A Closed U-BG-Filter and Completely Closed U-BG-Filter of a U-BG-BH-Algebra

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Abstract: In this study, we introduce the notion of closed U-BG-filter and completely closed U-BG-filter in U-BG-BH-algebra and observed that every closed and completely closed filter of a U-BG-BH-algebra is a closed and completely closed U-BG-filter. A necessary and sufficient condition is derived for every closed and completely closed U-BG-filter of U-BG-BH-algebra to become a closed or completely closed filter. Some properties of closed and completely closed U-BG-filter are studied with respect to homomorphism, Cartesian products and quotient U-BG-BH-algebra.

Key words: BH-algebra, U-BG-BH-algebra, filter, U-BG-filter, closed filter, completely closed filter, homomorphism, Cartesian products and quotient U-BG-BH-algebra

INTRODUCTION

Deeba (1980) introduced the notion of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via a filter. Also, Deeba and Thaheem (1990) studied a filters in BCK-algebra in 1990. Hoo (1991) was presented the filters in BCI-algebra. Meng (1996) introduced the notion of BCK-filter in BCK-algebra. Abbass and Dahham (2016) discussed the concept of completely closed filter of a BH-algebra and completely closed filter with respect to an element of BH-algebra. The notion of U-BG-BH-algebra was introduced and extensively studied by Abbass and Mahdi (2014). This class of U-BG-BH-algebra was introduced as a combination of the classes of BH-algebra and BG-algebra. Abbass and Hamza (2017) introduced the notion of U-BG-filter of U-BG-BH-algebra. In this study, the notion of closed U-BG-filter and completely closed U-BG-filter of U-BG-BH-algebra are introduced. Some researchers have studied the filters in a practical way different from what we study in our research, for example, by Hameed and Purushothaman. Also, by Jeyachitra and Manickam, researchers proposed and developed a simple and new reconfigurable millimeter-wave photonic transversal filter featuring high quality windowing property.

MATERIALS AND METHODS

In this study, some basic concepts about a BG-algebra, BH-algebra, associative BH-algebra, BH-ideal, regular subset of X, U-BG-BH-algebra, filter, U-BG-filter, subalgebra, normal subset and quotient U-BG-BH-algebra are given.

Definition 1; Kim and Kim (2008): A BG-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

- $x*x = 0$, for all $x \in X$
- $x*0 = x$, for all $x \in X$
- $(x*y)*(0*y) = x$, for all $x, y \in X$

Lemma 1; Kim and Kim (2008): Let $(X, *, 0)$ be a BG-algebra. Then:

- The right cancellation law holds in X , i.e., $x*y = z*y$ implies $x = z$,
- $0*(0*x) = x$, for all $x \in X$
- If $x*y = 0$, then $x = y$, for all $x, y \in X$
- If $0*x = 0*y$, then $x = y$ for all $x, y \in X$
- $(x*(0*x))*x = x$ for all $x \in X$

Definition 2; Jun et al. (1998): A BH-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

- $x*x = 0$, for all $x \in X$
- $x*y = 0$ and $y*x = 0$ imply $x = y$, for all $x, y \in X$
- $x*0 = x$, for all $x \in X$

Definition 3; Abbass and Mhadi (2014): A U-BG-BH-algebra is defined to be a BH-algebra X in which there exists a proper subset U of X such that:

- $0 \in U$, $|U| = 2$
- U is a BG-algebra

Definition 4; Baik and Park (2010): A nonempty subset S of a BH-algebra X is called a BH-subalgebra or subalgebra of X if $x*y \in S$ for all $x, y \in S$.

Definition 5; Abbass and Dahham (2012c): Let, X be a BH-algebra, a non-empty subset N of X is said to be normal of X if $(x*a)*(y*b) \in N$ for any $x*y$ and $a*b \in N$, for all $x, y, a, b \in X$.

Theorem 1; Abbass and Dahham (2012c): Every normal subset N of a BH-algebra X is a subalgebra of X .

Definition 6; Abbass and Dahham (2012a): A BH-algebra X is called an associative BH-algebra if $(x*y)*z = x*(y*z)$, For all $x, y, z \in X$.

