

Common Fixed Point Theorems for Weak (ψ, φ) -Contraction in B-Rectangular Metric Spaces

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Abstract: In this research, we extend the results of Roshan and eta. For four mappings in b-rectangular metric spaces.

Key words: Point theorems, weak (ψ, φ) , b-rectangular, Roshan and eta, -contraction, literature

INTRODUCTION

Fixed point and common fixed point theorems for (weak) contractive mappings have been studied by many researchers (Chandok, 2011, 2012; Chandok *et al.*, 2013; Ciric, 1974). Many researchers have recently obtained fixed point, common fixed point, coupled fixed point results in partially ordered metric space (Abbas *et al.*, 2011; Aydi and Karapinar, 2012; Bhaskar and Lakshmikantham, 2006; Chandok, 2011; Chandok, 2012; Nieto and Lopez, 2005; Nieto and Rodriguez-Lopez, 2007; Ran and Reurings, 2004) and other spaces (Aydi *et al.*, 2012; Chandok *et al.*, 2013; Olatinwo and Postolache, 2012; George *et al.*, 2015).

Definition 1; Abbas *et al.* (2011): A pair (f, g) is named weakly increasing where, $f, g: X \rightarrow X$ and (X, \leq) is a partially ordered set, if $f(x) \leq g(x)$ and $g(x) \leq fg(x)$, $\forall x \in X$. A pair (f, g) is named partially weakly increasing if $(x) \leq f(x)$, $\forall x \in X$.

Definition 2; Abbas *et al.* (2011): A mapping $f: X \rightarrow X$ where (X, \leq) is a partially ordered set is named annihilator of a mapping $g: X \rightarrow X$ if $f g(x) \leq x$, $\forall x \in X$. And a mapping f is named a dominating mapping if $x \leq f(x)$, $\forall x \in X$. If every two elements in Y are comparable where $Y \subset X$, then Y is named well ordered. A point $x \in K$ is a common (coincidence) fixed point of self mappings of a metric space (X, ρ) if $Tx = Sx = x$ (i.e., $Tx = Sx = u$) where, $\emptyset \neq K \subset X$.

Definition 3; Abbas *et al.* (2011): A function $T: X \rightarrow X$ where (X, ρ) is a metric space are named compatible if:

$$\lim_{n \rightarrow \infty} \rho(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = z$ for some $z \in X$.

Definition 4; Jungck and Rhoades (1998): A functions S and T are named weakly compatible if they commute at coincidence points of S and T (that is $STx = TSx = u$).

Definition 5: "Let, Ψ denote the family of all non decreasing and continuous function $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ s.t. $\lim_{n \rightarrow \infty} \psi^n(t) = 0 \forall t > 0$ where ψ^n denotes the n th iterate of ψ . It is easy to show that for each $\psi \in \Psi$, the following is satisfied:

- $\psi(t) < t \quad \forall t > 0$
- $\psi(0) = 0$

Definition 6; Djoudi and Nisse (2003): A function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is named an altering distance function if the following properties are satisfied:

- $\varphi(r) = 0 \Leftrightarrow r = 0$
- φ is non-decreasing and continuous

By Roshan *et al.* (2016), introduced b-rectangular or b-generalized metric space (b-gms) as follows:

Definition 7; Roshan *et al.* (2016): A function $\rho: X \times X \rightarrow [0, \infty)$ on a non empty set X is a generalized metric or b-rectangular metric [b-gms] with parameter $s \geq 1$ if the triangle inequality in definition of a metric is replaced with the (b-rectangular inequality):

$$\rho(x, y) \leq s[\rho(x, u) + \rho(u, v) + \rho(v, y)] \text{ (b-rectangular inequality)} \tag{1}$$

For all distinct points $x, y, u, v \in X$. The pair (X, ρ) is named a b-rectangular or a b-generalized metric space (b-gms).

MATERIALS AND METHODS

Lemma 1; Roshan et al. (2016): “Let (X, ρ) be a (b-gms) and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.”

By Roshan et al. (2016) introduced almost generalized weakly contractive mappings. The purpose of this research is to extend the results of Roshan for four mappings in partially b-rectangular metric space.

RESULTS AND DISCUSSION

Now, we will prove our result.

