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Hybrid Synchronization of Hyperchaotic Lorenz and Hyperchaotic Chen Systems via Active Control

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Abstract: This study investigates the hybrid synchronization of Hyperchaotic Lorenz and Chen Systems via adaptive control. Explicitly, researchers derive new results for the hybrid chaos synchronization of identical Hyperchaotic Lorenz Systems, identical Hyperchaotic Chen Systems and non-identical Hyperchaotic Lorenz and Chen Systems. In this study, Researchers assume that the parameters of both master and slave systems are unknown and we devise adaptive hybrid synchronization schemes using the estimates of parameters for both Master and Slave Systems. The adaptive hybrid synchronization results derived in this study are established using Lyapunov Stability Theory. Since, the Lyapunov exponents are not required for these calculations, the proposed Adaptive Control Method is very effective and convenient to achieve hybrid synchronization of identical and non-identical Hyperchaotic Lorenz and Chen Systems. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive hybrid synchronization schemes for the hyperchaotic systems addressed in this study.

Key words: Adaptive control, hyperchaos, hybrid synchronization, Hyperchaotic Lorenz System, Hyperchaotic Chen System, India

INTRODUCTION

Chaotic Systems are non-linear dynamical systems that are highly sensitive to initial conditions. This sensitivity is popularly known as the butterfly effect (Alligood *et al*., 1997). The chaos phenomenon was first observed in weather models by Lorenz (1963). Hyperchaotic System is usually defined as a Chaotic System having more than one positive Lyapunov exponent. The hyperchaos phenomenon was first observed by Rossler (1979). A Hyperchaotic System has the characteristics of high capacity, high security and high efficiency. Thus, it has the potential of broad applications in non-linear circuits, secure communications, lasers, neural networks, biological systems and so on.

Synchronization of Chaotic Systems is a phenomenon that may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect which causes the exponential divergence of the trajectories of two identical Chaotic Systems started with nearly the same initial conditions, synchronizing two Chaotic Systems is seemingly a very challenging research problem. In most of the chaos synchronization approaches, the master-slave or drive-response formalism is used. If a particular Chaotic System is called a master or drive system and another Chaotic System is called a slave or response system then the idea of chaos synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Chaos is an interesting non-linear phenomenon and it has been intensively and extensively studied in the last three decades. Chaos Theory has wide applications in several fields such as Physical Systems (Lakshmanan and Murali, 1996), Chemical Systems (Han *et al*., 1995), Ecological Systems (Blasius *et al*., 1999) and secure communications, etc. (Cuomo *et al*., 1993; Kocarev and Parlitz, 1995).

In most of the chaos synchronization approaches, the master-slave or drive-response formalism is used. If a particular Chaotic System is called the master or drive system and another Chaotic System is called the slave or response system then the idea of synchronization is to

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use the output of the master system to control the slave system so that the states of the slave system track the states of the master system asymptotically. Since, the seminal research by Pecora and Carroll (1990) on complete synchronization of Chaotic Systems, there has been significant interest paid in the chaos literature on the synchronization of Chaotic and Hyperchaotic Systems (Pecora and Carroll, 1990; Ott *et al*., 1990; Yang and Chua, 1999; Park and Kwon, 2003; Hua and Guan, 2003; Ho and Hung, 2002; Yassen, 2003, 2005; Tian *et al*., 2007; Sundarapandian, 2011a-f; Chen and Lu, 2002; Jia and Tang, 2009; Mascolo and Grassi, 1999; Tan *et al*., 2003; Zhang *et al*., 2005; Utkin, 1977; Slotine and Sastry, 1983; Sundarapandian and Sivaperumal, 2011a, b; Ge and Chen, 2004; Wang and Guan, 2006; Zhang and Zhu, 2008; Chiang *et al*., 2008; Jia, 2007; Yan and Li, 2006; Li *et al*., 2007; Sarasu and Sundarapandian, 2011).

In the last 2 decades, several approaches have been developed for chaos synchronization such as the OGY Method (Ott *et al*., 1990), Sampled-data Feedback Synchronization Method (Yang and Chua, 1999), the Time-delay Feedback Method (Park and Kwon, 2003; Hua and Guan, 2003), the Active Control Method (Ho and Hung, 2002; Yassen, 2005; Tian *et al*., 2007; Sundarapandian, 2011a, b), the Adaptive Control Method (Yassen, 2003; Chen and Lu, 2002; Jia and Tang, 2009; Sundarapandian, 2011c, d), the Backstepping Method (Mascolo and Grassi, 1999; Tan *et al*., 2003; Zhang *et al*., 2005), the Sliding Mode Control Method (Utkin, 1977; Slotine and Sastry, 1983; Sundarapandian and Sivaperumal, 2011a, b) and others.

So far, various synchronization methods have been developed such as the complete synchronization (Pecora and Carroll, 1990), the phase synchronization (Ge and Chen, 2004), the generalized synchronization (Wang and Guan, 2006), the anti-synchronization (Zhang and Zhu, 2008; Chiang *et al*., 2008; Sundarapandian, 2011g; Sundarapandian and Sivaperumal, 2011b), the projective synchronization (Jia, 2007) and the generalized projective synchronization (Yan and Li, 2006; Li *et al*., 2007; Sarasu and Sundarapandian, 2011). Complete Synchronization (CS) is characterized by the equality of state variables evolving in time while the Anti-Synchronization (AS) is characterized by the disappearance of the sum of relevant state variables evolving in time.

Projective Synchronization (PS) is characterized by the fact the master and slave systems could be synchronized up to a scaling factor whereas in Generalized Projective Synchronization (GPS), the responses of the synchronized dynamical states synchronize up to a constant scaling matrix α. It is easy to see that the complete synchronization and the anti-synchronization are the special cases of the generalized projective synchronization where the scaling matrix $\alpha = 1$ and $\alpha = -1$, respectively.

In hybrid synchronization of Chaotic Systems (Sundarapandian, 2011h), one part of the systems (for instance, the odd-numbered states) is completely synchronized while the other part (for instance, the even numbered states) is anti-synchronized so that Complete Synchronization (CS) and Anti-Synchronization (AS) co-exist in the Master-Slave Chaotic Systems. The co-existence of CS and AS is very useful in secure communication and chaotic encryption schemes.

