenient to introduce the following notation:

$$M_x(\tau_1, \tau_2) = \begin{bmatrix} I + \phi_1(\tau_2)\phi_2(\tau_1) \\ K \\ K\phi_2(\tau_1) \end{bmatrix}$$

$$N_x(\tau_2) = \begin{bmatrix} -\left(\int_0^{\tau_2} e^{A(\tau_2 - \tau)} B d\tau \right)^T & 1 - 1 \end{bmatrix}$$

Corollary 3: A closed orbit corresponding to a solution to (8)-(10) exists iff $\exists\ \tau_1 > 0$ and $\tau_2 > 0$ such that

$$N_x(\tau_2) \in \mathrm{Im}\,(M_x(\tau_1, \tau_2)) \tag{20}$$

Moreover, this closed orbit corresponds to a limit cycle solution iff $|\lambda_i| < 1$, $\forall_i \neq 1$, where λ_i are the eigenvalues of the matrix $D^x_{T/2} \in R^{n \times n}$

$$D^x_{T/2} = \phi_2(\tau_2)\, \phi_1(\tau_1) \tag{21}$$

with increasing index for non-increasing absolute values, i.e., $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$.

Proof: Inserting F_0, $F^{(2)}$ and $b^{(2)}$ given by (2), (4) and (7), in $D_{T/2}$, $M(\tau_1; \tau_2)$ and $N(\tau_1)$ from Theorem 3.2 yields

$$D_{T/2} = \begin{bmatrix} \phi_1(\tau_2)\,\phi_2(\tau_1) & \phi_1(\tau_2)\, f_2(\tau_1) \\ 0_{n \times n} & \phi_4(\tau_1 + \tau_2) \end{bmatrix} \tag{22}$$

$$M = \begin{bmatrix} I+\phi_1(\tau_2)\,\phi_2(\tau_1) & \phi_1(\tau_2)\,f_1(\tau_1) \\ 0_{n\times n} & I+\phi_4(\tau_1+\tau_2) \\ K\,\phi_2(\tau_1) & K(f_1(\tau_1)-\phi_4(\tau_1)) \end{bmatrix} \quad (23)$$

$$N = \begin{bmatrix} -\left(\int_0^{\tau_2} e^{A(\tau_2-\tau)} B d\tau\right)^T & 0_{n\times n} & 1-1 \end{bmatrix}^T$$

Condition (13) in Theorem 3.2 is fulfilled if a solution Z_0 to (17) exists. Writing $Z_0 = [X^T{}_0\, E^T{}_0]^T$, $X_0 \in R^n$, $E_0 \in R^n$ and evaluating part of (17) corresponding to the second lines in M and N given by (23) it follows that:

$$(I+\phi_4(\tau_1+\tau_2))\, E_0 = (I+e^{(A-LC)(\tau_1+\tau_2)})\, E_0 = 0$$

Since, (A − LC) is a Hurwitz matrix and $\tau_1 > 0$; $\tau_2 > 0$, it must hold that $E_0 = 0$. Taking the equations of motion given by (4) into account, it is clear that with $\in(t) = 0$, t = 0, it must hold that $\in(t) = 0$; $t \geq 0$. Consequently, (22) could be reduced to (21) and (20) follows from (17).

Analysis of closed orbit existence condition: The question of whether a limit cycle exists or not in the 2 above considered controller implementations boils down to the problem of solving system of Eq. 17 with respect to Z_0. The linear system of equations is overdetermined and τ_1, τ_2 have to be manipulated to make it consistent. The following observations can be readily made.

- Letting $\tau_1 \to 0$ leads to an inconsistent system due to the last 2 equations of (17).
- $Z_0 \neq 0$ since otherwise the equation $\tilde{K}\, Z_0 = 1$ cannot be satisfied.
- Assuming $\tau_2 = 0$ yields $(I + e^{F0\tau1})\, Z_0 = 0$ which cannot be fulfilled since $\det(I + e^{F0\tau1}) \neq 0$ and $Z_0 \neq 0$.
- Due to stability of F_0, $\lim_{\tau1\to\infty} \tilde{K}\, e^{F0\tau1} = 0$. Thus large values of τ_1 cause inconsistency of the system.
- For $\tau_2 \to \infty$ and a stable F, one gets.

$$Z_0 = -\lim_{\tau_2\to\infty} \int_0^{\tau_2} e^{F(\tau_2-\tau)} b\, d\tau = -F^{-1} b$$

To yield a solution to (17), 2 conditions have to be satisfied by choosing τ_1

$$\tilde{K}F^{-1} b = -1,$$
$$\tilde{K}e^{F_0\tau_1}\, F^{-1}\, b = 1$$

This is unlikely but cannot theoretically be ruled out.

