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Velocity Propagation of Waves at the Interface of Two Liquids for the Case of Large Wave Length

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Abstract: The surface waves occur at and near the free surface of an unbounded sheet of liquid which is irrotational incompressible and two-dimensional. To illustrate the waves at the interface of 2 liquids, the progressive waves and stationary waves on the surface of a canal and on a deep canal are also considered. For the convenience of our calculation, Bernoulli's equation is taken and then at steady state the velocity propagation is obtained. In fact, we have generalized velocity propagation when wave length is large compared with thickness between 2 layers.

Key words: Wave velocity, wave length, velocity propagation, bernoulli's equation, Schrodinger method

INTRODUCTION

Raisinghania (2000) discussed the surface waves occur at and near the free surface of an unbounded sheet of liquid where the depth is considerable compared to the wave length. For these wave the vertical acceleration is comparable with the horizontal and vertical directions (Coulson, 1968). We have discussed surface waves in different cases. A systematic perturbation approach, called the nonlinear Schrodinger method, which was reported by Trulsen *et al.* (2001) for second-order nonlinear waves on deep water and by Trulsen *et al.* (2001) for second-order nonlinear waves on finite depth. The kinematics of extreme waves in deep water was recently analyzed by Grue *et al.* (2003). They discovered that after proper normalization, the kinematics profiles under a variety of extreme wave crests fitted surprisingly well with a universal exponential profile e^{kz} where z is the vertical coordinate and k is a local wave number. The key to the successful collapse of data demonstrated by Grue *et al.* (2003) was to use the local trough-to-trough "wave period" as basis for normalization at the crest to be considered. Raisinghania (2000) also generalized the velocity of propagation when depths of the liquids is large compared with the wave length. Here we have generalized the velocity of propagation when wave length is large compared with thickness between two layers. We have also derived the equation of surface waves and surface waves in different cases.

EQUATION OF SURFACE WAVES

Let the x-axis be taken in the undisturbed surface in the direction of propagation of the waves and the y-axis vertically upwards. Taking the motion to be irrotational, incompressible and 2 dimensional as Clamond *et al.* (2003), the velocity potential φ exists such that throughout the liquid

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, \quad (1)$$

and at a fixed boundary $\partial\varphi/\partial x = 0$.

The pressure can be obtained from the Bernoull's equation (Jensen *et al*., 2004),

$$\frac{p}{\rho} = \frac{\partial\varphi}{\partial t} - gy - \frac{1}{2}q^2 + F(t) \quad (2)$$

Since, for free surface is a surface of pressure, p is constant, hence for the free surface

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\, v\frac{\partial p}{\partial y} = 0$$

Where, u and v are the velocity components on the free surface in x and y directions, respectively.

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But,

$$u = -\frac{\partial\varphi}{\partial x},\ v = -\frac{\partial\varphi}{\partial y}.$$

and at the free surface the relation 4 becomes

$$\frac{\partial p}{\partial t} - \frac{\partial\varphi}{\partial x}.\frac{\partial p}{\partial x} - \frac{\partial\varphi}{\partial y}.\frac{\partial p}{\partial y} = 0 \quad (3)$$

Let the motion be so small that the squares of small quantities may be omitted. Again without loss of generality we may include F (t) in φ and hence we may take F(t) = 0 in (2). Then (2) reduces

$$\frac{p}{\rho} = \frac{\partial\varphi}{\partial t} - gy. \quad (4)$$

Substituting the value of p from (4) in (3) we get

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial\varphi}{\partial x}.\frac{\partial^2\varphi}{\partial x\partial t} - \frac{\partial\varphi}{\partial y}\left(\frac{\partial^2\varphi}{\partial y\partial t} - g\right) = 0.$$

or, omitting the second and their terms which are of the same order as q^2, we get

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial y} = 0. \quad (5)$$

Condition (1) must be satisfied at the free surface.

If η the elevation of the free surface at time t above the point whose abscissa is x, the equation of the free surface is given by

$$\eta = f(x,t)$$
$$\text{or, } \eta - f(x,t) = 0.$$

But we know that if F(x, η, t) = η - f (x, t) = 0 be the boundary surface, then we must have

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial \eta} = 0.$$

i.e.,
$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v = 0 \quad (6)$$

Now ∂f/∂t is $\dot{\eta}$. Again ∂f/∂x or ∂η/∂x being the tangent of the slope of the free surface is small so that the second term in (1) can be omitted. Then (6) reduces to

$$\dot{\eta} = v = -\partial\varphi/\partial y. \quad (7)$$

Which holds at free surface.

Thus for the surface waves the velocity potential is a solution of Laplace's Eq. (1) which makes ∂φ/∂x = 0 as a fixed boundary and satisfies Eq. (5) and (7) at the free surface of the liquid.