In this study, researchers deploy the Adaptive Control Method to derive new results for the hybrid synchronization of identical Hyperchaotic Lorenz Systems (Gao *et al*., 2007), identical Hyperchaotic Chen Systems (Li-Xin *et al*., 2010) and non-identical Hyperchaotic Lorenz and Chen Systems. The adaptive synchronization results for the hybrid synchronization of Hyperchaotic Lorenz and Chen Systems are established by the Lyapunov Stability Theory (Hahn, 1967).

Systems description: The Hyperchaotic Lorenz System is a new hyperchaotic system derived from the Lorenz System by Gao *et al*. (2007). The Hyperchaotic Lorenz System is described by the 4D dynamics:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_4 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ \dot{x}_4 &= r x_2 x_3 \end{aligned} \tag{1}$$

Where x_i ($i = 1, 2, 3, 4$) are the states and σ, β, ρ, r are positive, constant parameters of the system. The system (1) is hyperchaotic when the parameter values are chosen as:

$$\sigma = 10,\ \beta = 8/3,\ \rho = 28 \text{ and } r = 0.1$$

Figure 1 shows the phase portrait of the Hyperchaotic Lorenz System (1). The Hyperchaotic Chen System is a new hyperchaotic system derived from the Chen System by Li-Xin *et al*. (2010). The Hyperchaotic Chen System is described by the 4D dynamics:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= 4x_1 - 10x_1 x_3 + c x_2 + 4x_4 \\ \dot{x}_3 &= x_2^2 - b x_3 \\ \dot{x}_4 &= -d x_1 \end{aligned} \tag{2}$$

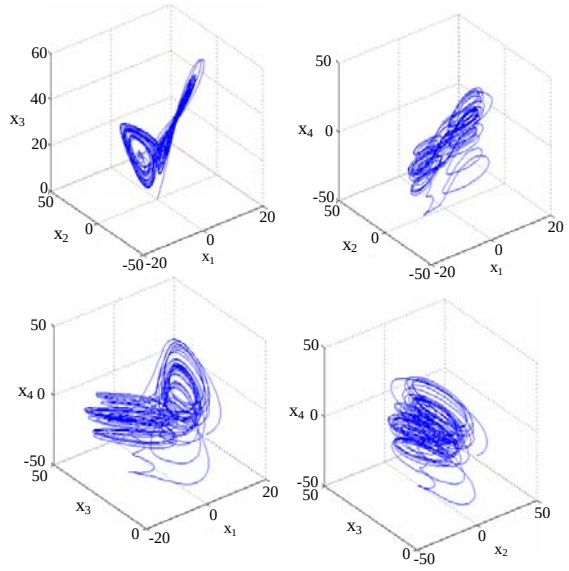

Fig. 1: Phase portrait of the Hyperchaotic Lorenz System

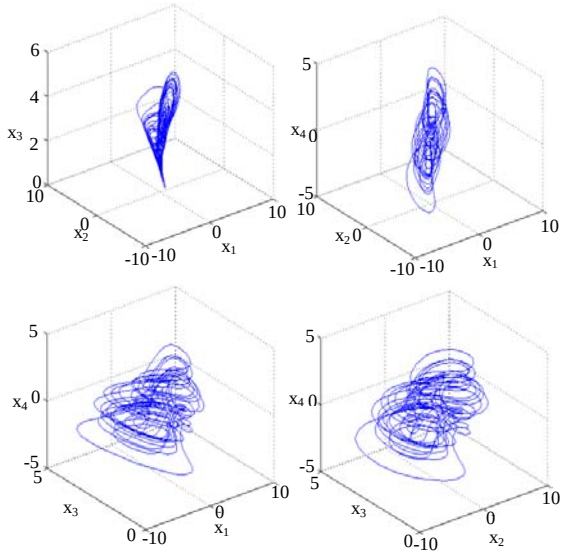

Fig. 2: Phase portrait of the Hyperchaotic Chen System

Where, x_i (i = 1, 2, 3, 4) are the states and a-d are positive, constant parameters of the system. The system (2) is hyperchaotic when the parameter values are taken as:

a = 35, b = 3, c = 21 and d = 2

Figure 2 shows the phase portrait of the Hyperchaotic Chen System (2).

ADAPTIVE HYBRID SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC LORENZ SYSTEMS

Theoretical results: In this research, researchers deploy the Adaptive Control Method to derive new results for the hybrid synchronization of identical Hyperchaotic Lorenz Systems (Gao *et al*., 2007). Thus, the master system is described by the Hyperchaotic Lorenz dynamics:

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1)\\ \dot{x}_2 &= \rho x_1 - x_2 - x_4 - x_1x_3\\ \dot{x}_3 &= x_1x_2 - \beta x_3\\ \dot{x}_4 &= rx_2x_3\end{aligned} \tag{3}$$

Where x_1-x_4 are the states and α, β, ρ, r are unknown parameters of the system. The slave system is described by the controlled Hyperchaotic Lorenz dynamics:

$$\begin{aligned}\dot{y}_1 &= \sigma(y_2 - y_1) + u_1\\ \dot{y}_2 &= \rho y_1 - y_2 - y_4 - y_1y_3 + u_2\\ \dot{y}_3 &= y_1y_2 - \beta y_3 + u_3\\ \dot{y}_4 &= ry_2y_3 + u_4\end{aligned} \tag{4}$$

Where y_1-y_4 are the states and u_1-u_4 are the adaptive controllers to be designed. The hybrid synchronization error is defined as:

$$\begin{aligned}e_1 &= y_1 - x_1, \quad e_2 = y_2 + x_2\\ e_3 &= y_3 - x_3, \quad e_4 = y_4 + x_4\end{aligned} \tag{5}$$

A simple calculation gives the error dynamics as:

$$\begin{aligned}\dot{e}_1 &= \sigma(e_2 - e_1 - 2x_2) + u_1\\ \dot{e}_2 &= \rho(e_1 + 2x_1) - e_2 - e_4 -\\ &\quad y_1y_3 - x_1x_3 + u_2\\ \dot{e}_3 &= -\beta e_3 + y_1y_2 - x_1x_2 + u_3\\ \dot{e}_4 &= r(y_2y_3 + x_2x_3) + u_4\end{aligned} \tag{6}$$

