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The Development of a Risk-Neutral Density Estimation Method

Hafizah Bahaludin and Mimi Hafizah Abdullah
Department of Computational and Theoretical Sciences,
Faculty of Science, International Islamic University Malaysia,
Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia

Abstract: The Risk-Neutral Density (RND) function is the distribution implied by option prices. Broadly, the approaches to extract RND can be classified into four categories; an underlying asset is assumed to follow a stochastic distribution, parametric techniques, semi- parametric techniques and smoothing a volatility function. Smoothing volatility function is a common practice in extracting the RND function. Theoretically, it can be estimated by differentiating the call prices twice with respect to the strike price if the continuous strike prices are available. This study focuses on the development of the risk-neutral density estimation by using the smoothing implied volatility smile method.

Key words: Options, risk-neutral density, extract, distribution, prices

INTRODUCTION

Market participants believe that information that is embedded in option prices can provide future expectation of the underlying asset and market sentiment. The option prices have significant information to predict future asset prices and to estimate the distribution of returns over possible values of asset prices. An option is a financial contract in which gives a right to the holder but not an obligation to buy or sell an underlying asset at a fixed price at a maturity date. Theoretically, prices of options today reflect the market situation in future and it can be considered as forward-looking information. In fact, option with different strike prices can be used to estimate the density function of the underlying asset.

Under risk-neutral valuation, the probability density contained in option prices is the risk-neutral density (hereafter RND). It means that the option price is equal to the present value of the discounted payoff calculated under RND given a risk-free interest rate. Note that, the risk-neutral probability distribution and the associated risk-neutral density function represent the forward looking prediction for the returns distribution of the underlying asset. These RNDs contain enormous information that are particularly useful for various application such as extraction of risk-aversion function (Bliss and Panigirtzoglou, 2004, Ait and Lo, 1998), monetary policy purposes (Bahra, 1997; Melick and Thomas, 1997; Abarca *et al.*, 2010) and asset allocation (Giamouridis and Skiadopoulos, 2009; Kostakis *et al.*, 2011).

A large and growing literature on the estimation method of RND from option prices can be classified into four categories. Firstly, the underlying asset is assumed to follow a certain stochastic process and RND is estimated from the process. Secondly, a parametric technique which assumed that the underlying asset is to follow a certain distribution such as mixed lognormal. Thirdly, a semi-parametric technique is used to estimate RND such as the implied binomial trees. Lastly, the RND estimation can be obtained using interpolation techniques in order to smooth a volatility function and numerically calculate the RND. This method also known as a non parametric technique. There are number of works that provide excellent and comprehensive review on the method of RND estimation. Jackwerth (1999) explained on the parametric and non-parametric techniques and proposed a semi parametric technique which is implied binomial trees. Bahra (1997) gave an overview of the five techniques related to the estimation of RND and discussed the advantages and drawbacks of each technique. The estimation techniques used are histogram, interpolation using observed call price, interpolating the volatility smile, parametric techniques which is two-lognormal mixture and finally assumed the underlying asset is follow the stochastics process. Another major study by Bliss and Panigirtzoglou (2002) claimed that the drawbacks of the different methods were either from parametric or non-parametric methods and they proposed an alternative way to extract the RND. Recently Figlewski pointed out the advantages and disadvantages of a certain technique and claimed that it is a common practice

Corresponding Author: Hafizah Bahaludin, Department of Computational and Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia

to use non-parametric techniques to interpolate and to smooth a volatility function. This paper differs from previous studies in which it presents non-parametric estimation methods specifically using smooth volatility function to extract the RND.

MATERIALS AND METHODS

Limitations of Black-Scholes-Merton Model: Although, Black-Scholes-Merton Model has been widely used to value options but there are some limitations to this model. Black-Scholes-Merton Model assumes that the underlying asset price follows a Geometric Brownian motion with a constant expected return and a constant drift. The asset price is lognormally distributed over the time. The only unobservable input based on this model is the volatility, in the asset's return. Nonetheless, the implied volatility can be inferred by taking the option price together with other parameters and inverting the volatility using the Black-Scholes-Merton Model. Empirical evidence found that implied volatility differs for each strikes price and maturity. This is in contrast with the assumption of Black-Scholes-Merton Model in which volatility is constant across maturity. In addition, implied volatility retrieved from the market creates a phenomenon called volatility smile.