Theorem 1: Let (X, \leq, ρ) be a complete ordered (b-gms) with parameter $s > 1$. Let, $I: X \rightarrow X$ be an increasing function in relation to \leq , suppose that I, J, S and T are self-mappings on X , the pairs (T, I) and (S, J) are a partially weakly increasing where I and J are dominating mappings and the mappings I and J are weak annihilators of T and S , respectively with $I(X) \subseteq T(X)$ and $J(X) \subseteq S(X)$. Further, assume that for any two comparable elements $u, v \in X$ and $\psi \in \Psi$ holds:

$$\left. \begin{aligned} \psi(sd(Iu, Jv)) &\leq \psi(M(u, v)) - \phi(M(u, v) + L\psi(N(u, v))) \\ \text{where} \\ M(u, v) &= \max\{\rho(Su, Tv), \rho(Su, Iu), \rho(Tv, Jv)\} \\ \text{and} \\ N(u, v) &= \min\{\rho(Su, Iu), \rho(Su, Jv), \rho(Tv, Iu), \rho(Tv, Jv)\} \end{aligned} \right\} (2)$$

If, for a non decreasing sequence $\{u_n\}$ where $u_n \leq v_n \forall n \geq 1$, $v_n \rightarrow t$ implies that $u_n \leq t$ and either: “ I and S are compatible, I or S is continuous and J, T are weakly compatible. J and T are compatible, J or T is continuous and I, S are weakly compatible, then I, J, S and T have a common fixed point in X . Moreover, the set of common fixed points of I, J, S and T is well ordered if and only if I, J, S and T have one and only one common fixed point in X ”.

Proof

Step I: We show that $\lim_{n \rightarrow \infty} \rho(v_n, v_{n+1}) = 0$. Let $u_0 \in X$ be an element, we construct the sequences $\{u_n\}$ and $\{v_n\}$ in X s.t.

$$v_{2n-1} = Iu_{2n-2} = Tu_{2n-1}, v_{2n} = Ju_{2n-1} = Su_{2n} \quad \forall n \geq 1 \quad (3)$$

By assumptions, we get:

$$\left. \begin{aligned} u_{2n-2} \leq Iu_{2n-2} = Tu_{2n-1} \leq ITu_{2n-1} \leq u_{2n-1}, \\ u_{2n-1} \leq Ju_{2n-1} = Su_{2n} \leq JSu_{2n} \leq u_{2n}, \quad \forall n \geq 1 \end{aligned} \right\} (4)$$

Then, $\forall n \geq 1$, we get $u_n \leq u_{n+1}$. Without loss of generality, we assume that $v_{2n} \neq v_{2n+1} \forall n \geq 1$. Now, we will prove that $\forall n \in \mathbb{N}$, we have:

$$\rho(v_{n+1}, v_{n+2}) < \rho(v_n, v_{n+1}) \quad (5)$$

By using the contrary that $\rho(v_{2n}, v_{2n+1}) \leq \rho(v_{2n+1}, v_{2n+2})$ for some $n \in \mathbb{N}$, since, v_{2n} and v_{2n+1} are comparable from (Eq. 2):

$$\begin{aligned} \psi(sp(v_{2n+1}, v_{2n+2})) &= \psi(sp(Iu_{2n}, Ju_{2n+1})) \leq \psi(M(u_{2n}, u_{2n+1})) - \\ &\phi(M(u_{2n}, u_{2n+1})) + L\psi(N(u_{2n}, u_{2n+1})) \end{aligned} \quad (6)$$

where:

$$\begin{aligned} M(u_{2n}, u_{2n+1}) &= \max\left\{ \begin{aligned} &\rho(Su_{2n}, Tu_{2n+1}), \rho(Su_{2n}, Iu_{2n}), \\ &\rho(Tu_{2n+1}, Ju_{2n+1}) \end{aligned} \right\} \\ &= \max\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2})\} \\ N(u_{2n}, u_{2n+1}) &= \max\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2})\} \end{aligned} \quad (7)$$

and:

$$\begin{aligned} N(u_{2n}, u_{2n+1}) &= \min\left\{ \begin{aligned} &\rho(Su_{2n}, Tu_{2n}), \rho(Su_{2n}, Ju_{2n+1}), \\ &\rho(Tu_{2n+1}, Iu_{2n}), \rho(Tu_{2n+1}, Ju_{2n+1}) \end{aligned} \right\} \\ &= \min\left\{ \begin{aligned} &\rho(v_{2n}, v_{2n+1}), \rho(v_{2n}, v_{2n+2}), \\ &\rho(v_{2n+1}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2}) \end{aligned} \right\} \\ N(u_{2n}, u_{2n+1}) &= \min\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2}), 0\} = 0 \end{aligned} \quad (8)$$