Let us now define the adaptive functions $u_1(t)$-$u_4(t)$ as:

$$\begin{aligned}u_1(t) &= -\hat{\sigma}(e_2 - e_1 - 2x_2) - k_1e_1\\ u_2(t) &= -\hat{\rho}(e_1 + 2x_1) + e_2 + e_4 +\\ &\quad y_1y_3 + x_1x_3 - k_2e_2\\ u_3(t) &= \hat{\beta}e_3 - y_1y_2 + x_1x_2 - k_3e_3\\ u_4(t) &= -\hat{r}(y_2y_3 + x_2x_3) - k_4e_4\end{aligned} \tag{7}$$

Where $\hat{\sigma}, \hat{\beta}, \hat{\rho}$ and $\hat{r}$ are estimates of σ, β, ρ and r, respectively and k_i (i = 1, 2, 3, 4) are positive constants. Substituting Eq. 7 into Eq. 6, the error dynamics simplifies to:

$$\begin{aligned}\dot{e}_1 &= (\sigma - \hat{\sigma})(e_2 - e_1 - 2x\ ) - k_1e_1\\ \dot{e}_2 &= (\rho - \hat{\rho})(e_1 + 2x_1) - k_2e_2\\ \dot{e}_3 &= -(\beta - \hat{\beta})e_3 - k_3e_3\\ \dot{e}_4 &= (r - \hat{r})(y_2y_3 + x_2x_3) - k_4e_4\end{aligned} \tag{8}$$

Let us now define the parameter estimation error as:

$$\begin{aligned} e_\sigma &= \sigma - \hat{\sigma}, \quad e_\beta = \beta - \hat{\beta} \\ e_\rho &= \rho - \hat{\rho}, \quad e_r = r - \hat{r} \end{aligned} \tag{9}$$

Substituting Eq. 9 into Eq. 8, researchers obtain the error dynamics as:

$$\begin{aligned} \dot{e}_1 &= e_\sigma(e_2 - e_1 - 2x_1) - k_1 e_1 \\ \dot{e}_2 &= e_\rho(e_1 + 2x_1) - k_2 e_2 \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \\ \dot{e}_4 &= e_r(y_2 y_3 + x_2 x_3) - k_4 e_4 \end{aligned} \tag{10}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used. Researchers consider the quadratic Lyapunov function defined by:

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\sigma^2 + e_\beta^2 + e_\rho^2 + e_r^2\right) \tag{11}$$

Which is a positive definite function on R^8. Researchers also note that:

$$\dot{e}_\sigma = -\dot{\hat{\sigma}}, \quad \dot{e}_\beta = -\dot{\hat{\beta}}, \quad \dot{e}_\rho = -\dot{\hat{\rho}}, \quad \dot{e}_r = -\dot{\hat{r}} \tag{12}$$

Differentiating Eq. 11 along the trajectories of Eq. 10 and noting Eq. 12, researchers find that:

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + \\ &e_\sigma\left[e_1(e_2 - e_1 - 2x_2) - \dot{\hat{\sigma}}\right] + \\ &e_\beta\left[-e_3^2 - \dot{\hat{\beta}}\right] + e_\rho\left[e_2(e_1 + 2x_1) - \dot{\hat{\rho}}\right] + \\ &e_r\left[e_4(y_2 y_3 + x_2 x_3) - \dot{\hat{r}}\right] \end{aligned} \tag{13}$$

In view of Eq. 13, the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{\sigma}} &= e_1(e_2 - e_1 - 2x_2) + k_5 e_\sigma \\ \dot{\hat{\beta}} &= -e_3^2 + k_6 e_\beta \\ \dot{\hat{\rho}} &= e_2(e_1 + 2x_1) + k_7 e_\rho \\ \dot{\hat{r}} &= e_4(y_2 y_3 + x_2 x_3) + k_8 e_r \end{aligned} \tag{14}$$

Where, k_5-k_8 are positive constants.

Theorem 1: The identical Hyperchaotic Lorenz Systems (3) and (4) with unknown parameters are globally and exponentially hybrid-synchronized by the adaptive control law (Eq. 7) where the update law for the parameter estimates σ, β, ρ, r is given by Eq. 14 and k_i (i = 1, 2, ..., 8) are positive constants. The errors for parameter estimates e_σ, e_β, e_ρ, e_r decay to zero exponentially as $t \to \infty$.

Proof: This result is a simple consequence of the Lyapunov Stability Theory. Researchers know that V as defined in Eq. 11 is a positive definite function on R^8. Substituting Eq. 14 into Eq. 13, researchers obtain:

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - \\ &k_5 e_\sigma^2 - k_6 e_\rho^2 - k_7 e_\beta^2 - k_8 e_r^2 \end{aligned} \tag{15}$$

Which is a negative definite function on R^8. Hence, by the Lyapunov Stability Theory (Hahn, 1967), it follows that $e_i(t) \to 0$ as $t \to \infty$ for i = 1, 2, 3, 4 and $e_\sigma \to 0$, $e_\beta \to 0$, $e_\rho \to 0$, $e_r \to 0$ as $t \to \infty$, exponentially. This completes the proof.

Numerical results: For the numerical simulations, the 4th-order Runge-Kutta Method with time-step $h = 10^{-8}$ is used to solve the two systems of differential Eq. 3 and 4 with the adaptive non-linear controller (Eq. 7) and update law of estimates (Eq. 14). Researchers take $k_i = 4$ for i = 1, 2,..., 8. The parameters of the Hyperchaotic Lorenz Systems are chosen so that the systems (3) and (4) are hyperchaotic, i.e.,

$$\sigma = 10, \ \beta = 8/3, \ \rho = 28 \text{ and } r = 0.1$$

The initial values of the parameter estimates are chosen as:

$$\hat{\sigma}(0) = 5, \ \hat{\beta}(0) = 3, \ \hat{\rho}(0) = 10 \text{ and } \hat{r}(0) = 6$$

The initial values of the master system (3) are chosen as:

$$x_1(0) = 2, \ x_2(0) = -5, \ x_3(0) = 6, \ x_4(0) = -12$$

The initial values of the slave system (4) are chosen as:

$$y_1(0) = 7, \ y_2(0) = 5, \ y_3(0) = 6, \ y_4(0) = 1$$

Figure 3 shows the hybrid synchronization of the Hyperchaotic Lorenz Systems (3) and (4). Figure 4 shows the time-history of the hybrid synchronization errors. Figure 5 shows the time-history of the parameter estimates $\hat{\alpha}, \hat{\beta}, \hat{\rho}, \hat{r}$. Figure 6 shows the time-history of the parameter estimation errors e_σ, e_β, e_ρ, e_r.