Theorem 2; Abbass and Dahham (2014): Let, X be an associative BH-algebra. Then the following proposition are hold:

- $0*x = x$, for all $x \in X$
- $x*y = y*x$, for all $x, y \in X$
- $x*(x*y) = y$, for all $x, y \in X$
- $(z*x)*(z*y) = x*y$, for all $x, y, z \in X$
- $x*y = 0 \Rightarrow x = y$, for all $x, y \in X$
- $(x*(x*y))*y = 0$, for all $x, y \in X$
- $(x*y)*z = (x*z)*y$, for all $x, y, z \in X$
- $(x*z)*(y*t) = (x*y)*(z*t)$, for all $x, y, z, t \in X$

Definition 7; Abbass and Mohammed (2013): A subset R of a BH-algebra X is said to be regular if it satisfies: $(\forall x \in R)(\forall y \in X)(x*y \in R \Rightarrow y \in R)$.

Definition 8; Jun et al. (1998): Let, X be a BH-algebra and $I(\neq \emptyset) \subseteq X$. Then, I is called an ideal of X if it satisfies:

- $0 \in I$
- If $x*y \in I$ and $y \in I \Rightarrow x \in I$, for all $x \in X$

Definition 9; Saeid et al. (2009): An ideal I of a BCH-algebra X is called a closed ideal of X if for every $x \in I$, we have $0*x \in I$. We generalize the concept of an ideal to a BH-algebra.

Definition 10: An ideal I of a BH-algebra X is called a closed ideal of X if: $0*x \in I$, for all $x \in I$.

Definition 11; Abass and Dahham (2012a): An ideal I of a BH-algebra X is called a completely closed ideal of X if: $x*y \in I$, for all $x, y \in I$.

Definition 12; Abbass and Mahdi (2016): Let, X be a BH-algebra and I be a subset of X . Then, I is called a BH-ideal of X if it satisfies the following conditions:

- $0 \in I$
- $x*y \in I$ and $y \in I$ imply $x \in I$
- $x \in I$ and $y \in X$ imply $x*y \in I$, $I*X \subseteq I$

Definition 13; Abbass and Mhadi (2014): A nonempty subset I of a U-BG-BH-algebra X is called a U-BG-ideal of X related to U if it satisfies:

- $0 \in I$
- $x*y \in I \Rightarrow x \in I$, for all $x \in U, y \in I$

Definition 14; Abbass and Dahham (2012b): A filter of a BH-algebra X is a non-empty F of X such that:

- F_1 : if $x \in F$ and $y \in F$, then $y*(y*x) \in F$ and $x*(x*y) \in F$
- F_2 : if $x \in F$ and $x*y = 0$ then $y \in F$. Further F is a closed filter if $0*x \in F$, for all $x \in F$. In sequel we shall denote $y*(y*x)$ by $x^{\wedge}y$

Definition 15; Abbass and Dahham (2012b): Let, X be a BH-algebra and F is a filter. Then, F is completely closed filter if $x*y \in F$, for all $x, y \in F$.

Definition 16; Abbass and Hamza (2017): A nonempty subset F of a U-BG-BH-algebra X is called a U-BG- filter of X , if it satisfies (F_1) and:

- F_3 : if $x \in F$ and $x*y = 0$ then $y \in F$. for all $y \in U$

Theorem 3; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra and F be a U-BG-filter of X such that $x*y \neq 0$, for all $y \notin F$ and $x \in F$. Then F is a filter of X .

Proposition 1; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra. Then, every filter of X is a U-BG-filter of X .

Proposition 2; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra and let $\{F_i, i \in \lambda\}$ be a family of U-BG-filters of X . Then, $\bigcap_{i \in \lambda} F_i$ is a U-BG-filter of X .

Proposition 3; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra and let $\{F_i, i \in \lambda\}$ be a chain of U-BG-filters of X . Then, $\bigcup_{i \in \lambda} F_i$ is a U-BG-filter of X .

