Weakly and Almost T-ABSO Fuzzy Submodules

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Abstract: Let, $\tilde{M}$ be a unitary R -module over R be a commutative ring with identity and let X be a fuzzy module of an R -module $\tilde{M}$. In this study, we present two concepts: the first concept is a weakly T-ABSO fuzzy submodule where a proper fuzzy submodule A of fuzzy module X of an R -module $\tilde{M}$ is called a weakly T-ABSO fuzzy submodule of X if whenever fuzzy singletons a_s, b_l of $R, x_v \in X, \forall s, l, v \in L$ and $0_1 \neq a_s b_l x_v \in A$ then either $a_s b_l \in (A :_R X)$ or $a_s x_v \in A$ or $b_l x_v \in A$. And the second concept is an almost T-ABSO fuzzy submodule where let R be an integral domain, X be fuzzy module of an R -module $\tilde{M}$ and A a proper fuzzy submodule of X . A is called an almost T-ABSO fuzzy submodule of X if for fuzzy singletons a_s, b_l of R and $x_v \in X$ with $a_s b_l x_v \in A - (A :_R X)A$, then either $a_s b_l \in (A :_R X)$ or $a_s x_v \in A$ or $b_l x_v \in A$. We study some basic properties and characterizations of weakly T-ABSO fuzzy submodules and almost T-ABSO fuzzy submodules. We present almost T-ABSO fuzzy submodules of X as a new generalization of T-ABSO fuzzy and weakly T-ABSO fuzzy submodules and relationships between them concepts are given.

Key words: T-ABSO fuzzy submodules, weakly T-ABSO fuzzy ideals, weakly T-ABSO fuzzy submodules, almost T-ABSO fuzzy ideal, almost T-ABSO fuzzy submodule, characterizations

INTRODUCTION

A prime submodule which play an important turn in the module theory over a commutative ring. This concept was generalized to prime fuzzy submodule by Rabi (2001). Sonmez *et al.* (2017) presented the concept of 2-absorbing fuzzy ideal which is a generalization of prime fuzzy ideal. Darani and Soheilnia (2012) presented the concept of 2-absorbing submodule where “a proper submodule N of $\tilde{M}$ is called 2-absorbing submodule of if whenever $a, b \in R, m \in \tilde{M}$ and $abm \in N$, then $a \in N$ or $b \in N$ or $m \in N$ or $ab \in (N : \tilde{M})$ ”. Hatam (2001) expand this concept where “let X be fuzzy module of an R -module $\tilde{M}$.”

A proper fuzzy submodule A of X is called T-ABSO fuzzy submodule if whenever a_s, b_l be F . Singletons of R and $x_v \in X, \forall s, l, v \in L$ such that $a_s b_l x_v \in A$ then either $a_s b_l \in (A :_R X)$ or $a_s x_v \in A$ or $b_l x_v \in A$ (Hatam, 2001). Presented the concept of a weakly prime fuzzy ideal while Badawi and Darani (2013) were studied the concept of a weakly 2-absorbing ideal where A proper ideal I of a commutative ring R is called a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$ (Badawi and Darani, 2013). A weakly prime submodule were presented by Atani and Farzalipour (2007) where “A proper submodule N of an R -module $\tilde{M}$ is called a weakly prime if for $a \in R$ and $m \in \tilde{M}$ with $0 \neq am \in N$, then either $m \in N$ or $a \in (N :_R \tilde{M})$ ”.

“Darani and Soheilnia (2012) were generalized of weakly prime submodule to weakly 2-absorbing submodule where” A proper submodule N of an R -module $\tilde{M}$ is called a weakly 2-absorbing of M if whenever $a, b \in R, m \in \tilde{M}$ and $0 \neq abm \in N$, then either $ab \in (N :_R \tilde{M})$ or $a \in N$ or $b \in N$ or $m \in N$ ” (Darani and Soheilnia, 2012). “A proper ideal I of R is said to be almost prime provided that $a, b \in R$ with $ab \in I^2$ imply that $a \in I$ or $b \in I$ ” (Bhatwadekar and Sharma, 2005) while A proper ideal I of R is said to be almost 2-absorbing ideal if whenever $a, b, c \in R$ with $abc \in I^2$, then either $ab \in I$ or $ac \in I$ or $bc \in I$ (Mohammad and Abu-Dawwas, 2016). Almost prime submodule studied by Khashan (2012) where “A proper submodule N of an R -module $\tilde{M}$ is called an almost prime submodule of $\tilde{M}$ if whenever $r \in R$ and $m \in \tilde{M}$ such that $rm \in N - (N : \tilde{M})N$, then either $m \in N$ or $r \in (N : \tilde{M})$ ”. Mohammad and Abu-Dawwas (2016) were generalization this notion to the almost 2-absorbing submodules where “let R be an integral domain, M be an R -module and N a proper submodule of $\tilde{M}$. N is called an almost 2-absorbing submodule of $\tilde{M}$ if $a, b \in R$ and $m \in \tilde{M}$ with $a bm \in N - (N : \tilde{M})N$ then either $ab \in (N : \tilde{M})$ or $a \in N$ or $b \in N$ ” (Mohammad and Abu-Dawwas, 2016).

In our study, we present the concepts of weakly prime fuzzy submodule, T-ABSO fuzzy submodule, weakly T-ABSO fuzzy ideal, weakly T-ABSO fuzzy submodule, almost T-ABSO fuzzy ideal and almost

T-ABSO fuzzy submodule and present a new basic properties, characterizations of these concepts and relationships between these concepts.

This study be composed of two sections: in section 1, we present and study the concept of weakly T-ABSO fuzzy ideal, weakly T-ABSO fuzzy submodule and we give many properties, characterizations and relationships between prime fuzzy submodule, weakly prime fuzzy submodule, T-ABSO fuzzy submodule and weakly T-ABSO fuzzy submodule.

In section 2, we present the concepts of almost prime fuzzy ideal, almost T-ABSO fuzzy ideal, almost prime fuzzy submodule and almost T-ABSO fuzzy submodule, so, many properties, characterizations and relationships between almost 2-absorbing submodule, T-ABSO fuzzy submodule and weakly T-ABSO fuzzy submodule are given. Note that, we denote to Fuzzy: F., Module: M., submodule: subm., $[0,1]$: L and otheriwise: o.w.

WEAKLY T-ABSO F. SUBM

In this section, we shall expand the concepts of weakly prime subm., weakly 2-absorbing ideal and weakly 2-absorbing subm. to weakly prime F. subm., weakly T-ABSO F. ideal, T-ABSO F. subm. and weakly T-ABSO F. subm and search some properties, characterizations and relations of weakly T-ABSO F. subm. with other concepts of F. subm. First, we shall fuzzify those concepts as follows:

Definition 2.1: A proper F. subm. A of FM X of an R-M $\tilde{M}$ is called weakly prime F. subm. if for F. singleton r_k of R and $x_v \subseteq X$ with $0_1 \neq r_k x_v \subseteq A$, then either $r_k \subseteq (A;_R X)$ or $x_v \subseteq A$ where:

$$0_1(y) = \begin{cases} 1 & y = 1 \\ 0 & y \neq 1 \end{cases}$$

The proposition specificates weakly prime F. subm. in terms of its level subm is given:

Proposition 2.2: Let A be F. subm. of FM X of an R-M $\tilde{M}$. Then A is a weakly F. subm. of X iff the level A_v is a weakly prime subm. of X_v , $\forall v \in L$.