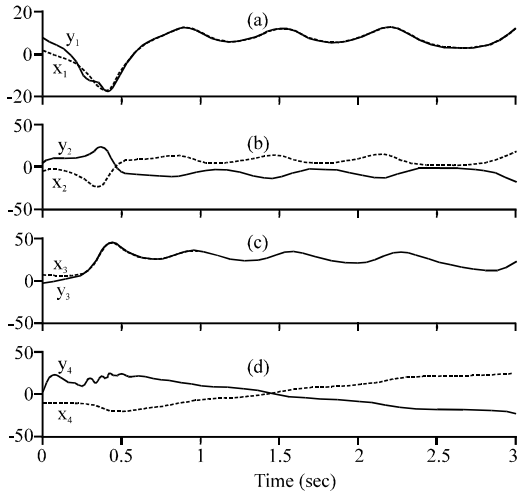

Fig. 3: a-d) Hybrid synchronization of the identical Hyperchaotic Lorenz Systems

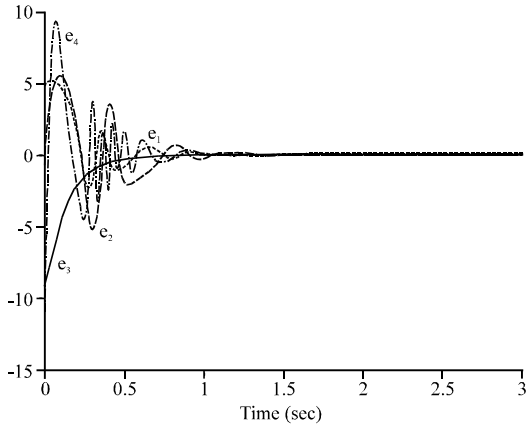

Fig. 4: Time history of the error states e_1-e_4

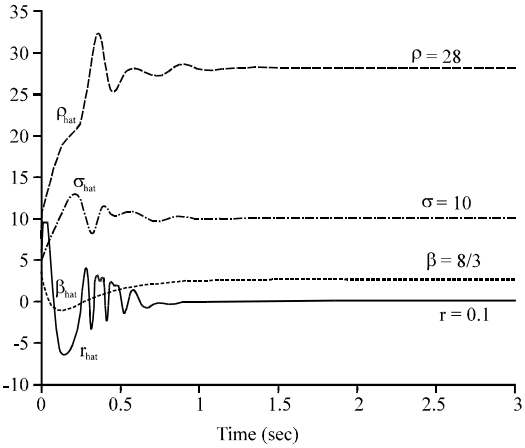

Fig. 5: Time history of the parameter estimates $\hat{\sigma}, \hat{\beta}, \hat{\rho}, \hat{r}$

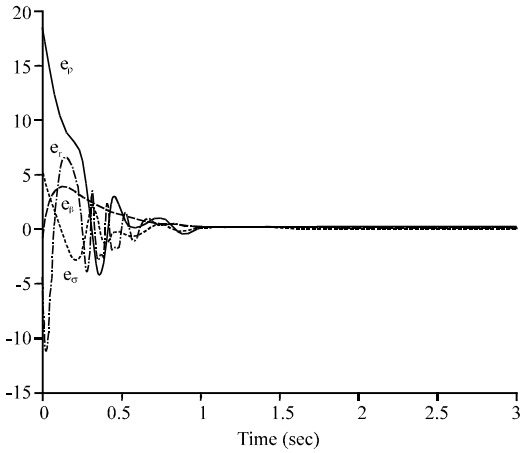

Fig. 6: Time history of the parameter estimation errors e_σ, e_β, e_ρ, e_r

ADAPTIVE HYBRID SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC CHEN SYSTEMS

Theoretical results: In this research, researchers deploy the Adaptive Control Method to derive new results for the hybrid synchronization of identical Hyperchaotic Chen System (Li-Xin *et al*., 2010). Thus, the master system is described by the Hyperchaotic Chen dynamics:

$$\dot{x}_1 = a(x_2 - x_1),\ \dot{x}_2 = 4x_1 - 10x_1x_3 + cx_2 + 4x_4$$
$$\dot{x}_3 = x_2^2 - bx_3,\quad \dot{x}_4 = -dx_1 \tag{16}$$

Where x_1-x_4 are the states and a-d are unknown parameters of the system. The slave system is described by the controlled hyperchaotic chen dynamics:

$$\dot{y}_1 = a(y_2 - y_1) + u_1,\ \dot{y}_2 = 4y_1 - 10y_1y_3 + cy_2 + 4y_4 + u_2$$
$$\dot{y}_3 = y_2^2 - by_3 + u_3,\quad \dot{y}_4 = -dy_1 + u_4 \tag{17}$$

Where y_1-y_4 are the states and u_1-u_4 are the adaptive controllers to be designed. The hybrid synchronization error is defined as:

$$e_1 = y_1 - x_1,\ e_2 = y_2 + x_2$$
$$e_3 = y_3 - x_3,\ e_4 = y_4 + x_4 \tag{18}$$

A simple calculation gives the error dynamics as:

$$\begin{aligned}
\dot{e}_1 &= a(e_2 - e_1 - 2x_2) + u_1\\
\dot{e}_2 &= 4(e_1 + 2x_1) + ce_2 + 4e_4 -\\
&\quad 10y_1y_3 - 10x_1x_3 + u_2\\
\dot{e}_3 &= -be_3 + y_2^2 - x_2^2 + u_3\\
\dot{e}_4 &= -d(e_1 + 2x_1) + u_4
\end{aligned} \tag{19}$$

