Differential Subordination and Superordination for Univalent Meromorphic Functions Involving Cho-Kwon-Srivastava Operator

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Abstract: Using a Cho-Kwon-Srivastava operator, we introduce and study some differential subordination and superordination results for univalent meromorphic functions in the punctured open unit disk U^* .

Key words: Univalent meromorphic function, subordination, super ordination, Cho-Kwon-Srivastava operator, admissible functions, investigating classes

INTRODUCTION

Let, $D(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $D[\sigma, n]$ be the subclass of $D(U)$ consisting of functions of the form $f(z) = \sigma + \sigma_n z^n + \sigma_{n+1} z^{n+1} + \dots$, with $D_0 = D[0, 1]$ and $D_1 = D[1, 1]$. Let f and g members of $D(U)$. The function f is said to be subordinate to g in U written $f < g$ or $f(z) < g(z)$ ($z \in U$), if there exists a Schwarz function $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g(w(z))$ ($z \in U$) (Bulboacă, 2005; Miller and Mocanu, 2000). In particular, if the function g is univalent in U , then, we have the following equivalence:

$$f(z) < g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U)$$

In order to prove the results, we shall need the following definition and theorem.

Definition (1.1); Miller and Mocanu (2000): Denote by Q the set of all functions that are analytic and injective on $\bar{U} \setminus E(q)$ where:

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty \right\}$$

And are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of Q for which $q(0) = \sigma$ be denoted by $Q(\sigma)$, $Q(0) = Q_0$ and $Q(1) = Q_1$.

Definition (1.2); Miller and Mocanu (2000): Let Ω be a set in $\mathbb{C} \setminus Q$ and let n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$ and:

$$\operatorname{Re} \left\{ \frac{t}{s} + 1 \right\} \geq k \operatorname{Re} \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, $k \geq n$. We write $\psi_1[\Omega, q]$ as $\Psi[\Omega, q]$. In particular when $q(z) = M \frac{Mz + \sigma}{M + \bar{\sigma}z}$ with $M > 0$ and $|\sigma| < M$, then $q(U) = U_M = \{w : |w| < M\}$, $q(0) = \sigma$, $E(q) = \emptyset$ and $q \in Q(\sigma)$. In this case, we set $\psi_n[\Omega, M, \sigma] = \Psi_n[\Omega, q]$ and in the special case when $\Omega = U_M$ the class is simply denoted by $\Psi_n[M, \sigma]$.

Definition (1.3); Miller and Mocanu (2003): Let Ω be a set in $\mathbb{C}$, $q \in D[\sigma, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi: \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$ and:

$$\operatorname{Re} \left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \operatorname{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\}$$

$z \in U$, $\zeta \in \partial U$, $m \geq n \geq 1$. We write $\psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

Theorem (1.4); Miller and Mocanu (2000): Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = \sigma$. If the analytic function $p(z) = \sigma + \sigma_n z^n + \sigma_{n+1} z^{n+1} + \dots$ ($z \in U$) satisfies the following inclusion relationship:

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$$

Then:

$$p(z) < q(z) (z \in U)$$

Theorem (1.5); Miller and Mocanu (2003): Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = \sigma$. If $p(z) \in Q(\sigma)$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U then:

$$\Omega \subset \left\{ \psi(p(z), zp'(z), z^2 p''(z); z) \right\}$$

Implies:

$$p(z) \prec q(z) (z \in U)$$

Let Σ denote the class of meromorphic functions $f(z)$ normalized by:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1)$$

Which are analytic in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. For function $f(z) \in \Sigma$ given by (Eq. 1) and $g(z) \in \Sigma$ defined by:

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (2)$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by:

$$(f * g)(z) = (g * f)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n a_n z^n \quad (3)$$

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by:

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} z^n \quad (4)$$

For $\beta \neq 0, -1, -2, \dots$ and $\alpha \in \mathbb{C} \setminus \{0\}$ where $\lambda_n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that:

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} {}_2F_1(1, \alpha, \beta; z)$$