Proof: ($\Rightarrow$) let $0 \neq ax \in \tilde{A}_v$ for each $a \in R$, $x \in X_v$, $\forall v \in L$, then $A(ax) \geq v$, hence $(ax)_v \subseteq A$, so that, $a_s x_k \subseteq A$ where $v = \min\{s, k\}$. But A is a weakly prime F. subm., then either $a_s \subseteq (A;_R X)$ or $x_k \subseteq A$, implies $a \in (A_v;_R X_v)$ or $x \in A_v$ where $(A;_R X)_v = (A_v;_R X_v)$ (Hatam, 2001). Thus A_v is a weakly prime subm. of X_v .

($\Leftarrow$) let $0_1 \neq a_s x_k \subseteq A$ for F. singleton a_s of R and $x_k \subseteq X$, $\forall s, k \in L$, then $0_1 \neq (ax)_v \subseteq A$ where $v = \min\{s, k\}$, hence,

$A(ax) \geq v$, so that, $ax \in A_v$. But A_v is a weakly prime subm., then either $a \in (A_v;_R X_v)$ or $x \in A_v$, hence, $a_s (A;_R X)$ or $x_k \subseteq A$, thus, A is weakly prime F. subm. of X.

Definition 2.3: A proper F. ideal I of acommutative Ring R is called weakly T-ABSO F. ideal if for F. singletons, a_s, b_l, r_k of R, $\forall s, l, k \in L$ such that $0_1 \neq a_s b_l x_k \subseteq I$, then either $a_s b_l \subseteq I$ or $a_s r_k \subseteq I$ or $b_l r_k \subseteq I$. The proposition specificates T-ABSO F. subm. in terms of its level subm. is given:

Proposition 2.5: Let A be T-ABSO F. subm. of F. M. X of an R-M $\tilde{M}$, if f the level subm. A_v is T-ABSO subm. of X_v , $\forall v \in L$ (Khalaf and Hannon, 2018).

Definition 2.6: A proper F. subm. A of F.M.X of an R-M $\tilde{M}$ is called a weakly T-ABSO F. subm. of X if whenever F. singletons a_s, b_l of R, $x_v \subseteq X$, $\forall s, l, v \in L$ and $0_1 \neq a_s b_l x_v \subseteq A$, then either $a_s b_l \subseteq (A;_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. The proposition specificates weakly T-ABSO F. subm. in terms of its level subm is given:

Proposition 2.7: Let A be F. subm. of F. M. X of an R-M $\tilde{M}$. Then A is a weakly T-ABSO F. subm. of X if f the level A_v is a weakly T-ABSO subm. of X_v , $\forall v \in L$.

Proof: By a similar on way to proof of proposition (2.5).

Remarks and examples 2.8:

- Prime F. subm. $\rightarrow$ weakly prime F. subm. $\rightarrow$ T-ABSO F. subm.
- Weakly prime F. subm. $\rightarrow$ weakly T-ABSO F. subm.
- T-ABSO F. subm. $\rightarrow$ weakly T-ABSO F. subm.

However, the converse incorrect, for example:

Let $X: Z_8 \rightarrow L$ such that:

$$X(y) = \begin{cases} 1 & \text{if } y \in Z_8 \\ 0 & \text{o.w.} \end{cases}$$

It is obvious that X be F. M. of Z-M Z_8 . Let, $A: Z_8 \rightarrow L$ such that:

$$A(y) = \begin{cases} v & \text{if } y \in (\bar{0}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$$

It is obvious that A is F. subm. of X. Now, $A_v = (\bar{0})$ and $X_v = Z_8$ as Z-M. where $A_v = (\bar{0})$ is not T-ABSO subm. since, $2.2.(\bar{2}) = (\bar{0})$ but $2.(\bar{2}) \neq (\bar{0})$ and $2.2 \notin (A_v;_R X_v) = 8_z$ while A_v is a weakly T-ABSO subm., so that, A is not T-ABSO F. subm. and it is weakly T-ABSO F. subm. (4) AF. subm. A is weakly prime F. subm. if f A is T-ABSO F. subm. and $(A;_R X)$ is a prime F. ideal. However, if A is T-ABSO F. subm. and $(A;_R X)$ is not a

prime F. ideal then A is not necessary weakly prime F. subm. for example: Let $X:Z \rightarrow L$ such that $x(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o.w.} \end{cases}$.

It is obvious that X be FM of Z-M Z. Let $A:Z \rightarrow L$ such that:

$$A(y) = \begin{cases} v & \text{if } y \in 4Z \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$$

It is obvious that A is F. subm. of X. Now, $A_v = 4Z$ and $X_v = Z$ as Z-M. where $A_v = 4Z$ is T-ABSO subm. since, $2.21 \in A_v = 4Z$ and $2.2 \in A_v$ but A_v is not weakly prime subm. since, $0 \neq 2.2 \in A_v$ but $2 \notin A_v$. So that, A is T-ABSO F. subm. but it is not weakly prime F. subm.

Theorem 2.9: Let R be a commutative ring and let X be F.M. of an R-M. $\tilde{M}$. Then the intersection of each pair of distinct weak prime F. subm. of X is weakly T-ABSO F. subm.

Proof: Let A and B be two distinct weak prime F. subm. of X. Suppose that F. singletons a_s, b_l of R and $x_v \in X$ such that $0_1 \neq a_s b_l x_v \in A \cap B$ but $0_1 \neq a_s x_v \notin A \cap B$ and $0_1 \neq b_l x_v \notin A \cap B$. Then $0_1 \neq a_s x_v \notin A$, $0_1 \neq b_l x_v \notin A$, $0_1 \neq a_s x_v \notin B$ and $0_1 \neq b_l x_v \notin B$ these are impossible, since, A and B are weak prime F. subm. So, suppose that $0_1 \neq a_s x_v \notin A$ and $0_1 \neq b_l x_v \notin B$. Since, $0_1 \neq a_s b_l x_v \in A$ and $0_1 \neq a_s b_l x_v \in B$, then $b_l \in (A:_{\tilde{R}} X)$ and $a_s \in (B:_{\tilde{R}} X)$. So that, $a_s b_l \in (A:_{\tilde{R}} X) \cap (B:_{\tilde{R}} X) = (A \cap B:_{\tilde{R}} X)$. Thus, $A \cap B$ is a weakly T-ABSO F. subm. of X.

Theorem 2.10: Let R be a commutative ring, X be F.M. of an R-M. $\tilde{M}$ and A be a weakly T-ABSO F. subm. of X. If A is not T-ABSO F. subm. then $(A:_{\tilde{R}} X)^2 A = 0_1$.