Surface waves in different cases: To illustrate the above theory, we consider the following cases

Case 1: Progressive waves on the surface of a canal. Consider the propagation of simple harmonic waves of the type

$$\eta = a\sin(mx - nt), \quad (8)$$

at the surface of canal of uniform depth h and having parallel vertical wall. Let the free surface be along the x-axis (i.e., y = 0), so that equation of the bottom is y = - h. Then we must find φ satisfying Eq. (8) and subjected to the following boundary conditions

$$\frac{\partial\varphi}{\partial y} = 0 \text{ at } y = -h,$$

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial y} = 0 \text{ at } y = 0, \quad (9)$$

$$v = \frac{\partial\eta}{\partial t} = -\frac{\partial\varphi}{\partial y} \text{ at } y = 0 \quad (10)$$

Using Eq. (8) and (10) gives ∂φ/∂y = an cos(mx-nt) at y =0. We have,

$$\varphi = D\cosh\{m(y+h)\}\cos(mx - nt). \quad (11)$$

Again, using (9) and (11) gives

$$n^2 = gm\tanh(mh) \quad (12)$$

Let the velocity of propagation, c = n/m and wave length, λ = 2π/m.

Then (12) reduces to $c^2 = g/m\tanh(mh)$ and $c^2 = g\lambda/2\pi\tanh(2\pi h/2\lambda)$.

Case 2: Progressive waves on a deep canal: If the depth h of the canal is sufficiently great in comparison with λ for e^{-mh} to be neglected, then in case 1 we must have B = 0. Thus,

$$\varphi = Ae^{mh}\cos(mx - nt). \quad (13)$$

Taking $n^2 = gm$ and $c^2 = g\lambda/2\pi$.

We now determine the constant A of Eq. (13) in terns of the amplitude a of the wave. Using Eq. (8) and (13), the boundary condition Eq. (10) gives na = mA, so that

$$\varphi = (na/m)\, e^{my} \cos(mx - nt) \text{ and}$$
$$\varphi = (ga/m)\, e^{my} \cos(mx - nt)$$

The velocity components of the particles are:

$$u = -\partial\varphi/\partial x = nae^{my} \sin(mx - nt) \text{ and}$$
$$v = -\partial\varphi/\partial y = -\, nae^{my} \cos(mx - nt).$$

Following the procedure of case 1, we obtain in the case for the displacement (x', y') of a particle form its mean position (x, y) $x' = ae^{my} \cos(mx - nt)$, $y' = ae^{my} \sin(mx - nt)$ and hence the path of the particle is a circle, $x^2 + y^2 = (ae^{my})^2$ of radius (ae^{my}), which decreases with depth of a particle under consideration.

Case 3: Stationary waves on the surface of canal: Consider a stationary wave of the type

$$\eta = a \sin(mx) \cos(nt), \qquad (14)$$

at the surface of canal of uniform depth h and having parallel vertical walls. Let the free surface be along the x-axes (i.e. y = 0), so that the equation of the bottom (rigid boundary) is y = -h. The we must find φ satisfying Eq. (1) and subjected to the following boundary conditions

$$\frac{\partial\varphi}{\partial y} = 0 \text{ at } y = -h \qquad (15)$$

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial y} = 0, \text{ at } y = 0$$

$$v = \frac{\partial\eta}{\partial t} = -\frac{\partial\varphi}{\partial y}, \text{ at } y = 0. \qquad (16)$$

using Eq. (14) and (16) gives $\partial\varphi/\partial y = an \sin(mx) \sin(nt)$ at y = 0. Also we have,

$$\varphi = D \cosh\{m(y+h)\} \sin(mx)\sin(nt), \qquad (17)$$

where D is a constant.

Again, using Eq. (15) and (17) gives

$$n^2 = gm \tanh(mh) \qquad (18)$$

Let

$$c = n/m \text{ and } \lambda = 2\pi/m.$$

denote the velocity of propagation and the wave length, respectively. Then (18) reduces to:

$$c^2 = \frac{g}{m} \tanh(mh).$$
$$c^2 = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right)$$

Case 4: Stationary waves on a deep canal: If the depth h of the canal is sufficiently great in comparison with λ for e^{-mh} to be neglected, then in case 3 we must have B = 0. Thus, we have,

$$\varphi = Ae^{my} \sin(mx) \sin(nx). \qquad (19)$$

Taking

$$n^2 = gm \text{ and } C^2 = g\lambda/2\pi.$$

Here we also determine the constant A of Eq. (19) in terms of the amplitude of the wave, using Eq. (14) and (19), the boundary condition Eq. (16) gives na = mA, so that

$$\varphi = \frac{na}{m} e^{my} \sin(mx) \frac{\sin(t)}{n} \sin(t) \text{ and}$$
$$\varphi = \frac{ga}{n} e^{-my} \sin(mx) \sin(nt).$$

The velocity components of the particles one

$$u = \frac{\partial\varphi}{\partial x} = -nae^{my} \cos(mx) \sin(nt) \text{ and}$$
$$v = \frac{\partial\varphi}{\partial y} - = -nae^{my} \sin(mx) \sin(nt).$$

Following the preceding procedure we obtain in this case

$$x' = ae^{my} \cos(mx) \cos(nt),\ y' = x' = ae^{my} \sin(mx) \cos(nt).$$

Hence, $y'/x' = \tan(mx)$ shows that the path of the particle is a straight line.