Where:

$${}_2F_1(1, \alpha, \beta; z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Is the well-known Gaussian hypergeometric function. Let us put:

$$q_{\lambda, \mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} z^n, (\lambda > 0, \mu \geq 0)$$

Corresponding to the functions $\tilde{\phi}(\alpha, \beta; z)$ and $q_{\lambda, \mu}(z)$ and using the Hadamard product for $f(z) \in \Sigma$ we defined a linear operator $L(\alpha, \beta, \lambda, \mu)$ (Ghanim and Darus, 2011a, b) on Σ by:

$$L(\alpha, \beta, \lambda, \mu)f(z) = (f(z) * \tilde{\phi}(\alpha, \beta; z) * q_{\lambda, \mu}(z))$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} \sigma_n z^n \quad (5)$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava (2002, 2003); Liu (2001; 2003); Liu and Srivastava (2001; 2004a, b); Cho and Kim (2007). For a function $f(z) \in L(\alpha, \beta, \lambda, \mu)$ we define:

$$I_{\alpha, \beta, \lambda}^{\mu, 0} = L(\alpha, \beta, \lambda, \mu)f(z)$$

And for $k = 1, 2, \dots$:

$$I_{\alpha, \beta, \lambda}^{\mu, 0} = z \left(I_{\alpha, \beta, \lambda}^{k-1} L(\alpha, \beta, \lambda, \mu)f(z) \right)' + \frac{2}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} n^k \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} \sigma_n z^n \quad (6)$$

Not that if $n = \beta, k = 0$ the operator $I_{\alpha, n, \lambda}^{\mu, 0}$ have been introduced by Cho *et al.* (2004) for $\mu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It was known that the definition of the operator $I_{\alpha, n, \lambda}^{\mu, 0}$ was motivated essentially by Choi *et al.* (2002) for analytic functions which includes a simpler integral operator studied earlier by Noor (1999) and others Liu (2001); Liu and Noor (2002). Note also the operator $I_{\alpha, \beta}^{0, k}$ have been recently introduced and studied by Ghanim and Darus (2010a, b; 2011). To our best knowledge, the recent work regarding $I_{\alpha, n, \lambda}^{\mu, 0}$ was charmingly studied by Piejko and Sokol (2008). In the same direction, we will study for the operator $I_{\alpha, \beta, \lambda}^{\mu, k}$ given in Eq. 6. Now, it follows from Eq. 5 and 6 that:

$$z \left(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \right)' = \lambda I_{\alpha, \beta, \lambda}^{\mu, k} f(z) - (\lambda+1) I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \quad (7)$$

$$z \left(I_{\alpha, \beta, \lambda}^{\mu, k} f(z) \right)' = \alpha I_{\alpha+1, \beta, \lambda}^{\mu, k} f(z) - (\alpha+1) I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \quad (8)$$

SUBORDINATION RESULTS INVOLVING THE OPERATOR

Definition (2.1): Let Ω be a set in $\mathbb{C}$, $q(z) \in Q_1 \cap D[q(0), 1]$. The class of admissible functions $\phi_n[\Omega, q]$ consists of those function $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

Whenever:

$$u = q(\zeta), v = \frac{k\zeta q'(\zeta) + (\lambda+1)q(\zeta)}{\lambda}$$

$$\operatorname{Re} \left\{ \frac{\lambda^2 w - (\lambda+1)^2 u}{\lambda v - (\lambda+1)u} - 2\lambda \right\} \geq k \operatorname{Re} \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}$$

$$z \in U, \zeta \in \partial U \setminus E(q), k \geq 1$$

Theorem (2.2): Let $\phi \in \Phi_n[\Omega, q]$. If $f(z) \in A$ satisfies:

$$\{\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) : z \in U\} \subset \Omega \quad (9)$$