Proof: Suppose that $(A:_{\tilde{R}} X)^2 A \neq 0_1$. We will show that A is T-ABSO F. subm. Let $a_s b_l x_v \in A$ for F. singletons a_s, b_l of R and $x_v \in X$. If $a_s b_l x_v \neq 0_1$ then either $a_s b_l \in (A:_{\tilde{R}} X)$ or $a_s x_v \in A$ or $b_l x_v \in A$, since, A is a weakly T-ABSO F. So, suppose that $a_s b_l x_v = 0_1$. Let $a_s b_l A \neq 0_1$, say $a_s b_l y_h \neq 0_1$ for some F. singleton $y_h \in A$. Hence, $0_1 \neq a_s b_l y_h = a_s b_l (x_v + y_h) \in A$. Since, A is weakly T-ABSO F. subm., we have $a_s b_l \in (A:_{\tilde{R}} X)$ or $a_s (x_v + y_h) \in A$ or $b_l (x_v + y_h) \in A$. Then $a_s b_l \in (A:_{\tilde{R}} X)$ or $a_s x_v \in A$ or $b_l x_v \in A$. Thus, we may assume that $a_s b_l A = 0_1$. If $a_s x_v (A:_{\tilde{R}} X) \neq 0_1$, then there exists $r_k \in (A:_{\tilde{R}} X)$ such that $a_s r_k x_v \neq 0_1$. Hence, $0_1 \neq a_s r_k x_v = a_s (b_l + r_k) x_v \in A$. Since, A is weakly T-ABSO F. subm., we get $a_s (b_l + r_k) \in (A:_{\tilde{R}} X)$ or $a_s x_v \in A$ or $(b_l + r_k) x_v \in A$. Thus, $a_s b_l \in (A:_{\tilde{R}} X)$ or $a_s x_v \in A$ or $b_l x_v \in A$. So, we can assume that $a_s x_v (A:_{\tilde{R}} X) = 0_1$. By a similar on way, we can assume that $b_l x_v (A:_{\tilde{R}} X) = 0_1$. Since, $(A:_{\tilde{R}} X)^2 A \neq 0_1$, there exist $c_n, d_m \in (A:_{\tilde{R}} X)$ and $z_u \in A$ with $c_n d_m z_u \neq 0_1$. If $a_s d_m z_u \neq 0_1$, then $0_1 \neq a_s d_m z_u = a_s (b_l + d_m) (x_v + z_u) \in A$, hence, $a_s (b_l + d_m) \in (A:_{\tilde{R}} X)$ or $a_s (x_v + z_u) \in A$ or $(b_l + d_m) (x_v + z_u) \in A$. So that, $a_s b_l \in (A:_{\tilde{R}} X)$ or $a_s x_v \in A$ or $b_l x_v \in A$. Then, we can assume that $a_s d_m z_u = 0_1$. By in a similar way, we can assume that $c_n d_m$

$x_v = 0_1$ and $c_n b_l z_u = 0_1$. Hence, from $0_1 \neq c_n d_m z_u = (a_s + c_n) (b_l + d_m) (x_v + z_u) \in A$, we have $(a_s + c_n) (b_l + d_m) \in (A:_{\tilde{R}} X)$ or $(a_s + c_n) (x_v + z_u) \in A$ or $(b_l + d_m) (x_v + z_u) \in A$. Thus, $a_s b_l \in (A:_{\tilde{R}} X)$ or $a_s x_v \in A$ or $b_l x_v \in A$. Therefore, A is T-ABSO F. subm.

Recall that "A subm. N of an R-M $\tilde{M}$ is called a nilpotent subm. if $(N:_{\tilde{R}} M)^k N = 0$ for some $k \in \mathbb{Z}^+$ " (Ali, 2008). We shall fuzzify this concept as follows:

Definition 2.11: A F. subm. A of FM X of an R-M $\tilde{M}$ is called a nilpotent F. subm. if $(A:_{\tilde{R}} X)^n A = 0_1$ for some $n \in \mathbb{Z}^+$.

Corollary 2.12: Let R be a commutative ring and X be F. M. of an R-M. $\tilde{M}$. Suppose that A be a weakly T-ABSO F. subm. of X that is not T-ABSO F. subm., then:

- A is a nilpotent F. subm.
- If X is a multiplication FM then $A^3 = 0_1$

The definitions of multiplication F.M. (Hatam, 2001), faithful F.M. (Badawi and Darani, 2013), finitely generated and cancellation FM (Hadi and Hamil, 2011).

Lemma 2.13: Let A be F. subm. of a finitely generated faithful multiplication (and so cancellation) F.M. X of an R-M $\tilde{M}$. Then, we have $(IA:_{\tilde{R}} X) = I(A:_{\tilde{R}} X)$ for every F. ideal I of R.

Proof: Since, X is a multiplication FM then $I(A:_{\tilde{R}} X)X = IA = (IA:_{\tilde{R}} X)X$. So that, $(IA:_{\tilde{R}} X) = I(A:_{\tilde{R}} X)$, since, X is cancellation FM.

Proposition 2.14: Let, X be a faithful multiplication FM of an R-M $\tilde{M}$ and let A be a weakly T-ABSO F. subm. of X. If A is not T-ABSO F. subm. then $A \subseteq X-R(0_1)$.

Proof: Assume that A is not T-ABSO F. subm. By theorem (2.10), $(A:_{\tilde{R}} X)^2 A = 0_1$. By lemma (2.13), then $(A:_{\tilde{R}} X)^3 \subseteq ((A:_{\tilde{R}} X)^2 A:_{\tilde{R}} X) = (0_1:_{\tilde{R}} X) = 0_1$, since, X is faithful, hence $(A:_{\tilde{R}} X)^3 = 0_1$. If $r_k \in (A:_{\tilde{R}} X)$, then $^3_k \subseteq 0_1$ and so, $r_k \subseteq \sqrt{0_1}$. Hence $(A:_{\tilde{R}} X) \subseteq \sqrt{0_1}$. Thus, $A = (A:_{\tilde{R}} X)X \subseteq \sqrt{0_1} X = -R(0_1)$. The definition of a cyclic F.M. (Hatam, 2001).

Proposition 2.15: Let, R be a commutative ring, X be a faithful cyclic FM of an R-M $\tilde{M}$ and A be a weakly T-ABSO F. subm. of X then A is a weakly T-ABSO F. subm. of X if $f(A:_{\tilde{R}} X)$ is a weakly T-ABSO F. ideal of R.

Proof: ($\Rightarrow$) Let, $0_1 \neq a_s b_l r_k \in (A:_{\tilde{R}} X)$ for F. singletons a_s, b_l, r_k of R. Suppose that $a_s b_l \notin (A:_{\tilde{R}} X)$ and $b_l r_k \notin (A:_{\tilde{R}} X)$. Hence, $0_1 \neq a_s b_l r_k x_v \in A$ for F. singleton $x_v \in X$. If $a_s b_l r_k x_v = 0_1$, then $a_s b_l r_k \in (0_1:_{\tilde{R}} X) = 0_1$ this is impossible. Since, A is a weakly T-ABSO F. subm. and $a_s b_l \notin (A:_{\tilde{R}} X)$ and $b_l r_k \notin (A:_{\tilde{R}} X)$, then $a_s r_k \in (A:_{\tilde{R}} X)$. Thus, $(A:_{\tilde{R}} X)$ is a weakly T-ABSO F. ideal of R.

($\Leftarrow$) suppose that $(A;_R X)$ is a weakly T-ABSOF. ideal of R and let $0_1 \neq a_s b_1 x_s \subseteq A$ for F. singletons a_s, b_1 of R and $x_s \subseteq X$. Since, X is a cyclic F. M., then there exists F. singleton r_k of R with $x_v = r_k y_h$ for each F. singleton $y_h \subseteq X$. Hence, $0_1 \neq a_s b_1 r_k y_h \subseteq A$, then $0_1 \neq a_s b_1 r_k \subseteq (A;_R y_h) = (A;_R X)$. Since, $(A;_R X)$ is a weakly T-ABSOF. ideal, then either $a_s b_1 \subseteq (A;_R X)$ or $a_s r_k \subseteq (A;_R X)$ or $b_1 r_k \subseteq (A;_R X)$. Therefore, $a_s b_1 \subseteq (A;_R X)$ or $a_s x_s \subseteq A$ or $b_1 x_s \subseteq A$. Thus, A is weakly T-ABSOF. subm. Now, we give two lemmas which are needed in the next theorem.

Lemma 2.16: Let, A be a weakly T-ABSOF. subm. of F.M.X of an R-M. $\dot{M}$ and F. singletons a_s, b_1 of R. If for some F. subm. B of X $a_s b_1 B \subseteq A$ and $0_1 \neq 2 a_s b_1 B$, then $a_s b_1 \subseteq (A;_R X)$ or $a_s B \subseteq A$ or $b_1 B \subseteq A$.