Let us define the adaptive functions $u_1(t)$-$u_4(t)$ as:

$$\begin{aligned} u_1(t) &= -\hat{a}(e_2 - e_1 - 2x_2) - k_1 e_1 \\ u_2(t) &= -4(e_1 + 2x_1) - \hat{c}e_2 - 4e_4 + \\ &\quad 10y_1y_3 + 10x_1x_3 - k_2e_2 \\ u_3(t) &= \hat{b}e_3 - y_2^2 + x_2^2 - k_3e_3 \\ u_4(t) &= \hat{d}(e_1 + 2x_1) - k_4e_4 \end{aligned} \tag{20}$$

Where $\hat{a}-\hat{d}$ are estimates of a-d, respectively and k_i, (i = 1, 2, 3, 4) are positive constants. Substituting Eq. 20 into Eq. 19, the error dynamics simplifies to:

$$\begin{aligned} \dot{e}_1 &= (a-\hat{a})(e_2 - e_1 - 2x_2) - k_1e_1 \\ \dot{e}_2 &= (c-\hat{c})e_2 - k_2e_2 \\ \dot{e}_3 &= -(b-\hat{b})e_3 - k_3e_3 \\ \dot{e}_4 &= -(d-\hat{d})(e_1 + 2x_1) - k_4e_4 \end{aligned} \tag{21}$$

Let us now define the parameter estimation error as:

$$\begin{aligned} e_a &= a - \hat{a},\ e_b = b - \hat{b} \\ e_c &= c - \hat{c},\ e_d = d - \hat{d} \end{aligned} \tag{22}$$

Substituting Eq. 22 into Eq. 21, researchers obtain the error dynamics as:

$$\begin{aligned} \dot{e}_1 &= e_a(e_2 - e_1 - 2x_2) - k_1e_1 \\ \dot{e}_2 &= e_ce_2 - k_2e_2 \\ \dot{e}_3 &= -e_be_3 - k_3e_3 \\ \dot{e}_4 &= -e_d(e_1 + 2x_1) - k_4e_4 \end{aligned} \tag{23}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov Approach is used. Researchers consider the quadratic Lyapunov function defined by:

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right) \tag{24}$$

which is a positive definite function on R^8. Researchers also note that:

$$\begin{aligned} \dot{e}_a &= -\dot{\hat{a}},\ \dot{e}_b = -\dot{\hat{b}} \\ \dot{e}_c &= -\dot{\hat{c}},\ \dot{e}_d = -\dot{\hat{d}} \end{aligned} \tag{25}$$

Differentiating Eq. 24 along the trajectories of Eq. 23 and noting Eq. 25, researchers find that:

$$\begin{aligned} \dot{V} = &-k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 + \\ &e_a\left[e_1(e_2 - e_1 - 2x_2) - \dot{\hat{a}}\right] + \\ &e_b\left[-e_3^2 - \dot{\hat{b}}\right] + e_c\left[e_2^2 - \dot{\hat{c}}\right] + \\ &e_d\left[-e_4(e_1 + 2x_1) - \dot{\hat{d}}\right] \end{aligned} \tag{26}$$

In view of Eq. 26, the estimated parameters are updated by the following law:

$$\begin{aligned} &\dot{\hat{a}} = e_1(e_2 - e_1 - 2x_2) + k_5e_a,\ \dot{\hat{b}} = -e_3^2 + k_6e_b \\ &\dot{\hat{c}} = e_2^2 + k_7e_c,\ \dot{\hat{d}} = -e_4(e_1 + 2x_1) + k_8e_d \end{aligned} \tag{27}$$

Where k_5-k_8 are positive constants.

Theorem 2: The identical Hyperchaotic Chen Systems (16) and (17) with unknown parameters are globally and exponentially hybrid-synchronized by the adaptive control law (20) where the update law for the parameter estimates a-d is given by Eq. 27 and k_i (i = 1, 2,..., 8) are positive constants. The errors for parameter estimates e_a-e_d decay to zero exponentially as $t \to \infty$.

Proof: This result is a simple consequence of the Lyapunov Stability Theory. Researchers know that V as defined in Eq. 24 is a positive definite function on R^8. Substituting Eq. 27 into Eq. 26, researchers obtain:

$$\begin{aligned} \dot{V} = &-k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - \\ &k_5e_\sigma^2 - k_6e_\rho^2 - k_7e_\beta^2 - k_8e_r^2 \end{aligned} \tag{28}$$

which is a negative definite function on R^8. Hence by the Lyapunov Stability Theory (Hahn, 1967), it follows that $e_i(t) \to 0$ as $T \to \infty$ for i = 1, 2, 3, 4 and $e_a \to 0$, $e_b \to 0$, $e_c \to 0$, $e_d \to 0$ as $t \to \infty$, exponentially. This completes the proof.

Numerical results: For the numerical simulations, the 4th-order Runge-Kutta Method with time-step $h = 10^{-8}$ is used to solve the two systems of differential Eq. 16 and 17 with the adaptive nonlinear controller (Eq. 20) and update law of estimates (Eq. 27).

Researchers take $k_i = 4$ for i = 1, 2,..., 8. The parameters of the Hyperchaotic Chen Systems are chosen so that the systems (16) and (17) are hyperchaotic, i.e.,

$$a = 35,\ b = 3,\ c = 21,\ d = 2$$

The initial values of the parameter estimates are chosen as:

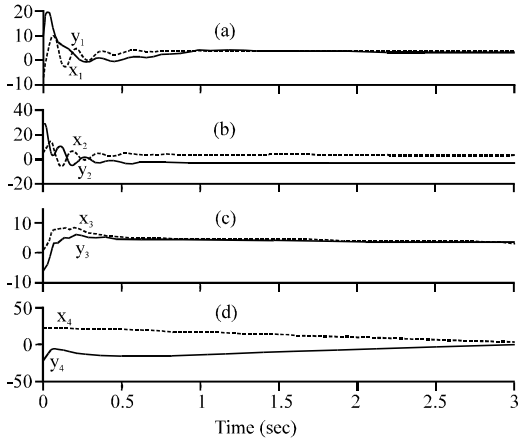

Fig. 7: a-d) Hybrid synchronization of the identical Hyperchaotic Chen Systems

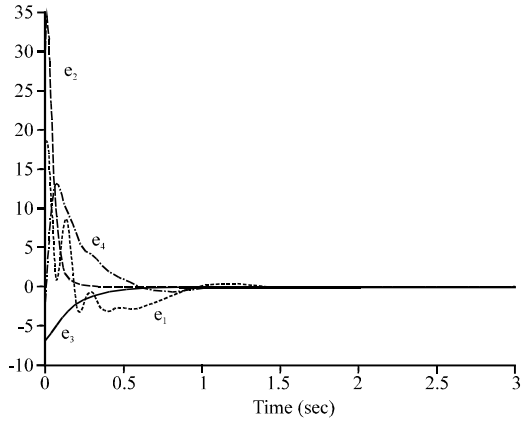

Fig. 8: Time history of the error states e_1-e_4

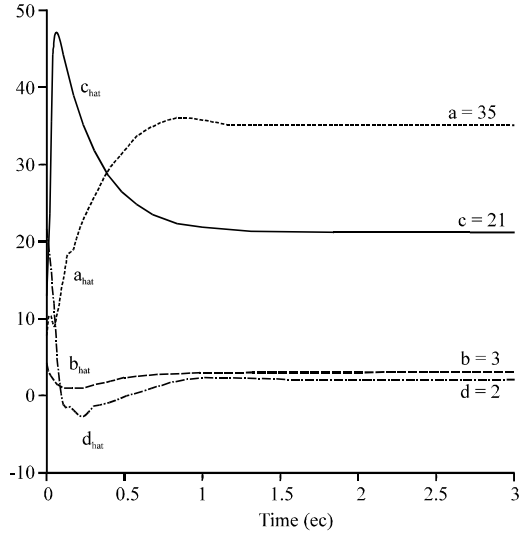

Fig. 9: Time history of the parameter estimates $\hat{a}-\hat{d}$

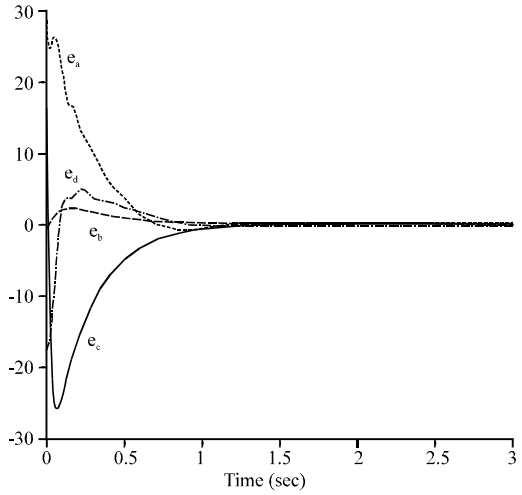

Fig. 10: Time history of the parameter estimation errors e_a-e_d

$$\hat{a}(0) = 7,\ \hat{b}(0) = 4,\ \hat{c}(0) = 1,\ \hat{d}(0) = 20$$

The initial values of the master system (16) are chosen as:

$$x_1(0) = -6,\ x_2(0) = 17,\ x_3(0) = 23,\ x_4(0) = -8$$

The initial values of the slave system (17) are chosen as:

$$y_1(0) = 12,\ y_2(0) = 17,\ y_3(0) = -6,\ x_4(0) = -22$$

Figure 7 shows the hybrid synchronization of the Hyperchaotic Chen Systems (16) and (17). Figure 8 shows the time-history of the hybrid synchronization errors. Figure 9 shows the time-history of the parameter estimates $\hat{a}-\hat{d}$. Figure 10 shows the time-history of the parameter estimation errors e_a-e_d.

ADAPTIVE HYBRID SYNCHRONIZATION OF HYPERCHAOTIC LORENZ AND HYPERCHAOTIC CHEN SYSTEMS

Theoretical results: In this study, researchers derive results for the adaptive hybrid synchronization of non-identical Hyperchaotic Systems viz. Hyperchaotic Lorenz System (Gao *et al*., 2007) and Chen System (Li-Xin *et al*., 2010). Thus, the master system is described by the Hyperchaotic Lorenz dynamics:

$$\dot{x}_1 = \sigma(x_2 - x_1),\ \dot{x}_2 = \rho x_1 - x_2 - x_4 - x_1 x_3$$
$$\dot{x}_3 = x_1 x_2 - \beta x_3,\ \dot{x}_4 = r x_2 x_3 \quad (29)$$

Where x_1-x_4 are the states and σ, β, ρ, r are unknown parameters of the system. The slave system is described by the controlled hyperchaotic chen dynamics:

$$\begin{aligned}\dot{y}_1 &= a(y_2 - y_1) + u_1\\ \dot{y}_2 &= 4y_1 - 10y_1y_3 + cy_2 + 4y_4 + u_2\\ \dot{y}_3 &= y_2^2 - by_3 + u_3\\ \dot{y}_4 &= -dy_1 + u_4\end{aligned} \tag{30}$$

Where y_1-y_4 are the states, a-d are unknown parameters of the system and u_1-u_4 are the adaptive controllers to be designed. The hybrid synchronization error is defined as:

$$\begin{aligned}e_1 &= y_1 - x_1,\ e_2 = y_2 + x_2\\ e_3 &= y_3 - x_3,\ e_4 = y_4 + x_4\end{aligned} \tag{31}$$

A simple calculation gives the error dynamics as:

$$\begin{aligned}\dot{e}_1 &= a(y_2 - y_1) - \sigma(x_2 - x_1) + u_1\\ \dot{e}_2 &= 4y_1 - 10y_1y_3 + cy_2 + 4y_4 +\\ &\quad \rho x_1 - x_2 - x_4 - x_1x_3 + u_2\\ \dot{e}_3 &= y_2^2 - by_3 - x_1x_2 + \beta x_3 + u_3\\ \dot{e}_4 &= -dy_1 + rx_2x_3 + u_4\end{aligned} \tag{32}$$

Let us define the adaptive functions $u_1(t)$-$u_4(t)$ as:

$$\begin{aligned}u_1(t) &= -\hat{a}(y_2 - y_1) + \hat{\sigma}(x_2 - x_1) - k_1e_1\\ u_2(t) &= -4y_1 + 10y_1y_3 - \hat{c}y_2 - 4y_4 -\\ &\quad \hat{\rho}x_1 + x_2 + x_4 + x_1x_3 - k_2e_2\\ u_3(t) &= -y_2^2 + \hat{b}y_3 + x_1x_2 - \hat{\beta}x_3 - k_3e_3\\ u_4(t) &= \hat{d}y_1 - \hat{r}x_2x_3 - k_4e_4\end{aligned} \tag{33}$$