Volatility smile indicates that the volatility of out of the money and in the money options are higher than that of the at the money options. This tends to make the future option prices different from the actual market option prices. It turns out that option prices have high probability of being in the money in the future and the deep out of the money options become expensive compared to the option prices calculated using the Black Scholes-Merton Model. This result produces in fatter tails of the true RND function compared with that of the lognormal RND function. Bahra (1997) pointed out that the convexity of the volatility smile indicates the degree to which the market RND function differs from the lognormal RND.

Extracting the risk-neutral density from option prices: Standard notation used are of the Black-Scholes-Merton model which are the variables C and P as the European call and European put prices, respectively is the price S_0 of the underlying asset; K is the strike price; γ is the continuously compounded risk-free rate; σ is the underlying asset price volatility and T is the time to maturity of the option. Additionally, $f(s_T)$ is the RND function and $F(S_T)$ is the risk-neutral distribution function.

The value of a European call option is the discounted expected value of the payoff on the expiration date, T. This calculation is under a risk-neutral probability and discounting it with the risk free rate:

$$C(T,K) = e^{-rT}\int_0^{\infty}(S_T - K, 0) f(S_T) dS_T \tag{1}$$

The partial derivative of C (T, K) with respect to the strike price, K yields:

$$\frac{\partial C(T,K)}{\partial K} = e^{-rt}\int_K^{\infty} f(S_T) dS_T = e^{-rt}\left[1 - F(K)\right] \tag{2}$$

Solving the risk-neutral distribution gives:

$$F(K) = e^{rt}\left[\frac{\partial C(T,K)}{\partial K}\right] + 1 \tag{3}$$

The approximation of Eq. 3 can be obtained using finite differences of option prices observed at discrete exercise prices in the market. Let there be option prices available for maturity T at N different exercise prices with $k_1, ..., K_N$ in an increasing order. Then, the approximation of centre $F(K_N)$ on is obtained by using the sequential strike prices K_{n-1}, K_n, K_{n+1}:

$$F(K_N) = \left[\frac{C_{n+1} - Cn_{-1}}{K_{n+1} - K_{n-1}}\right] + 1 \tag{4}$$

Taking the derivative a second time with respect to K in Eq. 4 yields the risk-neutral density function at K:

$$f(K) = e^{rt}\frac{\partial^2 C(T,K)}{\partial K^2} \tag{5}$$

In practice, the approximation of RND for a call option, $f(K_N)$ can be obtained as:

$$f(K_n) = e^{rt}\left[\frac{C_{n+1} - 2C_n + C_{n-1}}{(\Delta K)^2}\right] \tag{6}$$

While, the equations to calculate the RND puoption is as:

$$F(K) = e^{rt}\frac{\partial P(T,K)}{\partial K} \tag{7}$$

$$F(K_N) \approx e^{rt}\left[\frac{P_{n+1} - P_{n-1}}{K_{n+1} - K_{n-1}}\right] \tag{8}$$

$$f(K) = e^{rt} \frac{\partial^2 P}{\partial K^2} \quad (9)$$

$$f(k_n) \approx e^{rt} \frac{P_{n+1} - 2P_n + P_{n-1}}{(\Delta K)^2} \quad (10)$$

Breeden and Litzenberger (1978) stated that the RND can be extracted from a cross section of option prices. This RND approximation can be calculated if a continuum of option prices with the same time of maturity and strike prices range between zero until infinity are available on a single underlying asset. Unfortunately, in the actual market, options are only traded over a limited strike prices. The difference between strike prices is too widely spaced to estimate the density function using finite differences as introduced in Eq. 4 and 6. This shows that the extraction of a well behaved RND cannot be done using the initial data set. Researchers have suggested the use of interpolation techniques to fill in the option values between the available strike prices and smoothing the volatility smile in order to reduce the influence of noise.

RESULTS AND DISCUSSION

Techniques for estimating the risk-neutral density function: The RND function can be calculated by taking the second order derivative of the option prices with respect to the strike price if the strike price is continuous. However, in practice, the range of the strike prices is limited and subsequently the estimation of RND will not be accurate. Therefore, an interpolation technique is needed to fill in the gaps between the strike prices so that a well behaved RND function can be extracted.