Remark 1: Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be BH-algebra. A mapping $f: X \rightarrow Y$ is called a homomorphism if $f(x *_X y) = f(x) *_Y f(y)$ for any $x, y \in X$. A homomorphism f is called a monomorphism (resp., epimorphism) if it injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebra X and Y are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism f :

$X \rightarrow Y$. For any homomorphism $f: X \rightarrow Y$, the set $\{x \in X: f(x) = 0_Y\}$ is called the kernel of f , denoted by $\ker(f)$, the set $\{f(x): x \in X\}$ is called image of f , denoted by $\text{Im}(f)$. Notice that $f(0_X) = 0_Y$. By Jun *et al.* (1998) and the set $\{x \in X: f(x) = y, \text{ for some } y \in Y\}$ is preimage of f , denoted by $f^{-1}(Y)$ by Abbass and Mhadi (2014).

Theorem 4; Abbass and Hamza (2017): Let, $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a U-BG-BH- monomorphism and let F be a U-BG-filter of X . Then, $f(F)$ is $f(U)$ -BG-filter of Y .

Theorem 5; Abbass and Hamza (2017): Let, $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a U-BG-BH-isomorphism. If F is a U-BG-filter of Y . Then, $f^{-1}(F)$ is $f^{-1}(U)$ -BG-filter of X .

Proposition 4; Abbass and Hamza (2017): Let, X and Y be two U-BG-BH-algebras and $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a BH-homomorphism. Then, $\ker(f)$ is a U-B G-filter of X .

Remark 2; Kim and Kim (2008): Let $(X, *, 0)$ be a BG-algebra and let, N be a normal subalgebra of X . Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x * y \in N$ where $x, y \in X$. Then, it is easy to show $\sim_N$ is an equivalence relation on X . Denote the equivalence class containing x by $[x]_N$, i.e., $[x]_N = \{y \in X: x \sim_N y\}$ and let $X/N = \{[x]_N: x \in X\}$. If $*$ denoted on X/N by $[x]_N [y]_N = [xy]_N$. Then $(X/N, *, [0]_N)$ is a BG-algebra and it is called quotient BG-algebra of X by N , the researchers by Abbass and Dahham (2016), generalized this concept to BH-algebra to obtain $(X/N, *, [0]_N)$ quotient BH-algebra of X by N .

Theorem 6; Abbass and Hamza (2017): Let $(X, *, 0)$ be a U-BG- BH-algebra and N be a normal subalgebra, if F is a U-BG-filter in X , then, F/N is U/N -BG- filter of $(X/N, *, [0]_N)$.

Proposition 5; Abbass and Mhadi (2016): Let, X be a U-BG-BH-algebra. Then, every BH-ideal is a completely closed U-BG-ideal of X .

Theorem 7; Abbass and Dahham (2012): Let, N be a normal subalgebra of a BH-algebra X . Then, X/N is a BH-algebra.

Remark 3; Abbass and Hamza (2017): Let $\{(X_i, *, 0_i): i \in \lambda\}$ be a family of U_i -BG-BH-algebra. Define the Cartesian product of all $X_i, i \in \lambda$ to be the structure $\prod_{i \in \lambda} X_i = (\prod_{i \in \lambda} X_i, \otimes, (0_i))$ where, $\prod_{i \in \lambda} X_i$ is the set of tuples $\{(x_i): \text{for all } i \in \lambda \text{ and } x_i \in X_i\}$ and whose binary operation $\otimes$ is given by $(x_i) \otimes (y_i) = (x_i * y_i)$, for all $i \in \lambda$ and $x_i, y_i \in X_i$. Note that the binary operation $\otimes$ is componentwise.

Theorem 8; Abbass and Hamza (2017): Let $(\prod_{i \in \lambda} X_i, \otimes, (0_i))$ be a $\prod_{i \in \lambda} U_i$ -BG-BH-algebra. If $\{(F_i, *, 0_i): i \in \lambda\}$ be a family of U_i -BG-filter of X_i . Then $\prod_{i \in \lambda} F_i$ is a $\prod_{i \in \lambda} U_i$ -BG-filter of the product algebra $\prod_{i \in \lambda} X_i$.

Definition 17; Zhang *et al.* (2001): A BH-algebra X is said to be normal BH-algebra if it satisfies the following conditions:

- $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$
- $(x * y) * x = 0 * y$, for all $x, y \in X$
- $(x * (x * y)) * y = 0$ for all $x, y \in X$

RESULTS AND DISCUSSION

In this study, the notion of closed and completely closed U-BG-filter of U-BG-BH-algebra are introduced for our discussion, we shall link these notions with the notions which mentioned in preliminaries.