Then:

$$I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec q(z)$$

Proof: Define the analytic function $J(z)$ in U by:

$$J(z) = I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \quad (10)$$

In view of relation:

$$z(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z))' = \lambda I_{\alpha, \beta, \lambda}^{\mu, k} f(z) - (\lambda+1) I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \quad (11)$$

From Eq. 10 we have:

$$I_{\alpha, \beta, \lambda}^{\mu, k} f(z) = \frac{1}{\lambda} [zJ'(z) + (\lambda+1)J(z)] \quad (12)$$

Further, a simple computation shows that:

$$I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z) = \frac{1}{\lambda^2} [z^2 J''(z) + (2\lambda+3)zJ'(z) + (\lambda+1)^2 J(z)] \quad (13)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by:

$$u(r, s, t) = r, v(r, s, t) = \frac{1}{\lambda} [s + (\lambda+1)r]$$

$$w(r, s, t) = \frac{1}{\lambda^2} [t + (2\lambda+3)s + (\lambda+1)^2 r] \quad (14)$$

Let:

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi \left(r, \frac{1}{\lambda} [s + (\lambda+1)r], \frac{1}{\lambda^2} [t + (2\lambda+3)s + (\lambda+1)^2 r]; z \right) \quad (15)$$

The proof shall make use of theorem (Eq. 4). Using (Eq. 10, 12 and 13), from (Eq. 15), we obtain:

$$\psi(J(z), zJ'(z), z^2 J''(z); z) = \phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \quad (16)$$

Hence, (19) becomes:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) = \psi(J(z), zJ'(z), z^2 J''(z); z) \in \Omega \quad (17)$$

Note that:

$$\frac{t}{s} + 1 = \frac{\lambda^2 w - (\lambda+1)^2 u}{\lambda v - (\lambda+1)u} - 2\lambda$$

And since, the admissibility condition for $\phi \in \Phi_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in definition (1.2), hence, $\psi \in \Psi[\Omega, q]$ and by theorem (1.4) then $J(z) \prec q(z)$ or $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec q(z)$.

Theorem (2.3): Let $\phi \in \Phi_n[h, q]$ with $q(0) = 1$. If $f(z) \in A$ satisfies:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \prec h(z) \quad (18)$$

Then:

$$I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec q(z) (z \in U)$$

Our next result is an extension of theorem (2.2) to the case the behavior of $q(z)$ on ∂U is not known.

Corollary (2.4): Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_n[\Omega, q_p]$ for some $p \in (0, 1)$ where $q_p(z) = q(pz)$. If $f(z) \in A$ and:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \in \Omega$$

Then:

$$I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec q(z) (z \in U)$$

Proof: By theorem (2.2), we get $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec q(z)$. The result is now deduced from the following subordination relationship:

$$q_p(z) \prec q(z) (z \in U)$$

In the particular case $q(z) = Mz$, $M > 0$ and in view of definition (2.1), the class of admissible functions $\Phi_n[\Omega, q]$ denoted by $\Phi_n[\Omega, M]$ is described below.

Definition (2.5): Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_n[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that:

$$\phi \left(\text{Me}^{\theta}, \frac{1}{\lambda} [k + \lambda + 1] \text{Me}^{\theta}, \frac{1}{\lambda^2} [1 + [k(2\lambda + 3) + (\lambda + 1)^2] \text{Me}^{\theta}; z] \right) \notin \Omega \quad (19)$$

Whenever $z \in U$, $k \geq 1$, $\text{Re}(\text{Le}^{i\theta}) \geq (k-1)kM$ and $\theta \in \mathbb{R}$:

Corollary (2.6): Let $\phi \in \Phi_n[\Omega, M]$. If $f(z) \in A$ satisfy the following inclusion relationship:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \in \Omega$$

Then:

$$I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec Mz (z \in U)$$

In the special case $\Omega = q(U) = \{w: |w| < M\}$ the class $\Phi_n[\Omega, M]$ is simply denoted by $\Phi_n[M]$.