- Naturally, (17) has to be solved numerically to obtain the parameters of limit cycle τ_1, τ_2. An efficient method for its solution is provided in Appendix A and B.

APPENDIX A

Eigenvalues of matrix exponential products: Consider the following matrix product:

$$P(T_1, T_2) = \exp(A_1T_1)\exp(D^{-1}A_2D_{T2}) = \exp(A_1T_1)D^{-1}$$

Where,

$$A_1 = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}; A_2 = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}; D = \begin{bmatrix} d_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

and $\tau_1, \tau_2 \in R^+$, $a_1, a_2 \in R^-$, $d_1, d_2 \in R^1$. A direct evaluation of P yields

$$P = \frac{e^{(a_1\tau_1+a_2\tau_2)}}{2} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Where,

$$p_{11} = \frac{1}{d_2}((d_2-d_1)\cos(b_1\tau_1-b_2\tau_2) + (d_2+d_1)\cos(b_1\tau_1+b_2\tau_2))$$

$$p_{12} = \frac{1}{d_1}((d_2-d_1)\sin(b_1\tau_1-b_2\tau_2) + (d_1+d_2)\sin(b_1\tau_1+b_2\tau_2))$$

$$p_{21} = \frac{1}{d_2}((d_2-d_1)\sin(b_1\tau_1-b_2\tau_2) - (d_1+d_2)\sin(b_1\tau_1+b_2\tau_2))$$

$$p_{22} = \frac{1}{d_1}((d_1-d_2)\cos(b_1\tau_1-b_2\tau_2) + (d_2+d_1)\cos(b_1\tau_1+b_2\tau_2))$$

The matrix P is always nonsingular since

$$\det P = e^{2(a_1\tau_1+a_2\tau_2)} > 0$$

The eigenvalues of P are

$$\lambda(X) = \frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} - e^{2(a_1\tau_1+a_2\tau_2)}}$$

Where,

$$\xi = \frac{e^{a(a_1\tau_1 + a_2\tau_2)}}{2d_1d_2}((d_1 + d_2)^2 \cos(b_1\tau_1 + b_2\tau_2) - (d_1 - d_2)^2 \cos(b_1\tau_1 - b_2\tau_2))$$

The condition that the eigenvalues are real is equivalent to

$$4|d_1d_2| \le (d_1 + d_2)^2 \cos(b_1\tau_1 + b_2\tau_2) - (d_1 - d_2)^2 \cos(b_1\tau_1 - b_2\tau_2) \le 2(d_1^2 + d_2^2)$$

where the upper bound is always valid.

For complex eigenvalues, it is straightforward to show that

$$\sup|\lambda(X)| = e^{(a_1\tau_1 + a_2\tau_2)}$$

and therefore this quantity is independent of d_i, b_i, i = 1, 2. Thus, none of the eigenvalues of P can be on or outside of the unit circle for the specified parameter intervals.

For real eigenvalues, it applies

$$|\lambda(X)| < |\xi| \le e^{(a_1\tau_1 + a_2\tau_2)} \max(\left|\frac{d_1}{d_2}\right|, \left|\frac{d_2}{d_1}\right|)$$

and $|\lambda(X)|$ is unbounded as D approaches singularity.

For the special case $d_1 = d_2$, the left hand side inequality of the eigenvalue realness condition becomes

$$1 \le |\cos(b_1\tau_1 + b_2\tau_2)|$$

which can be satisfied only if τ_i, i = 1, 2 are chosen so that $|\cos(b_1\tau_1 + b_2\tau_2)| = 1$. This, in its turn, means that:

$$\lambda(X) = \pm e^{(a_1\tau_1 + a_2\tau_2)}$$

which is definitely inside the unit circle. Thus, the scaling of eigenvectors $d_1 \neq d_2$ is necessary for obtaining $|\lambda(X)| \geq 1$.

