Convergence Weibull Distribution to Normal Distribution by using Differential Geometry

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Abstract: In this research, we use the differential geometry to show that Weibull distribution converges to normal distribution by connecting between differential geometry and statistics. We apply the following equation:

\[ k = \frac{1}{\sqrt{E(G)}} \left( \frac{\partial}{\partial \alpha} \left( \frac{1}{E(G)} \right) \right) \frac{\partial}{\partial \alpha} \left( \frac{1}{E(G)} \right) \]

to calculate the Gaussian curvature (k) for Weibull distribution in the case of parametric lines are not orthogonal and show that if it is convergent to normal distribution by comparing the value of Gaussian curvature for Weibull distribution with the value of Gaussian curvature for the normal distribution.

Key words: Calculate, Gaussian, weibull, geometry, curvature, comparing

INTRODUCTION

This research present some interesting connections between statistics and differential geometry. Also, it contains the result of computing the Gaussian curvature of Weibull distribution in the case of parametric lines are not orthogonal. If the Gaussian curvature of Weibull distribution approaches to the Gaussian curvature of the normal distribution, we say that the distribution converges to normal distribution.

There are some researchers who worked in this field in end of twenty century and beginning of twenty one century. Kass and Vos (1997) used the following equation:

\[ \frac{1}{\sqrt{E(G-F^2)}} \left( \frac{\partial}{\partial u} \frac{\partial}{\partial v} \left( \frac{1}{E(G-F^2)} \right) \right) \]

to compute the Gaussian curvature of trinomial and t families. Chen (2014) used the equation:

\[ \frac{1}{4 \sqrt{E(G-F^2)}} \left( \frac{\partial}{\partial u} \frac{\partial}{\partial v} \left( \frac{1}{E(G-F^2)} \right) \right) \]

to compute the Gaussian curvature of gamma family of distributions in the case of parametric lines are orthogonal (F = 0) and showed that Gamma distribution converges to the normal distribution.

Metric Tensor for Weibull distribution (Barndorff-Nielsen et al., 1986; Kass and Vos, 1997; O’ Neill, 1966; Wikipedia, 2005): A random variable x has a Weibull distribution if its probability density function of form:

\[ f(x, u, v) = \frac{u x^{u-1} \left( \frac{1}{v} \right)^u}{v^u}, \quad x > 0 \]

Where:

\[ v \] : Scale parameter
\[ u \] : Shape parameter

\[ E = E \left( \frac{\partial^2 \ln f}{\partial u^2} \right) = \frac{u}{u^2} + \frac{\Gamma^{(2)}(\alpha)}{\Gamma^{(1)}(\alpha)} = \frac{1}{u^2} + \frac{\alpha}{u}, \quad \alpha > 0 \]
\[
F = E \left( \frac{\partial^2 \ln f}{\partial u \partial v} \right) = \frac{1}{v} \frac{\partial}{\partial v} \left( \frac{x}{v} \right) - \frac{u}{v} \frac{\partial}{\partial u} \left( \frac{x}{v} \right),
\]

\[
G = E \left( \frac{\partial^2 \ln f}{\partial v^2} \right) = \left( \frac{u}{v} \right)^2.
\]

The Gaussian curvature of the probability distribution (Do Carmo, 1976): First, we define the six well known Christoffel symbols as:

\[
\Gamma_{ii}^i = \frac{E_{x_i} - 2F_{x_i} + F V}{2(EG-F^2)}, \quad \Gamma_{ij}^i = \frac{E_{x_i x_j} - 2F_{x_i x_j} + F_{x_j} V}{2(EG-F^2)},
\]

\[
\Gamma_{i}^j = \frac{2E_{x_i} - E_{x_j} - E_{x_j} G_{x_i}}{2(EG-F^2)}, \quad \Gamma_{ii}^j = \frac{2G_{x_i} - G_{x_i} - G_{x_i} F_{x_i}}{2(EG-F^2)} \quad \Gamma_{ij}^j = \frac{2F_{x_i} - F_{x_j} - F_{x_j} G_{x_i}}{2(EG-F^2)}.
\]