Corollary (2.7): Let $\phi \in \Phi_n[M]$. If $f(z) \in A$ satisfies the following inclusion relationship:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \prec M$$

Then:

$$|I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)| < M$$

Definition (2.8): Let Ω be a set in $\mathbb{C}$, $q(z) \in Q \cap D[q(0), 1]$. The class of admissible functions $\Phi_{n,1}[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever:

$$u = q(\zeta), v = \frac{1}{\lambda} \left\{ \frac{k\zeta q'(\zeta)}{q(\zeta)} + \lambda q(\zeta) \right\}, (q(\zeta) \neq 0) \\ \text{Re} \left\{ \frac{v[2(\lambda+1) + \lambda(w-v)]}{v-u} + \lambda(v-2u) \right\} \geq k \text{Re} \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\} \\ z \in U, \zeta \in \partial U \cap E(q), k \geq 1$$

Theorem (2.9): Let $\phi \in \Phi_{n,1}[\Omega, q]$. If $f(z) \in A$ satisfies:

$$\left\{ \phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right); z \in U \right\} \subset \Omega \quad (20)$$

Then:

$$\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \prec q(z)$$

Proof: Define the analytic function $J(z)$ in U by:

$$J(z) = \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \prec q(z) \quad (21)$$

Then, by using (1.7), we get:

$$\frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)} = \frac{1}{\lambda} \left\{ \frac{zJ'(z)}{J(z)} + \lambda J(z) \right\} \quad (22)$$

Differentiating logarithmically (Eq. 22) and further computations show that:

$$\frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)} = \frac{1}{\lambda} \left\{ \frac{-2(\lambda+1) \frac{zJ'(z)}{J(z)} + \lambda J(z) + \frac{1}{\frac{zJ'(z)}{J(z)} + \lambda J(z)}}{\left[\frac{zJ'(z)}{J(z)} + \frac{z^2 J''(z)}{J(z)} - \left(\frac{zJ'(z)}{J(z)} \right)^2 + \lambda zJ'(z) \right]} \right\} \quad (23)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by:

$$u(r, s, t) = r, \quad v(r, s, t) = \frac{1}{\lambda} \left\{ \frac{s}{r} + \lambda r \right\} w(r, s, t) = \frac{1}{\lambda} \left\{ -2(\lambda+1) + \frac{s}{r} + \lambda r + \frac{1}{\frac{s}{r} + \lambda r} \left[\frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r} \right)^2 + \lambda s \right] \right\} \quad (24)$$

Let:

$$\psi(r, s, t; z) = \phi(u, v, w; z)$$

$$\phi \left(r, \frac{1}{\lambda} \left\{ \frac{s}{r} + \lambda r \right\}, \frac{1}{\lambda} \left\{ \frac{-2(\lambda+1) + \frac{s}{r} + \lambda r + \frac{1}{\frac{s}{r} + \lambda r} \left[\frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r} \right)^2 + \lambda s \right]}{\left[\frac{s}{r} + \lambda r \right]} \right\}; z \right) \quad (25)$$

Using (Eq. 21, 22 and 23), from (Eq. 25), it follows that:

$$\psi(J(z), zJ'(z), z^2 J''(z); z) = \phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right)$$

Hence, (Eq. 20) $\psi(J(z), zJ'(z), z^2 J''(z); z) \in \Omega$. The proof is complete if it can be shown that the admissibility condition for $\phi \in \Phi_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in definition (1.2). For this purpose, note that:

$$\frac{t}{s} + 1 = \frac{v[2(\lambda+1) + \lambda(w-v)]}{v-u} + \lambda(v-2u)$$

Hence, $\psi \in \Psi[\Omega, q]$ and by theorem (1.4), we have $J(z) \prec q(z)$ or:

$$\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \prec q(z)$$

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω , the class $\Phi_{n,1}[h(U), q]$ is written as $\Phi_{n,1}[h, q]$. The following result is an immediate consequence of theorem (2.9).