In fact, when $d_1 \neq d_2$, it is easy to show that $\forall|r| \geq 1$ there exist infinitely many combinations of τ_1, τ_2, d_1, d_2 such that $\lambda = r$. Consider the following case where τ_1 and τ_2 are chosen so that

$$\cos(b_1\tau_1 + b_2\tau_2) = 1, \cos(b_1\tau_1 - b_2\tau_2) = 0$$

In this case

$$\xi = \frac{k}{2d_1d_2}(d_1 + d_2)^2$$

Where,

$$k = e^{(a_1\tau_1 + a_2\tau_2)}$$

is a constant with $0 \le k \le 1$.

The corresponding eigenvalues are given by

$$\lambda(P) = k(p \pm \sqrt{p^2 - 1}) \quad \text{(A.1)}$$

Where,

$$p = \frac{(d_1 + d_2)^2}{4d_1d_2} = \frac{d_1}{4d_2} + \frac{1}{2} + \frac{d_2}{4d_1}, \; |p| > 1 \quad \text{(A.2)}$$

It is clear that $\forall|\lambda| \geq 1$, there is a solution p to (A.1) with $|p| \geq 1$. Moreover, from (A.2) it follows that all p, $|p| \geq 1$, could be achieved by an infinite number of combinations of d_1 and d_2 obeying

$$d_2 = -d_1(1 - 2p) \pm 2d_1\sqrt{p^2 - p}$$

APPENDIX B

Separable nonlinear least squares: Introduce the augmented vector of unknowns

$$\bar{Z}_0^T = [\tau_1 \quad \tau_2 \quad Z_0^T]$$

and the residual

$$\bar{r}(\bar{Z}_0) = N(\tau_2) - M(\tau_1, \tau_2)Z_0$$

To find a solution to (17) numerically, the following optimization problem can be considered

$$\bar{Z}_0^* = \arg \min_{\bar{Z}_0 \in R^{2n+2}} \bar{r}^T(\bar{Z}_0)\,\bar{r}(\bar{Z}_0)$$

The corresponding Jacobian matrix whose elements are defined as:

$$J_{ij} = \frac{\partial \bar{r}_i(\tilde{Z}_0)}{\partial \tilde{Z}_{0j}}$$

is given by

$$J = [J_{\tau 1} \quad J_{\tau 2} \quad -M(\tau_1, \tau_2)]$$

System of algebraic Eq. 17 depends linearly on Z_0 and nonlinearly on τ_1, τ_2 and thus is suitable for the application of separable nonlinear least squares method (Golub and Pereyra, 1973). The derivation in the sequel follows the lines of Nielsen (2000). Yet the formulae of the latter work cannot be readily applied due to the dependance of N on one of the optimization variables.

The structure of (17) implies

$$Z_0(\tau_1, \tau_2) = M+(\tau_1, \tau_2)\, N(\tau_2)$$

where ·+ denotes pseudoinverse. Thus, the duty times τ_1, τ_2 can be found numerically by solving the following optimization problem

$$(\tau_1^*, \tau_2^*) = \arg\min_{\tau_1,\tau_2} r^T(\tau_1,\tau_2)\, r(\tau_1,\tau_2)$$

Where

$$r(\tau_1,\tau_2) = N((\tau_2) - M(\tau_1,\tau_2) \quad M^+(\tau_1,\tau_2)\, N(\tau_2) \tag{B.1}$$

A closed orbit exists if and only if $\| r(\tau^*_1, \tau^*_2)\| = 0$. The corresponding Jacobian has the form

$$J(\tau_1,\tau_2) = \left[\frac{\partial r(\tau_1,\tau_2)}{\partial \tau_1} \quad \frac{\partial r(\tau_1,\tau_2)}{\partial \tau_2}\right]$$

where the columns can be evaluated from (B.1) as

$$\frac{\partial r}{\partial \tau_1} = -\frac{\partial M}{\partial \tau_1} Z_0 - M\frac{\partial Z_0}{\partial \tau_1}$$
$$\frac{\partial r}{\partial \tau_2} = \frac{\partial N}{\partial \tau_2} - \frac{\partial M}{\partial \tau_2} Z_0 - M\frac{\partial Z_0}{\partial \tau_2} \tag{B.2}$$

with the arguments dropped for brevity. By taking derivatives of the involved block matrices one obtains

$$\frac{\partial M}{\partial \tau_1} = \begin{bmatrix} \Phi_2(\tau_2)\, F_0\, \Phi_1(\tau_1) \\ 0 \\ K F_0\ \Phi_1(\tau_1) \end{bmatrix}; \frac{\partial M}{\partial \tau_2} = \begin{bmatrix} F\Phi_2(\tau_2)\, \Phi_1(\tau_1) \\ 0 \\ 0 \end{bmatrix}$$

and

$$\frac{\partial N^T}{\partial \tau_2} = \left[-(F\int_0^{\tau_2} \Phi_2(\tau_2-\theta) b\, d\theta + b)^T \ 0 \ 0\right]$$