Proof: Put $(A;_R X) = K$ and assume that $a_s b_1 \not\subseteq K$. Then it is sufficient to prove that $B \subseteq (A;_X a_s) \cup (A;_X b_1)$. Let r_k be an arbitrary F. singleton of B. If $0_1 \neq a_s b_1 r_k$ and $a_s b_1 \not\subseteq K$, then either $a_s r_k \subseteq A$ or $b_1 r_k \subseteq A$, since, A is a weakly T-ABSOF. subm. So that, $r_k \subseteq (A;_X a_s) \cup (A;_X b_1)$. Now, let $0_1 = a_s b_1 r_k$. Since, $0_1 \neq 2 a_s b_1 B$, then, $0_1 \neq 2 a_s b_1 x_s$ for some F. singleton $x_s \subseteq B$ and hence $0_1 \neq a_s b_1 x_s \subseteq A$. Since, A is a weakly T-ABSOF. subm. and $a_s b_1 \not\subseteq K$, then either $a_s x_s \subseteq A$ or $b_1 x_s \subseteq A$. Put $y_h = x_v + r_k$. Hence, $0_1 \neq a_s b_1 y_h \subseteq A$ and since, $a_s b_1 \not\subseteq K$, then either $a_s y_h \subseteq A$ or $b_1 y_h \subseteq A$. Now, we meditation three cases:

Case (1): $a_s x_s \subseteq A$ and $b_1 x_s \subseteq A$. Note that, $a_s y_h \subseteq A$ or $b_1 y_h \subseteq A$ and so, either $a_s r_k \subseteq A$ or $b_1 r_k \subseteq A$.

Case (2): $a_s x_s \subseteq A$ and $b_1 x_s \not\subseteq A$. On the contrary let $a_s r_k \not\subseteq A$. Hence, $a_s y_h \not\subseteq A$ and so, $b_1 y_h \subseteq A$. Thus, $a_s (y_h + x_s) \not\subseteq A$ and $b_1 (y_h + x_s) \not\subseteq A$. Since, A is a weakly T-ABSOF. subm. and $a_s b_1 \not\subseteq K$, then $0_1 = a_s b_1 (y_h + x_s) = 2 a_s b_1 x_s$, this is impossible. Thus, $a_s r_k \subseteq A$.

Case (3): $a_s x_s \not\subseteq A$ and $b_1 x_s \subseteq A$. Then, proof in a similar way case (2).

Lemma 2.17: Let I be F. ideal of R and A, B two F. subm. of F.M. X of an R-M such that $a_s IB \subseteq A$ where a_s be F. singleton of R. If A is a weakly T-ABSOF. subm. and $0_1 \neq 4 a_s IB$, then $a_s I \subseteq (A;_R X)$ or $a_s B \subseteq A$ or $IB \subseteq A$.

Proof: Let $a_s I \not\subseteq (A;_R X) = K$. Then $a_s b_1 \not\subseteq K$ for some F. singleton $b_1 \subseteq I$. We claim that there exists $r_h \subseteq I$ such that $0_1 \neq 4 a_s r_h B$ and $a_s r_h \not\subseteq K$. Since, $0_1 \neq 4 a_s IB$, then $0_1 \neq 4 a_s c_n B$ for some F. singleton $c_n \subseteq I$. If $a_s c_n \not\subseteq K$ or $0_1 \neq 4 a_s r_h B$, then by putting $r_h = c_n$ or $r_h = b_1$, we get the outcome. Therefore, let $a_s c_n \subseteq K$ and $4 a_s b_1 B = 0_1$. Then, $0_1 \neq 4 a_s (c_n + b_1) B \subseteq A$ and $a_s (c_n + b_1) \not\subseteq K$. So that, $r_h \subseteq I$ such that $0_1 \neq 4 a_s r_h B$ and $a_s r_h \not\subseteq K$. Then, $0_1 \neq 2 a_s r_h B$ and by lemma (2.16), we get $B \subseteq (A;_X a_s) \cup (A;_X r_h)$. If $a_s B \subseteq A$ there is nothing to prove. Therefore, suppose that $a_s B \not\subseteq A$ and hence, $r_h B \subseteq A$.

Now, we show that $I \subseteq (K;_R a_s) \cup (A;_R B)$. Let F. singleton $u_m \subseteq I$. If $0_1 \neq 2 a_s u_m B$, then by lemma (2.16), $a_s u_m \subseteq K$ or $a_s B \subseteq A$ or $u_m B \subseteq A$. But, we assumed $a_s B \not\subseteq A$, then $u_m \subseteq (K;_R a_s) \cup (A;_R B)$.

Now, suppose that $2 a_s u_m B = 0_1$. Hence, $0_1 \neq 2 a_s (r_h + u_m) B \subseteq A$ and by lemma (2.16), then either $a_s (r_h + u_m) \subseteq K$ or $a_s B \subseteq A$ or $(r_h + u_m) B \subseteq A$. Since, $a_s B \not\subseteq A$, so that, $(r_h + u_m) \subseteq (K;_R a_s) \cup (A;_R B)$. If $(r_h + u_m) \subseteq (A;_R B)$, then $u_m \subseteq (A;_R B)$ because $r_h \subseteq (A;_R B)$. Therefore, let $(r_h + u_m) \subseteq (K;_R a_s) \cup (A;_R B)$.

Meditation $2 a_s (r_h + u_m + r_h) B = 4 a_s r_h B \neq 0_1$ and $2 a_s (r_h + u_m + r_h) B \subseteq A$. Since, $a_s r_h \not\subseteq K$ and $a_s (r_h + u_m) \subseteq K$, then $a_s (r_h + u_m + r_h) \not\subseteq K$. Then by lemma (2.16), $B \subseteq (A;_X a_s) \cup (A;_X r_h + u_m + r_h)$. But, since, $r_h + u_m \not\subseteq (A;_R B)$ and $r_h \subseteq (A;_R B)$, then $(r_h + u_m + r_h) \not\subseteq (A;_R B)$, hence, $B \subseteq (A;_X a_s)$ this is impossible. Thus, $r_h + u_m \subseteq (A;_R B)$ and since $r_h \subseteq (A;_R B)$, then $u_m \subseteq (A;_R B)$. Then $I \subseteq (K;_R a_s) \cup (A;_R B)$. So that, $IB \subseteq A$, since, $a_s I \not\subseteq K$. The following theorem gives a characterization of weakly T-ABSOF. subm.

Theorem 2.18: Let I_1, I_2 be F. ideals of R and A, B be F. subm. of F. M. X of an R-M. $\dot{M}$. If A is a weakly T-ABSOF. subm., $0_1 \neq 1 I_2 B \subseteq A$ and $0_1 \neq 8 (I_1 I_2 + (I_1 + I_2) (A;_R X)) (B + A)$, then either $I_1 I_2 \subseteq (A;_R X)$ or $I_1 B \subseteq A$ or $I_2 B \subseteq A$.

Proof: Note that $0_1 \neq 8 (I_1 I_2 + (I_1 + I_2) (A;_R X)) (B + A) = 8 I_1 I_2 B + 8 I_1 I_2 A + 8 I_1 (A;_R X) B + 8 I_2 (A;_R X) B + 8 I_1 (A;_R X) A + 8 I_2 (A;_R X) A$. Therefore, one of the following various types is satisfied, $0_1 \neq 8 I_1 I_2 B$.