Theorem (2.10): Let $\phi \in \Phi_{n,1}[h(U), q]$ with $q(0) = 1$, if $f(z) \in A$ satisfies the following inclusion relationship:

$$\phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right) \prec h(z)$$

Then:

$$\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \prec q(z)$$

In the particular case $q(z) = Mz$, $M > 0$. The class of admissible function $\Phi_{n,1}[\Omega, q]$, denoted by $\Phi_{n,1}[\Omega, M]$ is described below.

Definition (2.11): Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{n,1}[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ such that:

$$\phi \left(\frac{Me^{i\theta}}{\lambda} \left\{ k + \lambda Me^{i\theta} \right\}, \frac{1}{\lambda} \left\{ -2(\lambda+1) + k + \lambda Me^{i\theta} + \frac{1}{kM + \lambda M^2 e^{i\theta}} \right\}; z \right) \notin \Omega \quad (26)$$

Whenever and.

Corollary (2.12): Let $\phi \in \Phi_{n,1}[\Omega, M]$. If $f(z) \in A$ satisfies:

$$\phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right) \in \Omega$$

Then:

$$\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \prec Mz (z \in U)$$

In the special case $\Omega = q(U) = \{w: |q| \leq M\}$, the class $\Phi_{n,1}[\Omega, M]$ is defined by $\Phi_{n,1}[M]$ and corollary (2.12) takes follows for:

Corollary (2.13): Let $\phi \in \Phi_{n,1}[M]$. If $f(z) \in A$ satisfies:

$$\left| \phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-3, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}; z \right) \right| \leq M$$

Then:

$$\left| \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \right| \leq M$$

SUPERORDINATION RESULTS INVOLVING THE OPERATOR $I_{\alpha, \beta, \lambda}^{\mu, k} f(z)$

In this section we obtain differential super ordination for the operator $I_{\alpha, \beta, \lambda}^{\mu, k} f(z)$. For this purpose the class of admissible functions is given in the following definition.

Definition (3.1): Let Ω be a set in $\mathbb{C}$, $q(z) \in D[q(0), 1]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_n[\Omega, q]$ consists of those function $\phi: \mathbb{C}^3 \times \overline{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega$$

Whenever:

$$u = q(z), \quad v = \frac{zq'(z) + m(\lambda+1)q(z)}{m\lambda}$$

$$\operatorname{Re} \left\{ \frac{\lambda^2 w - (\lambda+1)^2 u}{\lambda v - (\lambda+1)u} \right\} \leq \frac{1}{m} \operatorname{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\}$$

$$z \in U, \zeta \in \partial U, m \geq 1$$

Theorem (3.2): Let $\phi \in \Phi'_n[\Omega, q]$. If $f(z) \in A$, $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \in \Omega_1$ and $\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z)$ is univalent in then:

$$\Omega \subset \left\{ \phi \left(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z \right) : z \in U \right\} \quad (27)$$

Implies:

$$q(z) \prec I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)$$

Proof : Let $J(z)$ define by (2.2) and $\psi(z)$ define by (2.8). Since, $\phi \in \Phi'_n[\Omega, q]$ from (2.2) and (3.1) we have:

$$\Omega \subset \left\{ \psi(J(z), zJ'(z), z^2J''(z); z) : z \in U \right\}$$

From (2.7), we see that the admissibility condition for $\phi \in \Phi'_n[\Omega, q]$ is equivalent to the admissibility condition for

ψ as given in definition (1.2). Hence, $\psi \in \Psi'[\Omega, q]$ and by and by theorem (1.4), we have $q(z) \prec J(z)$ or $q(z) \prec I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)$.

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω , the class $\Phi'_{n,1}[h(U), q]$ is written as $\Phi'_{n,1}[h, q]$. The following result is an immediate consequence of theorem (3.2).

Theorem (3.3): Let $q(z) \in D$, $h(z)$ is analytic on U and $\phi \in \Phi'_n[\Omega, q]$. If $f(z) \in A$, $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \in Q_1$ and:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z)$$

Is univalent in then:

$$h(z) \prec \phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \quad (28)$$

Implies:

$$q(z) \prec I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)$$

Combining theorem (2.3) and (3.3), we obtain the following sandwich theorem.