From Eq. 6 and 8 and definitions of ψ and ϕ , we have:

$$\begin{aligned} \psi(sp(v_{2n+1}, v_{2n+2})) &< \psi(\max\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2})\}) \\ \text{i.e., } \rho(v_{2n+1}, v_{2n+2}) &< \frac{1}{s} \max\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2})\} \end{aligned} \quad (9)$$

If $\max\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2})\} = \rho(v_{2n+1}, v_{2n+2})$ then, we have:

$$\rho(v_{2n+1}, v_{2n+2}) < \frac{\rho(v_{2n+1}, v_{2n+2})}{s}$$

This is contradiction. Hence, $\max\{\rho(v_{2n}, v_{2n+1}), \rho(v_{2n+1}, v_{2n+2})\} = \rho(v_{2n}, v_{2n+1})$, therefore, Eq. 9 is that:

$$\rho(v_{2n+1}, v_{2n+2}) < \frac{1}{s} \rho(v_{2n}, v_{2n+1}) \quad (10)$$

Is the sequence $\rho(v_n, v_{n+1})$ is decreasing. Then, $\exists \epsilon \geq 0$ s.t. $\lim_{n \rightarrow \infty} \rho(v_n, v_{n+1}) = \epsilon$. Suppose that, $\epsilon > 0$ taking $n \rightarrow \infty$ in Eq. 10, we have $\epsilon < \epsilon/s$ which is contradiction (since, $s > 1$). Hence, $\epsilon = 0$ that is:

$$\lim_{n \rightarrow \infty} \rho(v_n, v_{n+1}) = 0 \tag{11}$$

Step II: We will prove $\{v_n\}$ is a b-gms Cauchy sequence in X . Assume that $\{v_n\}$ is not a Cauchy sequence in X . Then, $\exists \epsilon > 0$, we can find two subsequences $\{v_{m_k}\}$ and $\{v_{n_k}\}$ of $\{v_n\}$ and of such that n_k is the smallest index:

$$n_k > m_k > k \text{ and } \rho(v_{m_k}, v_{n_k}) \geq \epsilon \tag{12}$$

This means that:

$$\rho(v_{m_k}, v_{n_k}) < \epsilon \tag{13}$$

From Eq. 13 and taking limsup as $k \rightarrow \infty$, we have:

$$\limsup_{k \rightarrow \infty} \rho(v_{m_k}, v_{n_k}) \leq \epsilon \tag{14}$$

On the other hand, we have:

$$\rho(v_{m_k}, v_{n_k}) \leq sp(v_{m_k}, v_{m_{k+1}}) + sp(v_{m_{k+1}}, v_{n_{k-1}}) + sp(v_{n_{k-1}}, v_{n_k})$$

Taking limsup as $k \rightarrow \infty$ and using Eq. 11 and 12, we have:

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow \infty} \rho(v_{m_{k+1}}, v_{n_{k-1}}) \tag{15}$$

Using Eq. 1 again, we get the following:

$$\rho(v_{m_k}, v_{n_k}) \leq sp(v_{m_k}, v_{n_{k-2}}) + sp(v_{n_{k-2}}, v_{n_{k-1}}) + sp(v_{n_{k-1}}, v_{n_k})$$

Using Eq. 11 and 12 and taking limsup as $k \rightarrow \infty$, we have:

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow \infty} \rho(v_{m_k}, v_{n_{k-2}}) \tag{16}$$

From Eq. 2, we get:

$$\begin{aligned} \psi(sp(v_{2m_k+1}, v_{2n_{k-1}})) &= \psi(sp(Iu_{2m_k}, Ju_{2n_{k-2}})) \leq \\ &\leq \psi(M(u_{2m_k}, u_{2n_{k-2}})) - \phi(M(u_{2m_k}, u_{2n_{k-2}})) + \\ &L\psi(N(u_{2m_k}, u_{2n_{k-2}})) \end{aligned} \tag{17}$$