Hence, for some F. singleton $a_s \subseteq I_2$, we have $0_1 \neq 8 a_s I_1 B$. Thus, $0_1 \neq 4 a_s I_1 B$ and by lemma (2.17), then either $a_s I_1 \subseteq (A;_R X) = K$ or $a_s B \subseteq A$ or $I_1 B \subseteq A$. If $I_1 B \subseteq A$, then, we get the outcome. Therefore, we assume that $I_1 B \not\subseteq A$ and so, $a_s \subseteq (K;_R I_1) \cup (A;_R B)$. Now, we prove that $I_2 \subseteq (K;_R I_1) \cup (A;_R B)$. Let, $b_1 \subseteq I_2$. If $0_1 \neq 4 b_1 I_1 B$, then by lemma (2.17) and since, $I_1 B \not\subseteq A$, we have $b_1 \subseteq (K;_R I_1) \cup (A;_R B)$. Now, let $4 b_1 I_1 B = 0_1$, then $0_1 \neq 4 (a_s + b_1) I_1 B \subseteq A$. By lemma (2.17) and since, $I_1 B \not\subseteq A$ then $(a_s + b_1) \subseteq (K;_R I_1) \cup (A;_R B)$. We meditation the the following four cases:

Case 1: $(a_s + b_1) \subseteq (K;_R I_1)$ and $a_s \subseteq (K;_R I_1)$. Then $b_1 \subseteq (K;_R I_1)$.

Case 2: $(a_s + b_1) \subseteq (A;_R B)$ and $a_s \subseteq (A;_R B)$. Then, $b_1 \subseteq (A;_R B)$.

Case 3: $a_s \subseteq (K;_R I_1) / (A;_R B)$ and $(a_s + b_1) \subseteq (A;_R B) / (K;_R I_1)$. Then $(a_s + b_1 + a_s) \not\subseteq (K;_R I_1)$ and $(a_s + b_1 + a_s) \not\subseteq (A;_R B)$. So that, $(a_s + b_1 + a_s) \not\subseteq (K;_R I_1) \cup (A;_R B)$. We meditation $4(a_s + b_1 + a_s) I_1 B = 8 a_s I_1 B \neq 0_1$. By lemma (2.17) and since, $I_1 B \not\subseteq A$, then $(a_s + b_1 + a_s) \subseteq (K;_R I_1) \cup (A;_R B)$ this is impossible. Since, $a_s \subseteq (K;_R I_1) \cup (A;_R B)$ and $(a_s + b_1) \subseteq (K;_R I_1) \cup (A;_R B)$, one of the following holds:

- $a_s \subseteq (A;_R B)$ and $(a_s + b_1) \subseteq (A;_R B) / (K;_R I_1)$. Then $b_1 \subseteq (A;_R B)$
- $a_s \subseteq (K;_R I_1) / (A;_R B)$ and $(a_s + b_1) \subseteq (K;_R I_1)$. Then $b_1 \subseteq (K;_R I_1)$

Case 4: $(a_s + b_l) \subseteq (K; I_1) / (A; R; B)$ and $a_s \subseteq (A; R; B) / (K; I_1)$. By in a similar way of case (3), we have $b_l \subseteq (K; I_1) \cup (A; R; B)$. Thus, $I_2 \subseteq (K; I_1) \cup (A; R; B)$.

If $0_1 \neq 8I_1 I_2 A$ and $8I_1 I_2 B = 0_1$, then $0_1 \neq 8I_1 I_2 (B+A) \subseteq A$ and hence by part (1), $I_1 I_2 \subseteq (A; R; X)$ or $I_1 (B+A) \subseteq A$ or $I_2 (B+A) \subseteq A$. So that, $I_1 I_2 \subseteq (A; R; X)$ or $I_1 B \subseteq A$ or $I_2 B \subseteq A$.

Let, $0_1 \neq 8I_2 (A; R; X) B$ and $8I_1 I_2 B = 0_1$. Hence, $8I_2 (I_1 + (A; R; X)) B = 8I_2 (A; R; X) B \neq 0_1$. By part (1), then either $I_2 (I_1 + (A; R; X)) \subseteq (A; R; X)$ or $I_2 B \subseteq A$ or $(I_1 + (A; R; X)) B \subseteq A$, hence, $I_1 I_2 \subseteq (A; R; X)$ or $I_1 B \subseteq A$ or $I_2 B \subseteq A$. By in a similar way if, $0_1 \neq 8I_1 (A; R; X) B$, we get the outcome. Let $0_1 \neq 8I_2 (A; R; X) A$ and $8I_1 I_2 B = 8I_1 I_2 A = 8I_2 (A; R; X) B = 8I_1 (A; R; X) B = 0_1$.

Hence, $8I_2 (I_1 + (A; R; X)) (B+A) = 8I_2 (A; R; X) A \neq 0_1$ and by part (1), then $I_2 (I_1 + (A; R; X)) \subseteq (A; R; X)$ or $I_2 (B+A) \subseteq A$ or $(I_1 + (A; R; X)) (B+A) \subseteq A$. So that, $I_1 I_2 \subseteq (A; R; X)$ or $I_1 B \subseteq A$ or $I_2 B \subseteq A$. Obvious if $0_1 \neq 8I_1 (A; R; X)$, we get the outcome.

ALMOST T-ABSO F. SUBM.

In this section, we shall expand the concepts of almost prime subm, almost 2-absorbing ideal and almost 2-absorbing subm. to almost prime F. subm. almost T-ABSO F. ideal and almost T-ABSO F. subm. We present an almost T-ABSO F. subm. as a generalization of T-ABSO F. subm. and weakly T-ABSO F. subm. and study some basic properties, characterizations and relationships of almost T-ABSO F. subm., T-ABSO F. subm. and weakly T-ABSO F. subm. We shall fuzzify these concepts as follows:

Definition 3.1: A proper F. ideal I of R is said to be almost prime F. if whenever F. singletons a_s, b_l of R such that $a_s b_l \subseteq I - I^2$, then either $a_s \subseteq I$ or $b_l \subseteq I$.

Definition 3.2: A proper F. subm. A of F.M. X of an R-M is called an almost prime F. subm. of X if whenever F. singletons a_s of R and $x_v \subseteq X$ such that $a_s x_v \subseteq A - (A; R; X) A$ then either $a_s \subseteq (A; R; X)$ or $x_v \subseteq A$.

Definition 3.3: A proper F. ideal I of R is said to be almost T-ABSO F. ideal if whenever F. singletons a_s, b_l, r_k of R such that $a_s b_l r_k \subseteq I - I^2$, then either $a_s b_l \subseteq I$ or $a_s r_k \subseteq I$ or $b_l r_k \subseteq I$.

Definition 3.4: Let R be an integral domain, X be F.M. of an R-M $\tilde{M}$ and A a proper F. subm. of X . A is called an almost T-ABSO F. subm. of X if for F. singletons a_s, b_l of R and $x_v \subseteq X$ with $a_s b_l x_v \subseteq A - (A; R; X) A$, then either $a_s b_l \subseteq (A; R; X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. The proposition specificates an almost T-ABSO F. subm. in terms of its level subm. is given:

Proposition 3.5: Let A be almost T-ABSO F. subm. of F.M. X of an R-M $\tilde{M}$, iff the level subm. A_v is almost T-ABSO subm. of $X_v, \forall v \in L$.

Proof: $(\Rightarrow)$ let $abx \in A_v - (A_v; R; X_v) A_v$ for each $a, b \in R$ and $x \in X_v$, hence, $abx \in (A - (A; R; X) A)_v$ then $(A - (A; R; X) A) (abx) \geq v$, so, $(abx)_v \subseteq A - (A; R; X) A$ implies that where $v = \min \{s, l, k\}$. Since, A be almost T-ABSO F. subm., then either $a_s b_l \subseteq (A; R; X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Hence, $(ab)_v \subseteq (A; R; X)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$, so that, $ab \in (A_v; R; X_v)$ or $ax \in A_v$ or $bx \in A_v$ where $(A; R; X)_v = (A_v; R; X_v)$ by Hatam, (2001). Thus, A_v is T-ABSO subm. of X_v .

