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Effect of Polynomial Shift in the Method for Finding the Largest Eigenvalue of a Polynomial

[1]N.F. Ibrahim and [2]N.A. Mohamed
[1]Marine Management Sciences Research Group, School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Malaysia
[2]Department of Mathematics, Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris, 35900 Proton City, Tanjung Malim, Perak, Malaysia

Abstract: A new method for finding the largest eigenvalue of a generalised nonnegative polynomial was introduced in 2014 by Ibrahim. The method was proven to be convergent for weakly irreducible polynomials. In the method, an irreducible polynomial is shifted such that it becomes primitive. However, it is unknown what is the optimal shift and the effect of the step length to the method. In this study we examine the effect of the step length to the method.

Key words: Spectral radius, eigenvalue, polynomial, optimal shift, largest eigenvalue

NTRODUCTION

Eigenvalue plays an important roles in marine energy system, ship structure (Soares and Fricke, 2011), aquatic conservation (Jacobi and Jonsson, 2011) and many others. In marine energy system for example, a mathematics equation is modeled based on the induction generator, transmission line and grid. The marine energy system is then analysed using the eigenvalues of the modeled system. In order to solve that kind of problem, a method was proposed in (Ibrahim, 2014). It can solve general nonnegative irreducible polynomials problem. The solution of eigenvalue problem of nonnegative irreducible polynomial is eventually the largest eigenvalue of the polynomial or also known as the spectral radius. The method shifts an irreducible polynomial to become a primitive polynomial. A primitive polynomial ensures the iterations to converge to the largest eigenvalue. In this paper, we examine what happen when we use different step lengths.

MATERIALS AND METHODS

Iterative method: An iterative method for finding the largest eigenvalue of nonnegative irreducible polynomials was proposed in (Ibrahim, 2014). This method was an extension of the method by Wood and ONeill (Varga, 1965). Wood and O'Neill presented a method for finding the largest eigenvalue of matrices which has some features that are similar to the power method (Varga, 1965). Many of important properties of nonnegative matrices been generalised to tensors such as the Perron Frobenius Theorem and minimax theorem (Chang *et al*., 2008). This is significance because tensor is much wider class than matrix. The term "tensor" is basically applied to data in three or more dimensions. It is also referred to as higher-order tensor or as a multi-dimensional, multi-way or n-way array. A matrix is a tensor of order two. As a sequence to the generalisation of important properties of nonnegative matrices to nonnegative tensors, Ng *et al.* (2009) presented an iterative method for calculating the largest eigenvalue and the associated eigenvector for nonnegative square tensors. The numerical results in (Ng *et al*., 2009) show that the Ng-Qi-Zhou method is efficient however not always convergent, for irreducible tensors. Later, this method was proven to be convergent for primitive nonnegative square tensors in (Chang *et al*., 2011). Friedland *et al.* (2013), the convergence of the method under weakly primitive square tensors was established. An irreducible tensor is primitive but not vice versa.

Zhang and Qi showed that the method has a linear convergence rate for essentially positive tensors. An essentially positive tensor is primitive but the reverse is not valid. Ng *et al.* (2009) method of was improved in (Liu *et al*., 2010) and was proven to be convergent for an irreducible nonnegative tensor. This improved method resembled a version of the Collatz method by Wood and Neill for nonnegative matrices. Zhang established the linear convergence of the improved method for weakly

Corresponding Author: N.F. Ibrahim, Marine Management Sciences Research Group, School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Malaysia

positive tensors and Zhou established the Q-linear convergence of the improved method under weak irreducibility condition.

Another important developement in this area of research is that the Perron-Frobenius theorem also has been extended to homogeneous and monotone functions (Gaubert and Gunawardena, 2004), nonnegative multilinear forms (Friedland *et al*., 2013) and nonnegative polynomial maps (Friedland *et al*., 2013). This makes it possible for Ibrahim (2014) to extend the method of Wood and O'Neill (Varga, 1965) to a wider class that is general polynomials. The optimal shift for matrices was mentioned briefly in however, there was no discussion in literature for tensors and polynomials. The lack of research about the optimal shift of the polynomials in the Ibrahim's method (Ibrahim, 2014) has inspired this research.

Algorithm: Let: $R^n \rightarrow R$:

$$P(y) = \sum_{\alpha \in R_+^n} a_\alpha x^\alpha$$

where, $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_a = (\alpha_{a1}, ,\alpha_{an})$ and $x^a = (x_1^{a1}, ..., x_n^{an})$, be a generalized irreducible polynomial with nonnegative coefficients.

The following algorithm was given in Ibrahim (2014) for finding the spectral radius of nonnegative irreducible polynomials. This algorithm was proven to be convergent.