Where:

$$M(u_{2m_k}, u_{2n_{k-2}}) = \max \left\{ \begin{aligned} &\rho(Su_{2m_k}, Tu_{2n_{k-2}}), \rho(Su_{2m_k}, Iu_{2m_k}), \\ &\rho(Tu_{2n_{k-2}}, Ju_{2n_{k-2}}) \end{aligned} \right\}$$

and:

$$N(u_{2m_k}, u_{2n_{k-2}}) = \min \left\{ \begin{aligned} &\rho(Su_{2m_k}, Iu_{2m_k}), \rho(Su_{2m_k}, Ju_{2n_{k-2}}), \\ &\rho(Tu_{2n_{k-2}}, Iu_{2m_k}), \rho(Tu_{2n_{k-2}}, Ju_{2n_{k-2}}) \end{aligned} \right\}$$

From Eq. 3, we have:

$$M(u_{2m_k}, u_{2n_{k-2}}) = \max \left\{ \begin{aligned} &\rho(v_{2m_k}, v_{2n_{k-2}}), \rho(v_{2m_k}, v_{2m_{k+1}}), \\ &\rho(v_{2n_{k-2}}, v_{2n_{k-1}}) \end{aligned} \right\} \tag{18}$$

and:

$$N(u_{2m_k}, u_{2n_{k-2}}) = \min \left\{ \begin{aligned} &\rho(v_{2m_k}, v_{2m_{k+1}}), \rho(v_{2m_k}, v_{2n_{k-1}}), \\ &\rho(v_{2n_{k-2}}, v_{2m_{k+1}}), \rho(v_{2n_{k-2}}, v_{2n_{k-1}}) \end{aligned} \right\} \tag{19}$$

Taking limsup in Eq. 18 and 19 and using Eq. 11 and 14, we have:

$$\limsup_{k \rightarrow \infty} M(u_{2m_k}, u_{2n_{k-2}}) = \max \left\{ \limsup_{k \rightarrow \infty} \rho(v_{2m_k}, v_{2n_{k-2}}), 0, 0 \right\} \leq \epsilon$$

So, we have:

$$\limsup_{k \rightarrow \infty} M(u_{2m_k}, u_{2n_{k-2}}) \leq \epsilon \tag{20}$$

and:

$$\limsup_{k \rightarrow \infty} N(u_{2m_k}, u_{2n_{k-2}}) = 0 \tag{21}$$

Similarly by taking liminf in Eq. 18 and using (Eq. 11 and 16), we have:

$$\frac{\epsilon}{s} \leq \liminf_{k \rightarrow \infty} M(u_{2m_k}, u_{2n_{k-2}}) \tag{22}$$

Now, taking $\lim_{x \rightarrow \infty} \sup$ in Eq. 17 and from Eq. 15, 20 and 21, we have:

$$\begin{aligned} \psi\left(s \cdot \frac{\epsilon}{s}\right) &\leq \psi\left(s \limsup_{k \rightarrow \infty} \rho(v_{2m_{k+1}}, v_{2n_{k-1}})\right) \leq \\ &\psi\left(\limsup_{k \rightarrow \infty} M(u_{2m_k}, u_{2n_{k-2}})\right) - \liminf_{k \rightarrow \infty} \phi\left(M(u_{2m_k}, u_{2n_{k-2}})\right) \\ &\leq \psi(\epsilon) - \liminf_{k \rightarrow \infty} \phi\left(M(u_{2m_k}, u_{2n_{k-2}})\right) \end{aligned}$$

Which implies that:

$$\varphi\left(\liminf_{k \rightarrow \infty} \left(M(u_{2m_k}, u_{2n_k-2}) \right)\right) = 0$$

Hence, $\liminf_{k \rightarrow \infty} \left(M(u_{2m_k}, u_{2n_k-2}) \right) = 0$, a contradiction with (Eq. 22). Thus, $\{v_n\}$ is a b-g.m.s. Cauchy sequenc in X . Since, X is complete, $\exists z \in X$ s.t. $v_{n_k} \rightarrow z \forall k \in \mathbb{N}$, therefore, we have:

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} v_{2n+1} &= \lim_{n \rightarrow \infty} Tu_{2n+1} = \lim_{n \rightarrow \infty} Iu_{2n} = z \\ \lim_{n \rightarrow \infty} v_{2n+2} &= \lim_{n \rightarrow \infty} Su_{2n+2} = \lim_{n \rightarrow \infty} Ju_{2n+1} = z \end{aligned} \right\} \quad (23)$$

Suppose that S is continuous. Since, I and S are compatible, we get:

$$\lim_{n \rightarrow \infty} ISu_{2n+2} = \lim_{n \rightarrow \infty} SJu_{2n+2} = Sz \quad (24)$$

Also $v_{2n+1} \leq Ju_{2n+1} = Su_{2n+2}$. Now, using Eq. 23 and 24, we get:

$$\left. \begin{aligned} \psi(sd(ISu_{2n}, Ju_{2n+1})) &\leq \psi(M(Su_{2n}, u_{2n+1})) - \\ \varphi(M(Su_{2n}, u_{2n+1})) + L\psi(N(Su_{2n}, u_{2n+1})) \end{aligned} \right\} \quad (25)$$

Where:

$$M(Su_{2n}, u_{2n+1}) = \max \left\{ \begin{aligned} &\rho(S^2u_{2n}, Tu_{2n+1}), \rho(S^2u_{2n}, ISu_{2n}), \\ &\rho(Tu_{2n+1}, Ju_{2n+1}) \end{aligned} \right\} \quad (26)$$

and:

$$N(Su_{2n}, u_{2n+1}) = \min \left\{ \begin{aligned} &\rho(S^2u_{2n}, ISu_{2n}), \rho(S^2u_{2n}, Ju_{2n+1}), \\ &\rho(Tu_{2n+1}, ISu_{2n}), \rho(Tu_{2n+1}, Ju_{2n+1}) \end{aligned} \right\} \quad (27)$$

Letting $n \rightarrow \infty$ in Eq. 25 and using Eq. 26 and 27, we get:

$$\psi(sp(Sz, z)) < \psi(\max\{\rho(Sz, z), \rho(Sz, Sz), \rho(z, z)\})$$

We get $\rho(Sz, z) < \rho(Sz, z)$, this is contradiction, since, $(s > 1)$ which implies that $Sz = z$. Hence, z is a fixed point of S :

$$\left. \begin{aligned} \psi(sp(Iz, Ju_{2n+1})) &\leq \psi(M(z, u_{2n+1})) - \\ \varphi(M(z, u_{2n+1})) + L\psi(N(z, u_{2n+1})) \end{aligned} \right\} \quad (28)$$

Where:

$$M(z, u_{2n+1}) = \max \left\{ \begin{aligned} &\rho(Sz, Tu_{2n+1}), \rho(Sz, Iz), \\ &\rho(Tu_{2n+1}, Ju_{2n+1}) \end{aligned} \right\} \quad (29)$$

and:

$$N(z, u_{2n+1}) = \min \left\{ \begin{aligned} &\rho(Sz, Iz), \rho(Sz, Ju_{2n+1}), \\ &\rho(Tu_{2n+1}, Iz), \rho(Tu_{2n+1}, Ju_{2n+1}) \end{aligned} \right\} \quad (30)$$

Letting $n \rightarrow \infty$ in Eq. 28 and using Eq. 29 and 30, we have:

$$\left. \begin{aligned} \psi(sp(Iz, z)) &< \psi(\max\{\rho(z, z), \rho(z, Iz), \rho(z, z)\}) - \\ \varphi(\max\{0, \rho(z, Iz), 0\}) + \psi \left(\min \left\{ \begin{aligned} &\rho(z, Iz), \rho(z, z), \\ &\rho(z, Iz), \rho(z, z) \end{aligned} \right\} \right) &< \psi(\rho(z, Iz)) \end{aligned} \right\}$$