Since, E, F and G are functions of parameters (u, v). Now, we select four formulas that can be used to compute the Gaussian curvature of the distributions:

\[
K = \frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right],
\]

\[
K = -\frac{1}{2\sqrt{EG-F^2}} \left[ \frac{\partial}{\partial u} \left( \frac{G_{x_i} F_{x_j} - E_{x_i} E_{x_j}}{\sqrt{EG-F^2}} \right) \frac{\partial}{\partial v} \left( \frac{E_{x_i} F_{x_j} - G_{x_i} G_{x_j}}{\sqrt{EG-F^2}} \right) \right] - \frac{1}{4(EG-F^2)} \left| \begin{array}{ccc}
E_{x_i} & F_{x_i} & G_{x_i} \\
E_{x_j} & F_{x_j} & G_{x_j}
\end{array} \right|.
\]

\[
K = \frac{1}{D \sqrt{G}} \left[ \frac{\partial}{\partial u} \left( \frac{D_{x_i} \sqrt{E}}{G} \right) + \frac{\partial}{\partial v} \left( \frac{D_{x_i} \sqrt{E}}{G} \right) \right]
\]

\[
K = \frac{R_{1212}}{EG-F^2} = \frac{(12,12)}{EG-F^2}
\]

where, \( D^2 = EG-F^2 \)

\[
K = \frac{R_{1212}}{EG-F^2} = \frac{(12,12)}{EG-F^2}
\]

where \((12,12) = R_{1212} = \sum_{m=1}^{\infty} R_{m}^{m} g_{m2} \)

\[
R_{m}^{m} = \frac{\partial}{\partial u} \left( \Gamma_{i}^{j} \frac{\partial}{\partial u} \right) - \frac{\partial}{\partial u} \left( \Gamma_{i}^{j} \right) + \frac{\partial}{\partial v} \left( \Gamma_{i}^{j} \right) - \frac{\partial}{\partial v} \left( \Gamma_{i}^{j} \right), \text{ sum on } m
\]

where, the quantities of \( R_{ijk} \) are components of a tensor of the fourth order. Notice that \( g_{11}, g_{12}, \text{ and } g_{22} \) are simply tensor notation for E, F and G.

**Remark:** We can not use the equation (A) to compute the Gaussian curvature of Weibull distribution because we take the case that the parametric lines us v of this distribution are not orthogonal (F = 0). The Gaussian curvature for Weibull distribution: We use the equation (C) to compute the Gaussian curvature of this distribution. We can find that:

\[
E_{x} = \frac{2c_{u}}{u}, \quad E_{v} = 0, \quad F_{x} = 0, \quad F_{v} = \frac{c_{v}}{v}, \quad G_{x} = \frac{2u}{v}, \quad G_{v} = \frac{2u^{2}}{v^{3}}
\]

\[
\Gamma_{i}^{j} = \frac{G_{x_i} E_{x_j} - 2F_{x_i} E_{x_j} + F_{x_j} V}{2(EG-F^2)}
\]

\[
R_{m}^{m} = \left( \frac{c_{i}}{u} \right)^{2} \left( \frac{c_{j}}{v} \right)^{2} \left( \frac{c_{k} - c_{i}}{c_{j} - c_{k}} \right) \left( \frac{c_{m} - c_{i}}{c_{m} - c_{j}} \right) \left( \frac{c_{m} - c_{k}}{c_{m} - c_{j}} \right)
\]

\[
\Gamma_{i}^{j} = \frac{2E_{x_i} - E_{x_j} - E_{x_j} G_{x_i}}{2(EG-F^2)}
\]

\[
\Gamma_{ii}^{j} = \frac{2G_{x_i} - G_{x_i} - G_{x_i} F_{x_i}}{2(EG-F^2)}
\]

\[
\Gamma_{ij}^{j} = \frac{2F_{x_i} - F_{x_j} - F_{x_j} G_{x_i}}{2(EG-F^2)}
\]

\[
\Gamma_{ij}^{j} = \frac{2E_{x_i} - E_{x_j} - E_{x_j} G_{x_i}}{2(EG-F^2)}
\]

\[
\Gamma_{ii}^{j} = \frac{2G_{x_i} - G_{x_i} - G_{x_i} F_{x_i}}{2(EG-F^2)}
\]

\[
\Gamma_{ij}^{j} = \frac{2F_{x_i} - F_{x_j} - F_{x_j} G_{x_i}}{2(EG-F^2)}
\]

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Now, we take the right hand side of equation (C):

\[
K = \frac{1}{D} \left[ \frac{\partial}{\partial u} \left( D_{G_{12}} \right) - \frac{\partial}{\partial v} \left( D_{G_{22}} \right) \right] =
\]

\[
\frac{1}{\sqrt{EG-F^2}} \left[ \frac{\partial}{\partial u} \left( \sqrt{EG-F^2} \right) \right] =
\]

\[
\frac{1}{\sqrt{c_1}} \left[ \frac{\partial}{\partial v} \left( \sqrt{c_1} \right) \right] - \frac{1}{\sqrt{c_1}} \left[ \frac{\partial}{\partial v} \left( \frac{c_1}{v} \right) \right] = \frac{1}{1.644934067} = -0.607627101
\]

This result shows that Weibull distribution converges to the normal distribution.

**CONCLUSION**

By using equation (C), we find the Gaussian curvature for Weibull distribution equal to that means this distribution converges to normal distribution.

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**REFERENCES**