Algorithm 1: Ibrahim's Method (Ibrahim, 2014):

Step 0: Choose $x^1 \in R_{>0}^n$. Set $Q(x) = P(x)+\rho x^{(d)}$ and let k = 1

Step 1: Compute

$$Q_i(x^{(k)}) = \sum_{j \in Z_+^n} a_{ij} x^j + \rho x^{[d]}, \quad i = 1,2,...,n$$

$$\underline{\lambda}_k = \min_{i \in \{1,2,...,n\},\, x_i^{(k)}>0} \frac{Q_i(x^{(k)})}{(x_i^{(k)})^d}$$

$$\overline{\lambda}_k = \max_{x_i^{(k)}>0} \frac{Q_i(x^{(k)})}{(x_i^{(k)})^d}$$

Step 2: If then let $\lambda = \overline{\lambda}_k$ and stop. Otherwise, compute

$$x^{(k+1)} = \frac{\left(Q(x^{(k)})\right)^{[\frac{1}{d}]}}{\left\| \left(Q^{(k)}\right)^{[\frac{1}{d}]} \right\|_p}$$

Replace k by and k+1 go to Step 1.

RESULTS AND DISCUSSION

Numerical tests was performed using Matlab to observe the changes in the polynomial shift. Let A be an m-ordern -dimensional square tensor.

$$A = \left(a_{i_1 i_2 \ldots i_m}\right), a_{i_1 i_2 \ldots i_m} \in R, 1 \le i_1, i_2, \ldots, i_m \le n$$

Define order -dimensional column vectors Ax^{m-1} as:

$$Ax^{m-1} = \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii2\ldots i_m} x_{i_2} \ldots x_{i_m} \right)_{1 \le i \le n}$$

We use the test function $Q(x) = Ax^{m-1}+\rho x^{(d)}$:

$$Q_i(x) = \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii2\ldots i_m} x_{i_2} \ldots x_{i_m} + \rho_i^{(d)} \right)_{1 \le i \le n}$$

The entry of the tensor is randomly generated with so that is nonnegative irreducible polynomials with degree. In the test, we use d = 7. We set the shift as $\rho = 1\times10^{-6}$, $\rho = 1\times10^{-4}$, $\rho = 1\times10^{-2}$, $\rho = 1$, $\rho = 1\times10^2$, $\rho = 1\times10^4$, $\rho = 1\times10^6$, $\rho = 1\times10^{10}$, $\rho = 1\times10^{12}$, $\rho = 1\times10^{14}$. Table 1-3 and Fig. 1-3 we present the results. From Table 1 and Fig. 1, the number of iterations are the same when $\rho = 1\times10^{-6}$, $\rho = 1\times10^{-4}$, $\rho = 1\times10^{-2}$, $\rho = 1$. The number of iteration is increasing starting from $\rho = 1\times10^2$ to $\rho = 1\times10^{14}$.

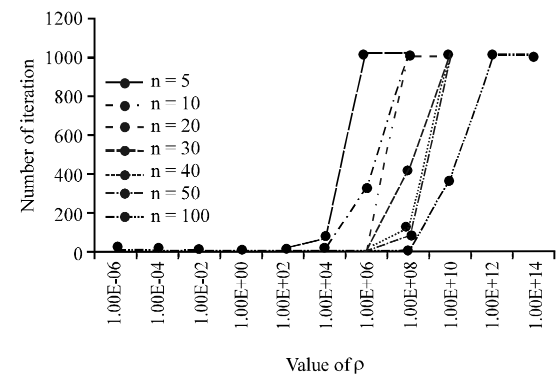

Fig. 1: Number of iterations when the value of ρ between 1×10^{-6} and 1×10^{14}

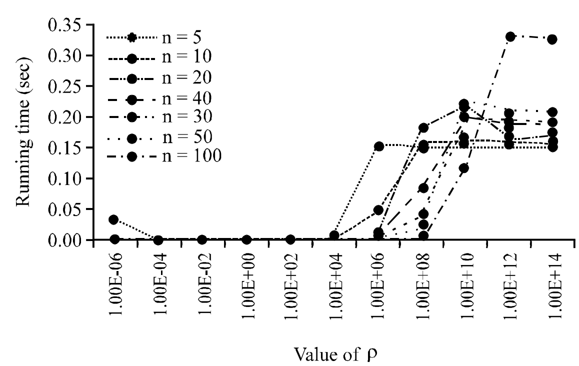

Fig. 2: Running time in seconds when the value of ρ between 1×10^{-6} and 1×10^{14}

Table 1: Number of iterations when $1\times10^{-6}\leq\rho\leq1\times10^{14}$ and $5\leq n\leq100$

ρ	n = 5	n = 10	n = 20	n = 30	n = 40	n = 50	n = 100
1.00E-06	6	6	6	5	5	5	5
1.00E-04	6	6	6	5	5	5	5
1.00E-02	6	6	6	5	5	5	5
1.00E+00	6	6	5	5	5	5	5
1.00E+02	8	7	6	6	6	6	6
1.00E+04	54	11	7	6	6	6	6
1.00E+06	1000	314	31	13	9	8	6
1.00E+08	1000	1000	1000	417	143	66	12
1.00E+10	1000	1000	1000	1000	1000	1000	357
1.00E+12	1000	1000	1000	1000	1000	1000	1000
1.00E+14	1000	1000	1000	1000	1000	1000	1000