We get, $sp(Iz, z) < \rho(z, Iz)$. We have $Iz = z$. Since, $I(X) \subseteq T(X)$, $\exists w \in X$ s.t. $Sz = z = Iz = Tw$, assume that $Jw \neq Tw$. Since, $z \leq Iz = Tw \leq ITw \leq w$ implies $z \leq w$ from (Eq. 2), we obtain:

$$\left. \begin{aligned} \psi(sp(Tw, Jw)) &= \psi(sp(Iz, Jw)) \leq \psi(M(z, w)) - \\ \varphi(M(z, w)) + L\psi(N(z, w)) \end{aligned} \right\} \quad (31)$$

where:

$$\left. \begin{aligned} M(z, uw) &= \max\{\rho(Sz, Tw), \rho(Sz, Iz), \rho(Tw, Jw)\} \\ &= \max\{0, 0, \rho(Tw, Jw)\} \\ &= \rho(Tw, Jw) \end{aligned} \right\} \quad (32)$$

and:

$$\left. \begin{aligned} N(z, w) &= \min\{\rho(Sz, Iz), \rho(Sz, Jw), \rho(Tw, Iz), \rho(Tw, Jw)\} \\ &= \min\{0, \rho(Sz, Jw), 0, \rho(Tw, Jw)\} \\ &= 0 \end{aligned} \right\} \quad (33)$$

Using Eq. 32 and 33 in Eq. 31, we get:

$$sp(Tw, Jw) < \rho(Tw, Jw)$$

which is contradiction. Therefor, $Tw = Jw$. Since, J and T are weakly compatible, hence, $Jz = JIz = JTz = TJz = Tz = Iz = z$. Thus, z is a coincidence point of T and J . Now, $u_{2n} \leq Iu_{2n}$ and $Iu_{2n} \rightarrow z$ implies $u_{2n} \leq z$. Thus, from Eq. 2, we get:

$$\left. \begin{aligned} \psi(sp(Iu_{2n}, Jz)) &\leq \psi(M(u_{2n}, z)) - \varphi(M(u_{2n}, z)) + L\psi(N(u_{2n}, z)) \end{aligned} \right\} \quad (34)$$

where:

$$M(u_{2n}, z) = \max\{\rho(Su_{2n}, Tz), \rho(Su_{2n}, Iu_{2n}), \rho(Tz, Jz)\} \quad (35)$$

and:

$$N(u_{2n}, z) = \min \left\{ \begin{array}{l} \rho(Su_{2n}, Iu_{2n}), \rho(Su_{2n}, Jz), \\ \rho(Tz, Iu_{2n}), \rho(Tz, Jz) \end{array} \right\} \quad (36)$$

Using Eq. 35 and 36 in Eq. 34 and taking limit as $n \rightarrow \infty$, we have:

$$\begin{aligned} sp(z, Jz) &< \max \{ \rho(z, Jz), \rho(z, z), \rho(Tz, Jz) \} = \\ &\max \{ \rho(z, Jz), 0, 0 \} sp(z, Jz) < \rho(z, Jz) \end{aligned}$$

This is contradiction, hence, $Jz = z$, therefore, we have $Iz = Jz = Sz = Tz = z$. In the same way, the result satisfies when (ii) be available. Now, assume that the set of common fixed points of T, S, I and J is well ordered. We shall show that common fixed point of T, S, I and J is single. Suppose that and $Tz_1 = Sz_1 = Iz_1 = Jz_1 = z_1$ and $Tz_2 = Sz_2 = Iz_2 = Jz_2 = z_2$ where $z_1 \neq z_2$. Then, from Eq. 2, we have:

$$\begin{aligned} \psi(sp(z_1, z_2)) &= \psi(sp(Iz_1, Jz_2)) \leq \psi(M(z_1, z_2)) - \\ &\phi(M(z_1, z_2)) + L\psi(N(z_1, z_2)) \end{aligned} \quad (37)$$

where:

$$\begin{aligned} M(z_1, z_2) &= \max \{ \rho(Sz_1, Tz_2), \rho(Sz_1, Iz_1), \rho(Tz_2, Jz_2) \} \\ &= \max \{ \rho(z_1, z_2), 0, 0 \} \\ &= \rho(z_1, z_2) \end{aligned} \quad (38)$$

and:

$$\begin{aligned} N(z_1, z_2) &= \min \{ \rho(Sz_1, Iz_1), \rho(Sz_1, Jz_2), \rho(Tz_2, Iz_1), \rho(Tz_2, Jz_2) \} \\ &= \min \{ 0, \rho(z_1, z_2), \rho(z_2, z_1), 0 \} \\ &= 0 \end{aligned} \quad (39)$$