To evaluate the rightmost terms in (B.2), the derivatives of Z_0 with respect to τ_1 and τ_2 are necessary. Recall now that Z_0 satisfies the system of normal equations

$$M^T M Z_0 = M^T N$$

Taking derivatives of the equation above yields

$$\frac{\partial M^T M}{\partial \tau_1} Z_0 + M^T M \frac{\partial Z_0}{\partial \tau_1} = \frac{\partial M^T}{\partial \tau_1} N$$
$$\frac{\partial M^T M}{\partial \tau_2} Z_0 + M^T M \frac{\partial Z_0}{\partial \tau_2} = \frac{\partial M^T}{\partial \tau_2} N + M^T \frac{\partial N}{\partial \tau_2}$$

Now rearrange the equations leaving only the terms involving derivatives of Z_0 in the left-hand side

$$M^T M\frac{\partial Z_0}{\partial \tau_1} = \frac{\partial M^T}{\partial \tau_1} N - \left(\frac{\partial M^T}{\partial \tau_1} M + M^T \frac{\partial M}{\partial \tau_1}\right)$$
$$Z_0 = \frac{\partial M^T}{\partial \tau_1} r - M^T \ J_{\tau 1}$$
$$M^T M\frac{\partial Z_0}{\partial \tau_2} = \frac{\partial M^T}{\partial \tau_2} N + M^T \frac{\partial N}{\partial \tau_2} - \left(\frac{\partial M^T}{\partial \tau_2} M + M^T \frac{\partial M}{\partial \tau_2}\right)$$
$$Z_0 = \frac{\partial M^T}{\partial \tau_2} r + M^T \ J_{\tau 2} \tag{B.3}$$

The vectors $\partial Z_0/\partial \tau_i$ i = 1, 2 can be found solving normal Eq. (B.3). Substituting them into (B.2) results in the sought expressions for the Jacobian matrix of the separated problem. Now (17) can be resolved by means of standard software for the Gauss-Newton method (or its Levenberg-Marquardt modification) using the obtained formulae for the Jacobian.

EXPONENTIAL STABILITY

One way to exclude the existence of limit cycle solutions is to prove global stability of the closed loop system. Since, the existence or non-existence of limit cycles says nothing about stability, this kind of information implies increased knowledge about closedloop system performance. In this study exponential stability is therefore, analysed making use of existing mathematical tools. It is also shown that this kind of analysis provides upper bounds on convergence rate.

Johansson (1999) has formulated the search for piece-wise quadratic Lyapunov functions for piecewise affine systems in terms of LMIs. A limitation arising in this approach with respect to analysis of anti-windup controllers is that the plant dynamics have to be stable. Global analysis for the case of open-loop unstable plants can be performed by means of more versatile technique suggested in Gonçalves *et al.* (2003) but not pursuedhere. The method of Johansson (1999) is suitable for the openloop stable plants considered in this study and a Lyapunov function of the form

$$V(z) = \begin{cases} \bar{z}^T \bar{P}_1 \bar{z}, \ z \in \mathcal{Z}_1 \\ z^T \bar{P}_0 z, \ z \in \mathcal{Z}_0 \\ \bar{z}^T \bar{P}_2 \bar{z}, \ z \in \mathcal{Z}_2 \end{cases} \tag{24}$$

where, $\bar{z} = [z^T \, 1]^T$ and z is a solution to (8)-(10), could be sought for using the theorem given below. It is asumed hat the trajectories could not end up in attractive sliding modes. To deal with sliding modes the analysis conditions have to be extended (Johansson, 1999).