Table 2: Running time in seconds when $1\times10^{-6}\leq\rho\leq1\times10^{14}$ and $5\leq n\leq100$

ρ	n = 5	n = 10	n = 20	n = 30	n = 40	n = 50	n = 100
1.00E-06	0.034741	0.001311	0.001136	0.000889	0.00096	0.001216	0.003782
1.00E-04	0.001268	0.001339	0.001112	0.001143	0.001037	0.001242	0.002381
1.00E-02	0.001379	0.001091	0.001258	0.000864	0.001116	0.001167	0.002164
1.00E+00	0.001033	0.001096	0.0011	0.001153	0.001144	0.00123	0.001577
1.00E+02	0.001435	0.001274	0.001489	0.001771	0.001382	0.002511	0.002015
1.00E+04	0.007901	0.001845	0.001167	0.001016	0.001427	0.001425	0.002738
1.00E+06	0.154388	0.046113	0.004849	0.002997	0.001777	0.002758	0.001664
1.00E+08	0.147597	0.155747	0.180052	0.082755	0.033505	0.013335	0.003442
1.00E+10	0.153302	0.162098	0.215079	0.193362	0.186332	0.222441	0.114848
1.00E+12	0.152532	0.157763	0.166809	0.186715	0.193021	0.209227	0.32675
1.00E+14	0.146636	0.155061	0.170515	0.189394	0.189765	0.206673	0.323734

Table 3: The differences between upper bound and lower bound $(|\bar{\lambda}_k - \underline{\lambda}_k|)$ of spectral radius at the final iteration

ρ	n = 5	n = 10	n = 20	n = 30	n = 40	n = 50	n = 100
1.00E-06	1.84E-04	2.88E-04	4.88E-04	6.53E-02	1.49E-01	3.11E-01	7.14E-01
1.00E-04	1.84E-04	2.88E-04	4.88E-04	6.54E-02	1.49E-01	3.11E-01	7.14E-01
1.00E-02	1.86E-04	2.88E-04	4.88E-04	6.54E-02	1.49E-01	3.11E-01	7.15E-01
1.00E+00	3.79E-04	3.98E-04	3.94E-02	7.59E-02	1.71E-01	3.80E-01	7.35E-01
1.00E+02	1.17E-04	2.11E-04	1.99E-02	2.70E-02	5.00E-02	8.94E-02	1.21E-01
1.00E+04	2.72E-04	1.90E-03	2.02E-03	7.61E-02	1.56E-01	3.25E-01	6.81E-01
1.00E+06	7.74E+01	2.29E-03	1.70E-02	2.76E-02	9.83E-02	7.34E-02	8.91E-01
1.00E+08	2.68E+03	1.06E+04	1.45E+02	8.94E-02	2.19E-01	4.18E-01	3.17E+00
1.00E+10	2.77E+03	1.78E+04	3.18E+05	1.12E+06	1.11E+06	5.45E+05	5.49E+00
1.00E+12	2.77E+03	1.79E+04	3.41E+05	1.63E+06	3.97E+06	1.08E+07	1.06E+08
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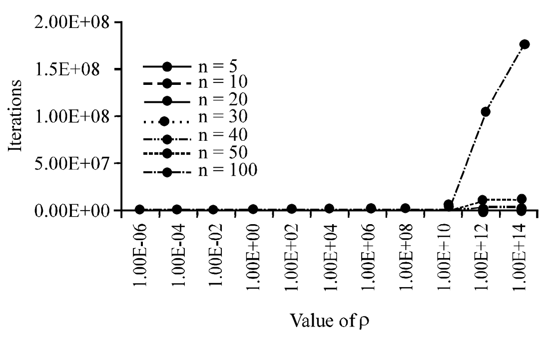

Fig. 3: The differences between upper bound and lower bound $(|\bar{\lambda}_k - \underline{\lambda}_k|)$ of spectral radius at the final iteration when the value of ρ between 1×10^{-6} and 1×10^{14}

In term of running time as we can see from the Fig. 2, for $\rho = 1\times10^{-6}$, $\rho = 1\times10^{-4}$, $\rho = 1\times10^{-2}$, $\rho = 1$, $\rho = 1\times10^{2}$ $\rho = 1\times10^{4}$ running time is acceptable but there is steep increase when the value of $\rho\geq1\times10^{2}$. Figure 3, it is clear that it efficient to use value of ρ between $\rho = 1\times10^{-6}$ and $\rho = 1\times10^{10}$. For the value bigger than, it is no longer efficient.

CONCLUSION

In this study, we did numerical tests to find out the most efficient polynomial shift in Algorithm 1. Algorithm 1 is an algorithm to find the largest eigenvalue of nonnegative irreducible polynomials. From the results, we compare the number of iterations, running time andthe differences between upper bound and lower bound $|\bar{\lambda}_k - \underline{\lambda}_k|$ of spectral radius at the final iteration. It is safe to conclude that the most efficient shift is However, more extensive testsis needed. We also need further study to explain whythe most efficient shift is $\rho\leq1$.

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