Using Eq. 38 and 39 in Eq. 37, we get $sp(z_1, z_2) < \rho(z_1, z_2)$, a contradiction, hence, $z_1 = z_2$. On the contrary, if T, S, I and J only one common fixed point to have then the set of common fixed point of I, J, S and T being singleton is well ordered. This completes the proof. If $I = J$, then we get the following result:

Corollary 1: Let (X, \leq, ρ) be a complete ordered b -rectangular metric space with parameter $s > 1$. Assume that I, S and T are self-mappings on X , the pairs (T, I) and (S, I) are partially weakly increasing with $I(X) \subseteq T(X)$ and $I(X) \subseteq S(X)$ and the dominating mapping I is a weak annihilator of T and S . Further, suppose that exists the function $\psi \in \Psi$ such that for any two comparable elements $x, y \in X$:

$$\psi(sp(Iu, Jv)) \leq \psi(M(u, v)) - \phi(M(u, v)) + L\psi(N(u, v))$$

where:

$$M(u, v) = \max \{ \rho(Su, Tv), \rho(Su, Iu), \rho(Tv, Iv) \}$$

and:

$$N(u, v) = \min \{ \rho(Su, Iu), \rho(Su, Iv), \rho(Tv, Iu), \rho(Tv, Iv) \}$$

holds. If, for a non decreasing sequence $\{v_n\}$ with $u_n \leq v_n \forall n \geq 1, y_n \rightarrow v$ implies $u_n \leq v$ and either, “ I, S are compatible, I or S is continuous and I, T are weakly compatible or I, T are compatible, I or T is continuous and I, S are weakly compatible. Then I, S and T have a common fixed point in X .” If $S = T$, then, we get the following result:

Corollary 2: Let (X, \leq, ρ) be a complete ordered b -rectangular metric space with parameter $s > 1$. Assume that are self-mappings on X , the pairs (T, I) and (T, J) are partially weakly increasing with $I(X) \subseteq T(X)$ and $J(X) \subseteq T(X)$ and the dominating mappings I are J a weak annihilator of T . Further, suppose that there exists function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$:

$$\psi(sp(Iu, Jv)) \leq \psi(M(u, v)) - \phi(M(u, v)) + L\psi(N(u, v))$$

where:

$$M(u, v) = \max \{ \rho(Tu, Tv), \rho(Tu, Iu), \rho(Tv, Jv) \}$$

and:

$$N(u, v) = \min \{ \rho(Tu, Iu), \rho(Tu, Jv), \rho(Tv, Iu), \rho(Tv, Jv) \}$$

holds. If, for any a non decreasing sequence $\{u_n\}$ with $u_n \leq u_n \forall n \geq 1, v_n \rightarrow v$ implies that $u_n \leq v$ and either, “ I, T are compatible, I or T is continuous, J and T are weakly compatible or”, J, T are compatible, J or T is continuous and I, T are weakly compatible. Then I, J and T have a common fixed point in X .”

Corollary 3: Let (X, \leq, ρ) be a complete ordered b -rectangular metric space with parameter $s > 1$. Assume that T and I are self-mappings on X , the pair (T, I) is partially weakly increasing with $I(X) \subseteq T(X)$ and the dominating mapping I is a weak annihilator of T . Further, suppose that exists the function $\psi \in \Psi$ such that for any two comparable elements $x, y \in X$:

$$\psi(sp(Iu, Jv)) \leq \psi(M(u, v)) - \phi(M(u, v)) + L\psi(N(u, v))$$

where:

$$M(u, v) = \max\{\rho(Tu, Tv), \rho(Tu, Iv), \rho(Tv, Iv)\}$$

and:

$$N(u, v) = \min\{\rho(Tu, Iv), \rho(Tu, Iv), \rho(Tv, Iv), \rho(Tv, Iv)\}$$

holds. If, for a non decreasing sequence $e\{u_n\}$ with $u_n \leq v_n \forall n \geq 1$, $v_n \rightarrow v$ indicates that $x_n \leq v$, further I, T are compatible, I or T is continuous, I and T are weakly compatible. Then I and T have a common fixed point in X.

CONCLUSION

Our results improves some fixed point theorems for (ψ, ϕ) -contraction in the literature.

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