$(\Leftarrow)$ Let $a_s b_l x_k \subseteq A - (A; R; X) A$ for F. singletons a_s, b_l of R and $x_k \subseteq X, \forall s, l, k \in L$, hence, $(abx)_v \subseteq A - (A; R; X) A$ where $v = \min \{s, l, k\}$, so that, $(A - (A; R; X) A) (abx) \geq v$, implies $abx \in A_v - (A_v; R; X_v) A_v$ but A_v is almost T-ABSO subm., then either $ab \in (A_v; R; X_v)$ or $ax \in A_v$ or $bx \in A_v$, since, $(A_v; R; X_v) = (A; R; X)_v$, hence, $ab \in (A; R; X)_v$. Hence, either $(ab)_v \subseteq (A; R; X)$ or $(ax)_v \subseteq A$ or $(bx)_v \subseteq A$, implies either $a_s b_l \subseteq (A; R; X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Thus, A be almost T-ABSO F. subm. of X .

Remark 3.6: Every T-ABSO F. subm. is weakly T-ABSO F. subm. and every weakly T-ABSO F. subm. is almost T-ABSO F. subm. However the converse incorrect, for example:

Let $X: Z_{24} \rightarrow L$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z_{24} \\ 0 & \text{o.w.} \end{cases}$

It is obvious that X be F. M. of $Z-M. Z_{24}$.

Let $A: Z_{24} \rightarrow L$ such that $A(y) = \begin{cases} v & \text{if } y \in (\overline{12}) \\ 0 & \text{o.w.} \end{cases} \forall v \in L$

It is obvious that A is F. subm. of X . Now, $A_v = (\overline{12})$ and $X_v = Z_{24}$ as $Z-M.$, then $(A_v; R; X_v) A_v = 12 Z(\overline{12}) = (\overline{12})$. So that, A_v is an almost T-ABSO F. subm. but A_v is not T-ABSO F. subm. since, $2.2, (\overline{3}) \in A_v, 2, (\overline{3}) \notin A_v$ and $2.2 \notin (A_v; R; X_v)$ and hence, A_v is not weakly T-ABSO F. subm. Thus, A is almost T-ABSO F. subm. but it is not T-ABSO F. subm. and it is not weakly T-ABSO F. subm.

Proposition 3.7: Let X be FM of an R-M $\tilde{M}$ and A be F. subm. of X . Then the following expressions are equipollent:

- A is an almost T-ABSO F. subm. of X .
- For F. singletons a_s, b_l of R with $a_s b_l \subseteq R - (A; R; X)$, $(A; R; X) \langle a_s b_l \rangle = (A; R; X) \langle a_s \rangle \cup (A; R; X) \langle b_l \rangle \cup ((A; R; X) A; (A; R; X) \langle a_s b_l \rangle)$
- For F. singletons a_s, b_l of R with $a_s b_l \subseteq R - (A; R; X)$, $(A; R; X) \langle a_s b_l \rangle = (A; R; X) \langle a_s \rangle$ or $(A; R; X) \langle a_s b_l \rangle = (A; R; X) \langle b_l \rangle$ or $(A; R; X) \langle a_s b_l \rangle = ((A; R; X) A; (A; R; X) \langle a_s b_l \rangle)$

Proof: (1) $\Rightarrow$ (2) If $a_s b_l \subseteq R - (A; R; X)$ and $x_v \subseteq (A; R; X) \langle a_s b_l \rangle$, then $a_s b_l x_v \subseteq A$. But if $a_s b_l x_v \notin (A; R; X) A$, then $a_s b_l x_v \subseteq A - (A; R; X) A$. Hence, $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Thus, $x_v \subseteq (A; R; X) \langle a_s \rangle$ or $x_v \subseteq (A; R; X) \langle b_l \rangle$. (2) $\Rightarrow$ (3) straight forward since, if F. subm. equals to the union of two F. subm. then it is one of them. (3) $\Rightarrow$ (1) Let $a_s b_l x_v \subseteq A - (A; R; X) A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. Suppose that $a_s b_l \notin (A; R; X)$, we prove that $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. By (3) $(A; R; X) \langle a_s b_l \rangle = (A; R; X) \langle a_s \rangle$ or $(A; R; X) \langle a_s b_l \rangle = (A; R; X) \langle b_l \rangle$ or $(A; R; X) \langle a_s b_l \rangle = ((A; R; X) A; (A; R; X) \langle a_s b_l \rangle)$. Since,

$a_s b_l x_v \notin (A :_R X)A$, then $x_v \notin ((A :_R X)A :_X \langle a_s b_l \rangle)$. Thus, $x_v \notin (A :_X \langle a_s \rangle)$ or $x_v \notin (A :_X \langle b_l \rangle)$. Hence, $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Recall "If N is a subm. of R -M. $\dot{M}$ and $r \in R$ then a subm. N_r of $\dot{M}$ is defined by $N_r = (N : r) = \{m \in \dot{M} : rm \in N\}$ ". (Ashour *et al.*, 2016). We shall fuzzify this concept as follows:

Definition 3.8: Let A be F. subm. of F.M. X of an R -M. $\dot{M}$ and F. singleton a_s of R , then F. subm. A_{a_s} of X is defined by $A_{a_s} = (A :_X a_s) = \{x_v \in X : a_s x_v \subseteq A\}$.

Theorem 3.9: Let X be F.M. of an R -M. $\dot{M}$ and A be a proper F. subm. of X . The following expressions are equipollent:

- A is an almost T-ABSO F. subm
- For F. singletons $a_s b_l$ of R such that $a_s b_l \subseteq (A :_R X)$,
 $A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A :_R X)A)_{a_s b_l}$

Proof: (1) $\Rightarrow$ (2) Let A be an almost T-ABSO F. subm. and suppose that $a_s b_l \subseteq (A :_R X)$, let F. singleton $x_v \subseteq a_s b_l$, then $a_s b_l x_v \subseteq A$. If $a_s b_l x_v \notin (A :_R X)A$, hence, $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$, so that, $x_v \subseteq A_{a_s}$ or $x_v \subseteq A_{b_l}$. If $a_s b_l x_v \subseteq A - (A :_R X)A$, hence, $x_v \subseteq ((A :_R X)A)_{a_s b_l}$. So that, $A_{a_s b_l} \subseteq A_{a_s} \cup A_{b_l} \cup ((A :_R X)A)_{a_s b_l}$. Since, $A_{a_s} \cup A_{b_l} \cup ((A :_R X)A)_{a_s b_l} \subseteq A_{a_s b_l}$. Then, $A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A :_R X)A)_{a_s b_l}$.

(2) $\Rightarrow$ (1) Let F. singletons $a_s b_l$ of R and $x_v \subseteq X$ such that $a_s b_l x_v \subseteq A - (A :_R X)A$. Suppose that $a_s b_l \notin (A :_R X)$, then $x_v \subseteq A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A :_R X)A)_{a_s b_l}$ but $a_s b_l x_v \notin (A :_R X)A$, so that, $x_v \subseteq A_{a_s}$ or $x_v \subseteq A_{b_l}$. Then $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Thus, A be an almost T-ABSO F. subm. of X .

Proposition 3.10: Let X be FM of an R -M. $\dot{M}$ and A be a proper F. subm. of X , then A is an almost T-ABSO F. subm. in X if f for any F. singletons a_s, b_l of R and F. subm. B of X such that $a_s b_l B - \{0_1\} \subseteq A - (A :_R X)A$, implies that $a_s b_l \subseteq (A :_R X)$ or $a_s B \subseteq A$ or $b_l B \subseteq A$.