Where $\hat{\sigma}, \hat{\beta}, \hat{\rho}, \hat{r}, \hat{a} - \hat{d}$ are estimates of σ, β, ρ, r, a-d, respectively and k_i (i = 1, 2, 3, 4) are positive constants. Substituting Eq. 33 into Eq. 32, the error dynamics simplifies to:

$$\begin{aligned}\dot{e}_1 &= (a - \hat{a})(y_2 - y_1) -\\ &\quad (\sigma - \hat{\sigma})(x_2 - x_1) - k_1e_1\\ \dot{e}_2 &= (c - \hat{c})y_2 + (\rho - \hat{\rho})x_1 - k_2e_2\\ \dot{e}_3 &= -(b - \hat{b})y_3 + (\beta - \hat{\beta})x_3 - k_3e_3\\ \dot{e}_4 &= -(d - \hat{d})y_1 + (r - \hat{r})x_2x_3 - k_4e_4\end{aligned} \tag{34}$$

Let us now define the parameter estimation error as:

$$\begin{aligned}e_a &= a - \hat{a},\ e_b = b - \hat{b}\\ e_c &= c - \hat{c},\ e_d = d - \hat{d}\\ e_\sigma &= \sigma - \hat{\sigma},\ e_\beta = \beta - \hat{\beta}\\ e_\rho &= \rho - \hat{\rho},\ e_r = r - \hat{r}\end{aligned} \tag{35}$$

Substituting Eq. 35 into Eq. 34, researchers obtain the error dynamics as:

$$\begin{aligned}\dot{e}_1 &= e_a(y_2 - y_1) - e_\sigma(x_2 - x_1) - k_1e_1\\ \dot{e}_2 &= e_cy_2 + e_\rho x_1 - k_2e_2\\ \dot{e}_3 &= -e_by_3 + e_\beta x_3 - k_3e_3\\ \dot{e}_4 &= -e_dy_1 + e_rx_2x_3 - k_4e_4\end{aligned} \tag{36}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov Approach is used. Researchers consider the quadratic Lyapunov function defined by:

$$V = \frac{1}{2}\left(\begin{array}{l} e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 +\\ e_c^2 + e_d^2 + e_\sigma^2 + e_\beta^2 + e_\rho^2 + e_r^2\end{array}\right) \tag{37}$$

Which is a positive definite function on R^{12}. Researchers also note that:

$$\begin{aligned}\dot{e}_a &= -\dot{\hat{a}},\ \dot{e}_b = -\dot{\hat{b}},\ \dot{e}_c = -\dot{\hat{c}},\ \dot{e}_d = -\dot{\hat{d}}\\ \dot{e}_\sigma &= -\dot{\hat{\sigma}},\ \dot{e}_\beta = -\dot{\hat{\beta}},\ \dot{e}_\rho = -\dot{\hat{\rho}},\ \dot{e}_r = -\dot{\hat{r}}\end{aligned} \tag{38}$$

Differentiating Eq. 37 along the trajectories of Eq. 36 and noting Eq. 38, we find that:

$$\begin{aligned}\dot{V} &= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 +\\ &\quad e_a\left[e_1(y_2 - y_1) - \dot{\hat{a}}\right] + e_b\left[-e_3y_3 - \dot{\hat{b}}\right] +\\ &\quad e_c\left[e_2y_2 - \dot{\hat{c}}\right] + e_d\left[-e_4y_1 - \dot{\hat{d}}\right] +\\ &\quad e_\sigma\left[-e_1(x_2 - x_1) - \dot{\hat{\sigma}}\right] + e_\beta\left[e_3x_3 - \dot{\hat{\beta}}\right] +\\ &\quad e_\rho\left[e_2x_1 - \dot{\hat{c}}\right] + e_r\left[e_4x_2x_3 - \dot{\hat{r}}\right]\end{aligned} \tag{39}$$

In view of Eq. 39, the estimated parameters are updated by the following law:

$$\begin{aligned}\dot{\hat{a}} &= e_1(y_2 - y_1) + k_5e_a,\ \dot{\hat{b}} = -e_3y_3 + k_6e_b\\ \dot{\hat{c}} &= e_2y_2 + k_7e_c,\ \dot{\hat{d}} = -e_4y_1 + k_8e_d\\ \dot{\hat{\sigma}} &= -e_1(x_2 - x_1) + k_9e_\sigma,\ \dot{\hat{\beta}} = e_3x_3 + k_{10}e_\beta\\ \dot{\hat{\rho}} &= e_2x_1 + k_{11}e_\rho,\ \dot{\hat{r}} = e_4x_2x_3 + k_{12}e_r\end{aligned} \tag{40}$$

Where, k_i (i = 5,..., 12) are positive constants.

Theorem 3: The non-identical hyperchaotic Lorenz System (29) and hyperchaotic Chen System (30) with unknown parameters are globally and exponentially hybrid-synchronized by the adaptive control law (33) where the update law for the parameter estimates σ, β, ρ, r, a-d is given by Eq. 40 and k_i (i = 1, 2,..., 12) are positive constants. The errors for parameter estimates e_σ, e_β, e_ρ, e_r, e_a-e_d decay to zero exponentially as $t\to\infty$.

Proof: This result is a simple consequence of the Lyapunov Stability Theory. We know that V as defined in Eq. 37 is a positive definite function on R^{12}. Substituting Eq. 40 into Eq. 39, researcherse obtain:

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_a^2 - k_6e_b^2 - k_7e_c^2 - k_8e_d^2 - k_9e_\sigma^2 - k_{10}e_\beta^2 - k_{11}e_\rho^2 - k_{12}e_r^2 \quad (41)$$

Which is a negative definite function on R^{12}. Hence by the Lyapunov Stability Theory (Hahn, 1967), it follows that $e_1(t)\to 0$ as $t\to 0$ for i = 1, 2, 3, 4 and all the parameter estimation errors e_a-e_d, e_σ, e_β, e_ρ, e_r tend to zero exponentially with time. This completes the proof.