Shimko (1993) was the first to propose fitting techniques to smooth the volatility smile by translating the quoted price into implied volatility. Then, the RND can be extracted from second derivatives of option prices with respect to strike price (Breeden and Litzenberger, 1978). The use of implied volatility instead of quoted price can eliminate a substantial amount of non linearity (Brunner and Hafner, 2003). Interpolating the quoted prices directly seems a straightforward and appealing solution but the method produces bad results such as recurrent violation of monotonicity and convexity conditions. This means that the implied volatility tends to be smoother than the option prices. Shimko (1993) fitted the implied volatility using a quadratic polynomial function, in which the implied volatility is the y-axis and strike price is the x-axis:

$$\sigma_i(K) = a_0 + a_1 K_1 + a_2 K_i^2 \quad \text{for } i = 1, \ldots, N \quad (11)$$

where, N is the number of observed strike prices. Campa *et al*. (1998) followed Shimko (1993) to fit the volatility function with respect to the strike price but by using a cubic spline interpolation. Cubic spline is used because it controls the flexibility of the volatility smile shape compared to that of a quadratic polynomial. The cubic spline function of the strike price is:

$$f_1(K_1) = a_1 + b_1(K - K_1) + c_1(K - K_1)^2 + d_1(K - K_1)^3 \quad (12)$$

Next, the matrix of a polynomial parameter of a cubic spline $\Theta = (a_i, b_i, c_i, d_i)$ has to be estimated. It can be achieved by minimizing the following objective function:

$$\min_{\Theta} \left\{ \sum_{i=1}^{N} W_i \left(\sigma_i - f_i(K_i, \Theta)^2 + \lambda \int_0^{+\infty} f^N(K;\Theta)^2 dK \right) \right\} \quad (13)$$

where, $\{K_i\}_{i=1}^n$ are the strike prices where the function is evaluated and $\{\sigma_i\}_{i=1}^n$ are the corresponding implied volatilities λ and is a smoothing parameter that controls the curvature of the spline. The first term of objective function is representing the goodness of fit and the second term measures the smoothness of the spline. The continuous series of implied volatilities with respect to the strike price can be obtained after estimating a cubic spline function parameter.

In a different study, Malz (1997) used a quadratic polynomial for fitting a volatility function but used an option delta rather than a strike price. Option delta can be defined as the rate of change of option price with respect to the price of the underlying asset. This approach is used to avoid grafting the tails onto the distribution. There are two ways to convert the strike prices into option delta either using' smile conversion or point conversion (Bu and Hadri, 2007). The smile conversion transforms the strike prices into the option delta by using the Black-Scholes-Merton Model. The call delta is given by:

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) = \Phi \left[\frac{\ln S_0 - \ln K + \left(r - q + \frac{\partial_K^2}{2} \right) T}{\partial_K \sqrt{T}} \right] \quad (14)$$

where, Φ(.) is the cumulative probability distribution function for a standardized normal distribution and σ_K is the implied volatility corresponding to the strike price.

The point conversion is using a single at the money implied volatility instead of using the implied volatility of each strike price to convert the strike prices into the deltas (Santos and Guerra, 2015; Bu and Hadri, 2007). The call delta is calculated using at-the-money implied volatility so that the ordering is same with the strike prices. If the implied volatility corresponding to each strike price is used to calculate the delta, it would create the kinks. The equation of point conversion is as follows:

$$\Delta = N(d_1) = \frac{\partial C}{\partial S} = \Phi\left[\frac{\ln S_0 - \ln K + \left(r - q + \frac{\partial^2_{ATM}}{2}\right)}{\partial_{ATM}\sqrt{T}}\right] \quad (15)$$

where, σ_{ATM} is the at-the-mone implied volatility. Bliss and Panigirtzoglou (2002) adopted the approaches of Campa *et al*., 1998) and combined the said method with the method suggested by Malz (1997). The cubic spline is fitted to the implied volatility across the call option delta instead of the strike price. By using the value of the delta which falls between 0 and 1, the windows of interpolation is reduced as compared to that of using strike price which extends to infinity. A small delta which approaches to zero represents a high strike price of an option and vice versa. The cubic spline function in delta is:

$$\sigma_i(\Delta) = a_i + b_i(\Delta - \Delta_i) + c_i(\Delta - \Delta_i)^2 + d_i(\Delta - \Delta_i)^3 \quad (16)$$