Waves at the interface of two liquids: Let a layer of fluid density ρ_3 and thickness h separates two fluids of densities ρ_1 and ρ_2 extending to infinity in opposite directions. Let c be the velocity of propagation of oscillatory waves at the interface of two liquids in the direction in which the liquids are moving.

We make the motion steady by superimposing on the whole mass the velocity-c. Thus the wave profile is

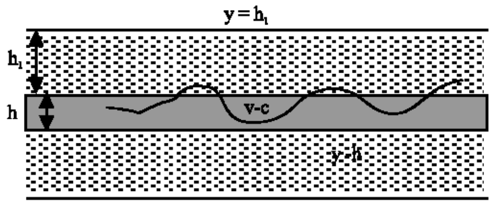

Fig. 1: Waves at the interface of the liquids

reduced to rest in space and the new velocities of liquids becomes v_1-c and v-c as shown in Fig. 1.

The velocity and stream function for the lower liquid moving with – (v-c) in the negative direction of x-axis are given by:

$$\varphi = -(v-c)\,x + D\cos h\{m(y+h)\}\,\cos(mx), \qquad (20)$$
$$\Psi = -(v-c)\,y - D\sin h\{m(y+h)\}\,\sin(mx). \qquad (21)$$

Similar, expression for the upper liquid may be deduced from Eq. (20) and (21) by replacing v by v_1 and h by $-h_1$. Thus, we get

$$\varphi_1 = -(v_1-c)\,x + D_1\cosh\{m(y-h_1)\}\cos(mx) \text{ and}$$
$$\Psi_1 = -(v_1-c)y - D_1\sin(hm)(y-h_1)\sin(mx).$$

Clearly the above expressions for Ψ and Ψ_1 make the boundaries y = -h, y = h_1

let

$$\eta = a\sin(mx). \qquad (22)$$

represent the displacement of the interface. If the liquids do not separate then (22) must be a streamline for both surfaces. This condition is satisfied by assuming the streamline to be $\Psi = \Psi_1 = 0$. neglecting the squares of small quantities (e.g., a^2), we thus obtain

$$-(v-c)\,a - D\sin(mh) = 0 \text{ and}$$
$$-(v_1-c)\,a + D^l\sin(mh_l) = 0.$$

From Bernoulli's equations, we obtain

$$\frac{p_3}{\rho_3} = gy + \frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\} = \text{constant}, \qquad (23)$$

$$\text{and } \frac{p_1}{\rho_1} = gy + \frac{1}{2}\left\{\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \left(\frac{\partial\varphi_1}{\partial y}\right)^2\right\} = \text{constant}. \qquad (24)$$

But at the interface y = η = a sin(mx), Hence neglecting a^2, Eq. (54) and (55) give

$$\frac{p_3}{\rho_3} + ga\sin(max) + \frac{1}{2}(v-c)^2$$
$$(1-2am\coth(mh)\sin(mx)) = \text{constant}$$

$$\frac{p_1}{\rho_1} + ga\sin(max) + \frac{1}{2}(v_1-c)^2$$
$$(1+2am\coth(mh_1)\sin(mx)) = \text{constant}$$

Since, the pressure is continuous across the interface, putting p = p_3 in above equations, subtracting and then equating to zero the coefficient of sin(mx), we obtain

$$g(\rho_3 - \rho_1) = (v-c)^2 m\rho_3\coth(mh) + (v_1-c)^2 m\rho_1\coth(mh_1) \qquad (25)$$

Equation (25) determines the velocity of propagation c of waves of wave length $2\pi/m$ at the interface. We; can also treat Eq. (25) as the condition for stationary waves at the interface of two streams.

Whose velocities are v-c and v_1-c. When the liquids are at rest (i.e. v = v_1 = 0), the wave velocity is given by

$$c^2 = \frac{g}{m}\,\frac{\rho_3-\rho_1}{\rho_3\coth(mh) + \rho_1\coth(mh)}$$

shows that when $\rho_1 > \rho_3$ the equilibrium position is unstable.

Let the liquids be at rest and the waves of length λ, large compared with h, that we may take coth(mh) = coth(mh_1) = 1, then we have

$$c^2 = \frac{g}{m}\,\frac{\rho_3-\rho_1}{\rho_3+\rho_1}$$

the finally,

$$c^2 = \frac{g\lambda}{2\pi}\left[\frac{(\rho_3-\rho_1)}{(\rho_3+\rho_1)}\right]$$

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