Proof: ($\Rightarrow$) Suppose that $a_s b_l \notin (A :_R X)A$, hence, $B \subseteq A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A :_R X)A)_{a_s b_l}$ but $a_s b_l B \notin (A :_R X)A$, so that, $B \subseteq A_{a_s}$ or $B \subseteq A_{b_l}$. Then $a_s B \subseteq A$ or $b_l B \subseteq A$.

($\Leftarrow$) Assume that $a_s b_l x_k \notin A - (A :_R X)$ for F. singletons a_s, b_l of R and $x_k \subseteq X$. Hence, $a_s b_l (x_k) - \{0_1\} \subseteq A - (A :_R X)A$, then $a_s b_l \subseteq (A :_R X)$ or $a_s (x_k) \subseteq A$ or $b_l (x_k) \subseteq A$. So that, $a_s b_l \subseteq (A :_R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$, thus, A is an almost T-ABSO F. subm. of X .

Theorem 3.11: Let X be a finitely generated faithful multiplication of an R -M. $\dot{M}$ and A be a proper subm. of X . The following expressions are equipollent:

- A is almost T-ABSO F. subm. in X
- $(A :_R X)$ is almost T-ABSO F. ideal in R
- $A = IX$ for some almost T-ABSO F. ideal I of R

Proof: (1) $\Rightarrow$ (2) Suppose A is almost T-ABSO F. subm. and let $a_s b_l x_k \subseteq A - (A :_R X) - (A :_R X)^2$ for F. singletons a_s, b_l, r_k of R . Hence, $a_s b_l r_k X - \{0_1\} \subseteq A - (A :_R X)A$. If $a_s b_l r_k X \subseteq (A :_R X)A$, then by lemma (2.13), $a_s b_l r_k \subseteq ((A :_R X)A :_R X) = (A :_R X)^2$ this is impossible. Since, A is almost T-ABSO F. subm., then either $a_s b_l \subseteq (A :_R X)$ or $a_s r_k X \subseteq A$ or $b_l r_k X \subseteq A$, so that, $a_s b_l \subseteq (A :_R X)$ or $a_s r_k \subseteq (A :_R X)$ or $b_l r_k \subseteq (A :_R X)$. Thus, $(A :_R X)$ is almost T-ABSO F. ideal in R .

(2) $\Rightarrow$ (1) Assume that $(A :_R X)$ is almost T-ABSO F. ideal in R and let $a_s b_l x_v \subseteq A - (A :_R X)A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. Hence, $a_s b_l ((x_v) :_R X) \subseteq ((a_s b_l x_v) :_R X) \subseteq (A :_R X)$. Also $a_s b_l ((x_v) :_R X) \notin (A :_R X)^2$ because if $((a_s b_l x_v) :_R X) \subseteq (A :_R X)^2 \subseteq ((A :_R X)A :_R X)$, hence, $a_s b_l (x_v) = a_s b_l ((x_v) :_R X)X$ this is impossible. Since, $(A :_R X)$ is almost T-ABSO F. ideal, then either $a_s b_l \subseteq (A :_R X)$ or $a_s ((x_v) :_R X) \subseteq (A :_R X)$ or $b_l ((x_v) :_R X) \subseteq (A :_R X)$. If $a_s ((x_v) :_R X) \subseteq (A :_R X)$, then $(a_s x_v) \subseteq a_s (x_v) = a_s ((x_v) :_R X)X \subseteq (A :_R X)X = A$, hence, $a_s x_v \subseteq A$. By in a similar way researchers get $b_l x_v \subseteq A$. So that, A is almost T-ABSO F. subm. in X . (2) $\Rightarrow$ (3) if we choose $I = (A :_R X)$, we get the outcome. The definition of maximal F. subm (Saifur, 2016).

Recall "A nonzero R -M.M is calld local if it has a largest proper subm (namely $\text{Rad}(M)$) that is its unique subm has a to be the radical (where $\text{Rad}(M)$ is an intersection of all maximal subm of M) (Clark *et al.*, 2006). Now, we shall expand this concept to local F.M. as follows:

Definition 3.12: A F.M. $X \neq 0_1$ of an R -M. $\dot{M}$ is called local F.M. if has a largest proper F. subm (namely $F-R(X)$) that is its unique maximal F. subm has to be the radical (where $F-R(X)$ is an intersection of all maximal F. subm of X).

Proposition 3.13: Let X be a local multiplication F.M. subm of an R -M. $\dot{M}$ with a unique maximal F. subm. K and $(K :_R X)K = 0_1$ then any proper F. subm of X is almost T-ABSO F. subm iff it is weakly T-ABSO F. subm.

Proof: ($\Rightarrow$) for any proper F. subm. A of X , $A \subseteq K(A :_R X)A = 0_1$, since, $(K :_R X)K = 0_1$. Suppose that $a_s b_l x_v \subseteq A - (A :_R X)A$ for F. Singletons a_s, b_l of R and $x_v \subseteq X$, then $0_1 \neq a_s b_l x_v \subseteq A$. Since, A is almost T-ABSO F. subm, then either $a_s b_l \subseteq (A :_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. So that, it is weakly T-ABSO F. subm. of X .

($\Leftarrow$) it is petty, since, every weakly T-ABSO F. Subm. Is almost T-ABSO F. subm. Now, we give two lemmas which are needed in the next theorem.

Lemma 3.14: Let X be F.M. of an R-M $\dot{M}$. A be an almost T-ABSO F. subm. of X and F. singletons a_s, b_i of R . If B is F. subm. of X such that $a_s, b_i \subseteq (A :_R X) A$ and $2 a_s b_i B \subseteq (A :_R X)$, then either $a_s b_i \subseteq (A :_R X)$ or $a_s B \subseteq A$ or $b_i B \subseteq A$.

Proof: By in a similar way to proof of lemma (2.16) but we replace $0_1 \neq a_s b_i r_k$ by $a_s b_i r_k \subseteq (A :_R X) A$ where $r_k \subseteq B$.

Lemma 3.15: Let, X be F.M. of an R-M. $\dot{M}$ A be an almost T-ABSO F. subm. Of X and F. singletons a_s, b_i of R . If I is F. ideal of R and B is F. subm. of X such that $a_i B \subseteq A$ and $4 a_i B \subseteq (A :_R X) A$ then either $a_i \subseteq (A :_R X)$ or $a_s B \subseteq A$ or $I B \subseteq A$.

Proof: By in a similar way to proof of lemma (2.17) but we replace $0_1 \neq 4 a_s I B$ by $4 a_s I B \subseteq (A :_R X) A$. The following theorem gives a characterization of almost T-ABSO F. subm.

Theorem 3.16: Let I_1, I_2 be F. ideals of R and A, B F. subm. of F.M. X of an R-M $\dot{M}$. If A is an almost T-ABSO F. subm. of X such that $I_1 I_2 B \subseteq A - (A :_R X) A$ and $8(I_1 I_2 + (I_1 + I_2)(A :_R X))(B + A) \subseteq (A :_R X)$ or $I_1 B \subseteq A$ or $I_2 B \subseteq A$.

Proof: By in a similar way to proof of theorem (2.18) but we replace $0_1 \neq 8(I_1 I_2 + (I_1 + I_2)(A :_R X))(B + A)$ by $8(I_1 I_2 + (I_1 + I_2)(A :_R X))(B + A) \subseteq (A :_R X) A$.

The product $AB = IJX$ where I, J are F. ideal of R and A, B are F. subm. of a multiplication FM of an R-M. $\dot{M}$ such that $A = IX$ and $B = JX$ (Atani and Farzalipour, 2007). By using this definition of products of F. subm, we give the following characterization of almost T-ABSO F. subm. under classes a finitely generated faithful multiplication FM.