The parameter $\Theta = (a_i, b_i, c_i, d_i)$ can be estimated by minimizing the objective function as follows:

$$\min_{\Theta}\left\{\sum_{i=1}^{N} w_i\left(\sigma_i - \hat{\sigma}(\Delta_i, \Theta)^2 + \lambda\int_0^{+\infty} \sigma^n(\Delta;\Theta)^2 d\Delta\right)\right\} \quad (17)$$

where, $\hat{\sigma}(\Delta_i, \Theta)$ is the fitted implied volatility and W_i represents the weight attributed to each observation. Bliss and Panigirtzoglou (2004, 2002)described the weight of a parameter as the source of a price error and proposed Vega (v) as the weighting factor. Vega measures the impact of changes in the underlying volatility of the option price. Specifically, Vega shows the change in the price of the option for every one percent change in the underlying volatility. Vega value is used as the weighted factor to fit the volatility function to the near the money options (Malz, 1997). Vega is given by:

$$V = \frac{\partial C}{\partial \sigma} = S_T N(d_1)\sqrt{T} \quad (18)$$

The value of a smoothing parameter, λ is chosen to be approximately close to 1 in which the smoothness is the main concern (Santos and Guerra, 2015; Lai, 2011). The value of the smoothing parameter can be decided as λ = 0.99 (Santos and Guerra, 2015; Lai, 2011). Similar to Campa *et al*. (1998), Bliss and Panigirtzoglou (2002) and Lai (2011) used a cubic spline to interpolate a volatility function. However, this study used moneyness of options as the independent variable instead of the strike price or option delta. Interpolation is performed using implied volatility versus moneyness (S/K) space. Moneyness can be defined as the probability of option being in the money and it is calculated as the ratio of (S/K) where S is the asset price and K is the strike price. This study claimed that by using the implied volatility as a function of moneyness, the observation is less dispersed and continuous as compared to a function that uses the strike price.

At this point, interpolation of implied volatility uses either low-order polynomial or splines. The drawback of splines is that it requires the interpolation to pass through each point (implied volatility). It is shows that the spline interpolation incorporates measurement errors and price noise. Besides, the selection of smoothing parameter value can be argued in which the spline curves should be similar with the observed data points. Instead of a cubic spline, Figlewski suggested the use of at least a fourth order polynomial to interpolate the implied volatility as a function of moneyness. In addition, to allow the densities to take into account of a more complex shape, Bliss and Panigirtzoglou (2002) suggested to use fourth order spline with a single knot placed of at the money options.

Generally, the procedure to interpolate a volatility function and to estimate the RND can be summarized as follows:

- Decide the independent variable: strike price, moneyness or delta
- Calculate the implied volatility and independent variable from the option prices
- Interpolate across the implied volatility in (implied volatility/strike) space or (implied volatility/moneyness) space or (implied volatility/delta) space. Choose an interpolation method using either a polynomial or a spline function
- Convert the delta values to strike price using this Eq. 19

$$K_i = \frac{S_T}{\exp\left[N^{-1}(\Delta_i e^{rt})\sigma(\Delta_i)\sqrt{T} - \left(\frac{\sigma(\Delta_i)^2}{2}\right)T\right]} \quad (19)$$

- Convert the implied volatility to call price using Black-Scholes-Merton Model

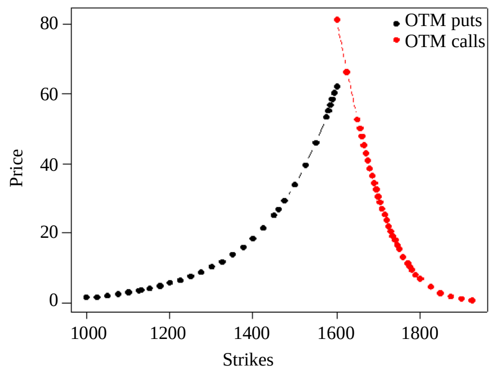

Fig. 1: Out of the money calls and puts options available on July 5, 2013

- Take the second derivative of a call function with respect to strike price and discount the call function using risk free rate to estimate the RND. Numerically it can be calculated using Eq. 6

S and P 500 index options on July 5, 2013 is used as an example. The index closed at 1631.89 and the options contract expired 121 days later on December 21, 2013. The options close price for each strike is calculated based on the midprice of bid and ask. The risk free rate is assumed to be 2.69% and the dividend is assumed to be 1.70%.