Theorem 4.1: If there exist a symmetric matrix T and non-negative scalars u_1, v_1, u_2, v_2, λ_0, λ_1 and λ_2, with

$$\begin{aligned} \bar{P}_1 &= \bar{H}_1^T \, T \, \bar{H}_1 \\ P_0 &= H_0^T \, T \, H_0 \\ \bar{P}_2 &= \bar{H}_2^T \, T \, \bar{H}_2 \end{aligned} \tag{25}$$

and

$$[\bar{H}_1 \; |H_0| \; \bar{H}_2] = \left[\begin{array}{cc|c|cc} \tilde{K} & -1 & & 0_{1\times 2n} & 0 \\ 0_{1\times 2n} & 0 & 0_{2\times 2n} & -\tilde{K} & -1 \\ I_{2n\times 2n} & 0 & I_{2n\times 2n} & I_{2n\times 2n} & 0 \end{array}\right] \tag{26}$$

such that

$$\begin{cases} 0 > \bar{F}^T \bar{P}_1 + \bar{P}_1 \bar{F} + u_1 Q_1 + 2\lambda_1 \bar{P}_1 \\ 0 < \bar{P}_1 - \upsilon_1 Q_1 \end{cases}$$
$$\begin{cases} 0 > F_0^T P_0 + P_0 F_0 + 2\lambda_0 P_0 \\ 0 < P_0 \end{cases} \tag{27}$$
$$\begin{cases} 0 > \bar{F}^T \bar{P}_2 + \bar{P}_2 \bar{F} + u_2 Q_2 + 2\lambda_2 \bar{P}_2 \\ 0 < \bar{P}_2 - \upsilon_2 Q_2 \end{cases}$$

Where,

$$\tilde{F} = \begin{bmatrix} F & b \\ 0_{1\times 2n} & 0 \end{bmatrix} \tag{28}$$

and

$$Q_1 = \begin{bmatrix} 0_{2n\times 2n} & \tilde{K}^T \\ \tilde{K} & -2 \end{bmatrix}, \ Q_2 = \begin{bmatrix} 0_{2n\times 2n} & -\tilde{K}^T \\ -\tilde{K} & -2 \end{bmatrix} \tag{29}$$

then, every trajectory z(t) tends to zero exponentially.

Proof: A proof is given by Johansson (1999).

In Theorem 4.1 $\bar{H}_1$, $\bar{H}_0$ and $\bar{H}_2$ are chosen to ensure continuity of the Lyapunov function while the matrices $\bar{F}$ are introduced to deal with the affine terms $b^{(p)}$ in (9) and (10). The relaxation terms u_iQ_i are selected to avoid conservatism by exploiting the fact that each affine dynamics is used in the limited regions Z_0, Z_1 and Z_2 of the state space, while the largest possible λ_0, λ_1 and λ_2 deliver a convergence rate measure.

Choosing Qi according to (29) is intuitive and takes advantage of the state space partitioning as follows. In region $\mathcal{Z}_1$, $\tilde{K}z - 1 \geq 0$ when $z \in \mathcal{Z}_1$ and $\tilde{K}z - 1 \leq 0$ when $z \notin \mathcal{Z}_1$. Since,

$$\bar{z}^T Q_1 \bar{z} = 2(\tilde{K}z - 1), \ \bar{z}^T (u_1 \ Q_1) \bar{z} \geq 0 \text{ when}$$
$$z \in \mathcal{Z}_1 \text{ and } \bar{z}^T (u_1 \ Q_1) \bar{z} \leq 0 \text{ when } z \notin \mathcal{Z}_1$$

Therefore, it might be easier to find a P_1 satisfying the first two LMIs in (27) for some $u_i > 0$ and $v_i > 0$.

Remark 6: In Johansson (1999) an algorithm to generate relaxation terms is given which suggest that the terms u_iQ_i and v_iQ_i above are replaced by terms of the form $\bar{E}_i^T U_i \bar{E}_i$ and $\bar{E}_i^T W_i \bar{E}_i$ where U_i and W_i are symmetric matrices with non-negative entries and the matrices $\bar{E}_i$ are given by the algorithm. The choices (29) for Q_i used here are equivalent to using search variables U_i and W_i in Theorem 4.1 in Johansson (1999) according to

$$U_i = \begin{bmatrix} 0 & u_i \\ u_1 & 0 \end{bmatrix}, \ W_i = \begin{bmatrix} 0 & \upsilon_i \\ \upsilon_i & 0 \end{bmatrix} \tag{30}$$

where, $u_1 > 0$, $v_1 > 0$. This simple choice of relaxation terms adds only one extra variable for each LMI and proved to be powerful enough when solving the LMIs in Theorem 4.1.