Numerical results: For the numerical simulations, the 4th-order Runge-Kutta Method with time-step $h = 10^{-8}$ is used to solve the two systems of differential Eq. 29 and 30 with the adaptive non-linear controller (Eq. 33) and update law of estimates (Eq. 40). Researchers take $k_i = 4$ for i = 1, 2,..., 12. The parameters of the Hyperchaotic Lorenz and Hyperchaotic Chen Systems are chosen so that the systems are hyperchaotic, i.e.,

$$\sigma = 10,\ \beta = 8/3,\ \rho,\ r = 0.1$$
$$a = 35,\ b = 3,\ c = 21,\ d = 2$$

The initial values of the parameter estimates are chosen as:

$$\hat{a}(0) = 1,\ \hat{b}(0) = 8,\ \hat{c}(0) = 10,\ \hat{d}(0) = 7$$
$$\hat{\sigma}(0) = 5,\ \hat{\beta}(0) = 14,\ \hat{\rho}(0) = 3,\ \hat{r}(0) = 11$$

The initial values of the master system (16) are chosen as:

$$x_1(0) = 2,\ x_2(0) = -5,\ x_3(0) = 4,\ x_4(0) = 1$$

The initial values of the slave system (17) are chosen as:

$$y_1(0) = 8,\ y_2(0) = -6,\ y_3(0) = -3,\ y_4(0) = -4$$

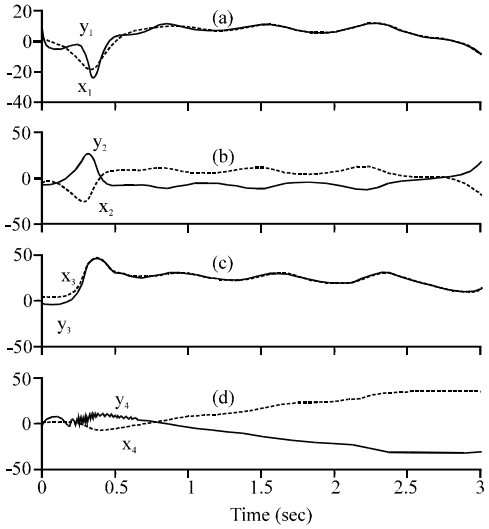

Fig. 11: a-d) Hybrid synchronization of the non-identical Hyperchaotic Lorenz and Chen Systems

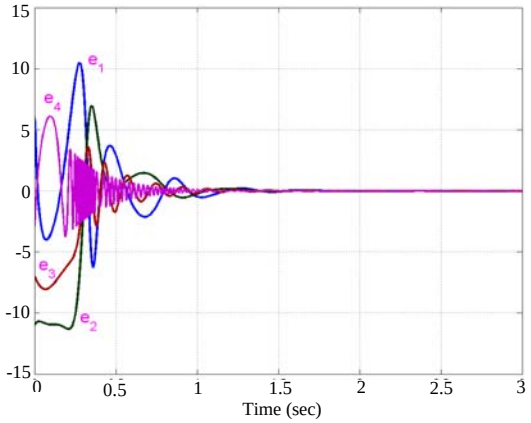

Fig. 12: Time history of the error states e_1-e_4

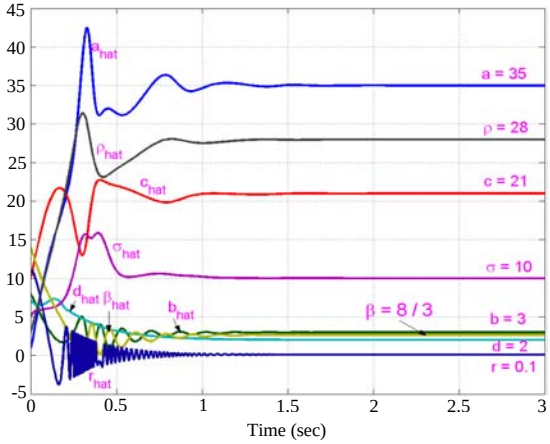

Fig. 13: Time history of the parameter estimates $\hat{a}$ – $\hat{d}$, $\hat{\sigma}$, $\hat{\beta}$, $\hat{\rho}$, $\hat{r}$

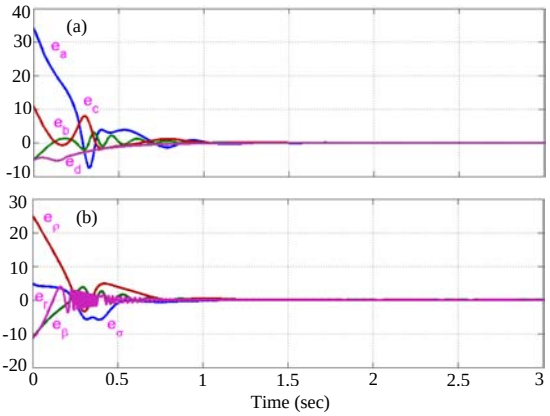

Fig. 14: a, b) Time history of the parameter estimation errors e_a-e_d, e_σ, e_β, e_ρ, e_r

Figure 11 shows the hybrid synchronization of the Hyperchaotic Lorenz and Hyperchaotic Chen Systems (16) and (17). Figure 12 shows the time-history of the hybrid synchronization errors. Figure 13 shows the time-history of the parameter estimates $\hat{a} - \hat{d}, \hat{\sigma}, \hat{\beta}, \hat{\rho}, \hat{r}$. Figure 14 shows the time-history of the parameter estimation errors e_a-e_d, e_σ, e_β, e_ρ and e_r.

CONCLUSION

In this study, researchers have deployed Adaptive Control Method for achieving adaptive hybrid synchronization of the identical Hyperchaotic Lorenz Systems, the identical Hyperchaotic Chen Systems and non-identical Hyperchaotic Lorenz and Hyperchaotic Chen Systems when the parameters of the systems are unknown. The adaptive hybrid synchronization results derived have been proved using the Lyapunov Stability Theory. Since, the Lyapunov exponents are not required for these calculations, the proposed Adaptive Control Method is very effective and convenient for achieving hybrid synchronization of the hyperchaotic systems addressed in this study. Numerical simulations are shown to demonstrate the effectiveness of the hybrid synchronization results for the hyperchaotic systems discussed in this study.

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