First, the observed prices that have midprice lower than \$0.50 is taken out to ensure the deep out of the money options are not involved. Only out the money (OTM) options are considered in which the options are most liquid (Bliss and Panigirtzoglou, 2004; Ait and Lo, 1998; Kostakis *et al.*, 2011; Bliss and Panigirtzoglou, 2002; Birru and Figlewski, 2012; Grith *et al.*, 2012; Ivanova and Gutierrez, 2014; Kempf *et al.*, 2015; Gutierrez and Humpreys, 2012). Gutierrez and Humpreys (2012) points out that the 81% options traded were OTM options and only 18% ITM options were traded. Also, the ATM options only used as a part of investment strategy such as straddles and strangles. Figure 1 shows the out of the money calls and puts options available on July 5, 2013.

Second, data from (strike/price) space is transformed into (delta/implied volatility) space. The implied volatilities and deltas are calculated using Black-Scholes-Merton Model. The smoothing spline is used in this example in which the smoothing parameter is 0.90 and the Vega is used as the weight of the parameter. The implied volatility has been fitted using the smoothing spline throughout the deltas as depicted in Fig. 2.

Then, the 5000 points of equally space delta from zero to one are transformed into strike price by using Eq. 18. Extract the implied volatility from the spline function and converted into call prices using Black-Scholes-Merton Model. The estimated of call and strike prices are shown in Fig. 3. Finally, compute the second derivative of call prices to obtain RND. Figure 4 presents the RND extracted by the smoothing spline on July 5, 2013.

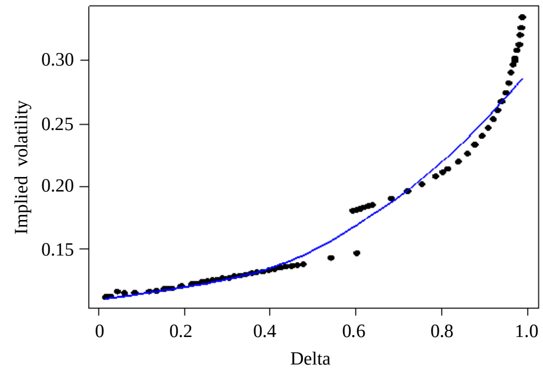

Fig. 2: The implied volatility has been fitted using smoothing spline

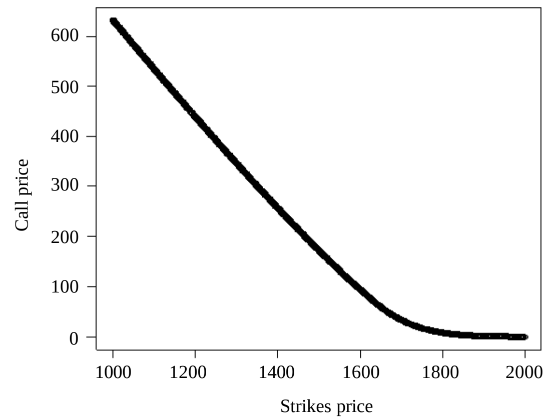

Fig. 3: The estimated call prices and strike prices

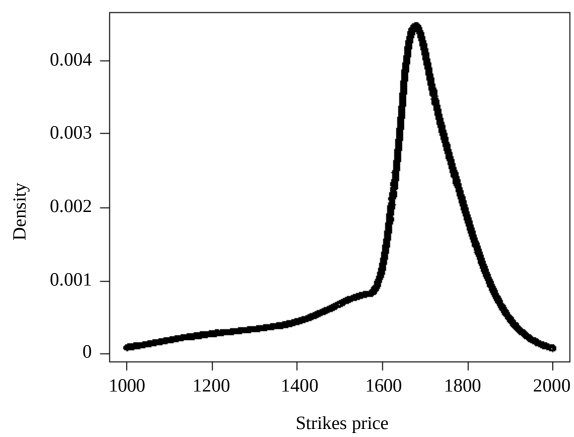

Fig. 4: Risk Neutral Density (RND) extracted by spline on July 5, 2013

CONCLUSION

This study gives an overview of the RND estimation using a smooth implied volatility smile methods. The RND can be derived from the second order derivatives of a call option function with respect to strike prices. The drawback of this approach is that it assumed the options' strike prices are continuous. In practice, only limited strike prices in discrete time are available. Considering this limitation, researchers have proposed several interpolation techniques to estimate RND accurately.

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