NUMERICAL EXAMPLE

The attention is now turned to a 3rd order dynamic system borrowed from Hippe and Wurmthaler (1999) and described by the transfer function

$$G(s) = \frac{2}{s^3 + 2s^2 + 2s + 2} \tag{31}$$

To avoid numerical problems a balanced state space realisation is used in standard LQ synthesis (Glad and Ljung, 2000) to calculate the optimal state feedback and Kalman filter gains K and L by minimisation of the following cost function

$$J = \lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} (x^T C^T QCx + u^T Ru)\, dt \qquad (32)$$

Using Q = 100000, R = 1 and assuming process noise w and measurement noise n to be white noise processes with spectral densities $\Phi_w = 10^2 \cdot I$ and $\Phi_n = 1$ gives

$$\begin{aligned} K &= [-214.7 \quad 196.5 \quad 92.01] \\ L &= [-12.04 \quad 5.868 \quad 0.419]^T \end{aligned} \qquad (33)$$

Maximum output of the actuator is set to one, i.e., $\alpha = 1$ in (1).

Applied control signal for estimation: From Corollary 3, a periodic solution is known to exist if there are duty times τ_1 and τ_2 satisfying (20). This condition has been investigated using unconstrained nonlinear minimisation of the form

$$\min_{\tau_1, \tau_2, x_0} \left\| Mx(\tau_1, \tau_2)X_0 - N_x(\tau_2) \right\|_2^2 \qquad (34)$$

Using a separable nonlinear least-squares method (Appendix), a minimum close to zero was found for

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \approx \begin{bmatrix} 0.0059 \\ 1.7987 \end{bmatrix},\; X_0 \approx \begin{bmatrix} 0.8332 \\ 0.9547 \\ -0.0836 \end{bmatrix} \qquad (35)$$

To investigate the stability of this periodic solution, the eigenvalues of the matrix $D^x_{T/2}$ given by (21) have been computed and the result is

$$[\lambda_1 \quad \lambda_2 \quad \lambda_3] = [-1.000 \quad -0.748 \quad 0.033] \qquad (36)$$

Hence, according to Corollary 3, the periodic solution given by (35), corresponds to a limit cycle. Figure 3 shows the states of the plant corresponding to a simulation of the closed-loop system with both plant and observer states initiated to $0.5X_0$ (i.e., $\epsilon(0) = 0$).

Computed control signal for estimation: In this case, limit cycles could in fact be excluded using exponential stability analysis according to Theorem 4.1. The LMIs in Theorem 4.1 are found to be feasible and with manual tuning, it was possible to achieve the maximum values 0.2, 1.2 and 0.2 for λ_i; i = 0, 1, 2. Without relaxation terms, the LMIs (27) are feasible but not strictly feasible.

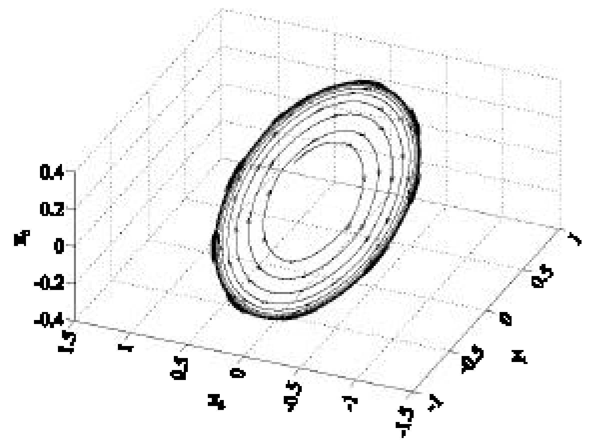

Fig. 3: The states of the plant corresponding to a simulation where both observer and plant states are initiated to $X_0/2$ with X_0 given by (35)

CONCLUSION

Existence of saturation-induced limit cycles is studied for 2 principally different controller implementations of observer-based state feedback controllers, one of which is typical to state-of-the-art anti-windup controllers. Sufficient conditions for exponential stability, as well as necessary and sufficient conditions for limit cycles using an exact representation of the non-linearity are provided for both implementations. It is proven that existence of limit cycles in the case of the actual (limited) control signal used for state estimation (anti-windup controller) is immaterial to the observer design and solely depends on plant properties and the feedback gain matrix. A numerical example demonstrates that while limit cycles exist in case of an implementation using for state estimation the control signal applied to the plant (anti-windup controller), global stability could be proven for the alternative implementation utilising the computed control signal. Finally, the condition for limit cycles is shown to involve the yet unsolved problem of spectral radius characterization of the product of 2 matrix exponentials.

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