Theorem 3.17: Let X be a finitely generated faithful multiplication F.M. of and R-M. $\dot{M}$ and A be a proper F. subm of X , then A is almost T-ABSO S. subm. in X iff whenever B, K and H are F. subm. of X such that $BKH - \{0_1\} \subseteq A - (A :_R X) A$, then either $BK \subseteq A$ or $BH \subseteq A$ or $KH \subseteq A$.

Proof: ($\Rightarrow$) Assume that A is almost T-ABSO F. subm. in X . by theorem (3.11), then $(A :_R X)$ is a almost T-ABSO F. ideal in R . We have $B = (B :_R X) X$, $K = (K :_R X) X$ and $H = (H :_R X) X$. Hence, $BKH = (B :_R X) (K :_R X) (H :_R X) X$. Assume that $BKH - \{0_1\} \subseteq A - (A :_R X) A$ but $BK \not\subseteq A$, $BH \not\subseteq A$ and $KH \not\subseteq A$.

So that, $(B :_R X) (K :_R X) \subseteq (A :_R X)$, $(B :_R X) (H :_R X) \subseteq (A :_R X)$ and $(K :_R X) (H :_R X) \subseteq (A :_R X)$. Since, $(A :_R X)$ is almost T-ABSO F. ideal in R , then $(B :_R X) (K :_R X) (H :_R X) \subseteq (A :_R X)$ or $(B :_R X) (K :_R X) (H :_R X) \subseteq (A :_R X)^2$.

If $(B :_R X) (K :_R X) (H :_R X) \subseteq (A :_R X)$, then $BKH = (B :_R X) (K :_R X) (H :_R X) X \subseteq BKH = (B :_R X) (K :_R X) (H :_R X) X$.

$X \subseteq (A :_R X)^2$ then $BKH = (B :_R X) (K :_R X) (H :_R X) X \subseteq (A :_R X)^2$. $X = (A :_R X) A$ this is impossible. Now, if $(B :_R X) (K :_R X) (H :_R X) \subseteq (A :_R X)^2$, then thus, $BK \subseteq A$ or $BH \subseteq A$ or $KH \subseteq A$.

($\Leftarrow$) To show that A is almost T-ABSO F. subm. in X , by theorem (3.11), it is sufficoent to show that $(A :_R X)$ is almost T-ABSO F. ideal in R . Let $a_s b_i r_k \subseteq (A :_R X) - (A :_R X)^2$ for F. singletons a_s, b_i, r_k of R . Hence, $a_s b_i r_k X - \{0_1\} \subseteq A - (A :_R X) A$. Put $a_s X = B$, $b_i X = K$ and $r_k X = H$, we have $BKH - \{0_1\} \subseteq A - (A :_R X) A$. By assumption, $BK \subseteq A$ or $BH \subseteq A$ or $KH \subseteq A$, so that, $a_s b_i X = K \subseteq A$ or $a_i r_k X \subseteq A$ or $b_i r_k X \subseteq A$. Then $a_s b_i \subseteq (A :_R X)$ or $a_i r_k \subseteq (A :_R X)$ or $b_i r_k \subseteq (A :_R X)$. Thus, $(A :_R X)$ is almost T-ABSO F. ideal in R , so that, A is almost T-ABSO F. subm. in X .

Corollary 3.18: Let A be a proper F. subm. of a finitely generated faithful multiplication F.M. X of an R-M. $\dot{M}$, then A is almost T-ABSO F. subm. in X iff whenever F. singletons x_v, y_h, z_m such that $x_v y_h z_m \subseteq A - (A :_R X) A$, then either $x_v, y_h \subseteq A$ or $x_v, z_m \subseteq A$ or $y_h z_m \subseteq A$.

Theorem 3.19: Let X be F.M. of an R-M. $\dot{M}$ and A be an almost T-ABSO F. subm. of X . Assume that F. singletons a_s, b_i of R and $x_v \subseteq X$ such that $a_s b_i x_v \subseteq (A :_R X) A$, $a_s b_i \subseteq (A :_R X)$, $a_s x_v \not\subseteq A$, $b_i x_v \not\subseteq A$. Then, $a_s b_i A \subseteq (A :_R X) A$.

Proof: Assume that $a_s b_i A \subseteq (A :_R X) A$. Hence, there exists $y_h \subseteq A$ such that $a_s b_i y_h \subseteq (A :_R X) A$, so that, $a_s b_i (x_v + y_h) \subseteq A - (A :_R X) A$. Since, A is an almost T-ABSO F. subm., then $a_s b_i A \subseteq (A :_R X)$ or $a_s (x_v + y_h)$ or $b_i (x_v + y_h) \subseteq A$, hence, $a_s b_i \subseteq (A :_R X)$ or $a_s x_v \subseteq A$ or $b_i x_v \subseteq A$ this is a discrepancy. Thus, $a_s b_i A \subseteq (A :_R X) A$.

Proposition 3.20: Let X be F.M. of an R-M $\dot{M}$, Y be an any F.M. of an R-M. $\dot{M}'$ and A be F. subm. of X . Then, A is an almost T-ABSO F. subm. of X iff $A \oplus Y$ is an almost T-ABSO F. subm. of $X \oplus Y$.

Proof: ($\Rightarrow$) Assume that A is an almost T-ABSO F. subm. of X . Let F. singletons $a_s b_i$ of R and $(x_v, y_h) \subseteq X \oplus Y$ such that $a_s b_i (x_v, y_h) \subseteq (A \oplus Y) - (A \oplus Y :_{R \oplus Y} X \oplus Y) (A \oplus Y)$, hence, $a_s b_i x_v \subseteq A - (A :_R X) A$ by $(A \oplus Y :_{R \oplus Y} X \oplus Y) = (A :_R X)$. Since, A is an almost T-ABSO F. subm. of X , then $a_s b_i \subseteq (A :_R X)$ or $a_s x_v \subseteq A$ or $b_i x_v \subseteq A$, so that, $a_s b_i \subseteq (A \oplus Y :_{R \oplus Y} X \oplus Y)$ or $a_s (x_v, y_h) \subseteq A \oplus Y$ or $b_i (x_v, y_h) \subseteq (A \oplus Y)$. Thus, $A \oplus Y$ is an almost T-ABSO F. subm. of $X \oplus Y$.

($\Leftarrow$) Assume that $A \oplus Y$ is an almost T-ABSO F. subm. of $X \oplus Y$. Let f. singletons $a_s b_i$ of R and $x_v \subseteq X$ such that $a_s b_i x_v \subseteq A - (A :_R X) A$. Hence, $a_s b_i (x_v, 0_1) \subseteq (A \oplus Y) - (A \oplus Y :_{R \oplus Y} X \oplus Y) (A \oplus Y)$. Since, $A \oplus Y$ is an almost T-ABSO F. subm. of $X \oplus Y$ that $a_s b_i \subseteq (A \oplus Y :_{R \oplus Y} X \oplus Y)$ or $a_s (x_v, 0_1) \subseteq A \oplus Y$ or $b_i (x_v, 0_1) \subseteq A \oplus Y$ that is $a_s b_i \subseteq (A :_R X)$ or $a_s x_v \subseteq A$ or $b_i x_v \subseteq A$. Thus, A is an almost T-ABSO F. subm. of X .

CONCLUSION

Through our research, we concluded to the concept F. prime subm. lead to the concept weakly F. prime subm. and through this we reached the concept weakly T-ABSO F. subm. One of the most important conclusions is the theorem (2.18). The other conclusion reached is almost T-ABSO F. subm. which was reached if A is a proper F. subm. of F. M. X then A is almost T-ABSO F. subm. in X iff is almost T-ABSO F. ideal and $A = IX$ for some almost T-ABSO F. ideal I of R.

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