Corollary (3.4): Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $h_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_n[h_2, q_2] \cap \Phi_n[h_1, q_1]$. If $f(z) \in A$, $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \in D \cap Q_1$ and:

$$\phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z)$$

Is univalent in U then:

$$h_1(z) \prec \phi(I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z), I_{\alpha, \beta, \lambda}^{\mu, k} f(z), I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z); z) \prec h_2(z)$$

Implies:

$$q_1(z) \prec I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \prec q_2(z)$$

Definition (3.5): Let Ω be a set in $\mathbb{C}$, $q(z) \in D[q(0), 1]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{n,1}[\Omega, q]$ consists of those function $\phi: \mathbb{C}^3 \times \overline{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \notin \Omega$$

Whenever:

$$u = q(z), \quad v = \frac{1}{\lambda} \left\{ \frac{kzq'(z)}{mq(z)} + \lambda q(z) \right\}, \quad (q(z) \neq 0)$$

$$\operatorname{Re} \left\{ \frac{v \left[2(\lambda+1) + \lambda(w-v) \right] + \lambda(v-2u)}{v-u} \right\} \leq \frac{1}{m} \operatorname{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\}$$

$$z \in U, \quad \zeta \in \partial U \setminus E(q), \quad m \geq 1$$

Now, we will give the dual result of theorem (2.9) for differential superordination.

Theorem (3.6): Let $\phi \in \Phi_{n,1}[\Omega, q]$. If $f(z) \in A$, $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \in Q_1$ and:

$$\phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right)$$

Is univalent in U , then:

$$\Omega \subset \left\{ \phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right) : z \in U \right\} \quad (29)$$

Implies:

$$q(z) \prec \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}$$

Proof: Let $J(z)$ define by (2.13) and $\psi(z)$ define by. Since, $\phi \in \Phi'_{n,1}[\Omega, q]$ from (Eq. 25) and (29) we have:

$$\Omega \subset \{ \psi(J(z), zJ'(z), z^2J''(z); z) : z \in U \}$$

From (Eq. 25), we see that the admissibility condition for $\phi \in \Phi'_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in definition (1.3). Hence, $\psi \in \Psi'[\Omega, q]$ and by and by theorem (1.5), we have $q(z) \prec J(z)$ or:

$$q(z) \prec \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}$$

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω , the class $\Phi'_{n,1}[h(U), q]$ is written as $\Phi'_{n,1}[h, q]$. The following result is an immediate consequence of theorem (3.6).

Theorem (3.7): Let $q(z) \in D$, $h(z)$ is analytic on U and $\phi \in \Phi'_{n,1}[\Omega, q]$. If $f(z) \in A$, $I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z) \in Q_1$ and:

$$\phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right)$$

Is univalent in U , then:

$$h(z) \prec \phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right) \quad (30)$$

Implies:

$$q(z) \prec \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}$$

Combining theorem (2.10) and (3.7), we obtain the following sandwich-type theorem.

Corollary (3.8): Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $q_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{n,1}[h_2, q_2] \cap \Phi_{n,1}[h_1, q_1]$. If

$$f(z) \in A, \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \in D \cap Q_1 \text{ and:}$$

$$\phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right)$$

Is univalent in U then:

$$h_1(z) \prec \phi \left(\frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}, \frac{I_{\alpha, \beta, \lambda}^{\mu-2, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu-1, k} f(z)}; z \right) \prec h_2(z)$$

Implies:

$$q_1(z) \prec \frac{I_{\alpha, \beta, \lambda}^{\mu, k} f(z)}{I_{\alpha, \beta, \lambda}^{\mu+1, k} f(z)} \prec q_2(z)$$

CONCLUSION

These results are obtained by investigating classes of admissible functions. Some of the results in this study would provide extensions of those given in earlier works.

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