Definition 18: A U-BG-filter F of a U-BG-BH-algebra X is called a closed U-BG-filter of X if: $0 * x \in F$ for all $x \in F$.

Example 1: Consider the U-BG-BH-algebra $X = \{0, 1, 2, 3, 4\}$ with binary operation “*” defined as follows Table 1: where, $U = \{0, 1, 2\}$, the U-BG-filter $F = \{1, 3\}$ is a closed U-BG-filter of X . But the U-BG-filter $F = \{1, 4\}$ is not a closed U-BG-filter, since, $4 \in F$ and $0 * 4 = 3 \notin F$.

Definition 19: A U-BG-filter F of a U-BG-BH-algebra X is called a completely closed U-BG-filter of X if: $x * y \in F$ for all $x, y \in F$.

Example 2: Consider the U-BG-BH-algebra X in example 1, the U-BG-filter $F = \{0, 1, 3\}$ is a completely closed U-BG-filter of X but the U-BG-filter $F = \{0, 1, 4\}$ is not a completely closed U-BG-filter of X , since, $1, 4 \in F$ but $1 * 4 = 2 \notin F$.

Remark 4: Let, X be a U-BG-BH-algebra. The filters $F = \{0\}$ and $F = X$ are completely closed U-BG-filters of X which are called a trivial completely closed U-BG-filters of X .

Proposition 6: Let, X be a U-BG-BH-algebra. Then, every closed filter of X is a closed U-BG-filter of X .

Proof: Directly by Proposition 1 and Definition 10.

Remark 5: The converse of Proposition 7 is not correct in general as in the following example.

Table 1: Closed U-BG-filter

*	0	1	2	3	4
0	0	1	2	3	3
1	1	0	1	3	2
2	2	2	0	1	2
3	3	3	1	0	0
4	4	3	1	4	0

Example 3: Consider the U-BG-BH-algebra X in Example (1), the U-BG-filter $F = \{0, 1, 3\}$ of X is a closed U-BG-filter of X but it is not a closed filter of X because F is not a filter of X , since, $3 \in F$, $3 * 4 = 0$ but $4 \notin F$.

Proposition 7: Let X be a U-BG-BH-algebra. Then every completely closed filter of X is a completely closed U-BG-filter of X .

Proof: Its directly by Definition (19) and Proposition (1).

Remark 6: The converse of Proposition (7) is not correct in general as in the following example.

Example 4: Consider the U-BG-BH-algebra X in example 1, the U-BG-filter $F = \{0, 1, 3\}$ of X is a completely closed U-BG-filter of X but it is not a completely closed filter of X because F is not a filter, since, $1 \in F$, $1 * 4 = 0$ but $4 \notin F$.

Proposition 8: Let, X be a U-BG-BH-algebra and F be a completely closed U-BG-filter of X . Then, $0 \in F$.

Proof : Let, F be a completely closed U-BG-filter of X and let $x \in F$. then $x * x \in F$ (since, F is a completely closed U-BG-filter of X). So, by using Definition (2) (I), $0 \in F$ (since, $x * x = 0$).

Proposition 9: Let, X be a U-BG-BH-algebra. Then, every completely closed U-BG-filter of X is a closed U-BG-filter of X .

Proof: Its directly from Proposition (8) and by applying Definition (19), we get $0 * x \in I$, so, I is a closed U-BG-filter of X .

Remark 7: The converse of Proposition (9) is not correct in general as in the following example.

Example 5: Consider a U-BG-BH-algebra, $X = \{0, 1, 2, 3, 4\}$ with binary operation “*” defined as follows Table 2: where, $U = \{0, 1, 2\}$. The U-BG-filter $F = \{0, 3, 4\}$ a closed U-BG-filter of X but it is not a completely closed U-BG-filter, since, $3, 4 \in F$ and $3 * 4 = 2 \notin F$.

Proposition 10: Let, X be a U-BG-BH-algebra and F be a completely closed U-BG-filter of X . Then, F is BH-algebra with the same binary operation on X and the constant 0.

Table 2: Closed U-BG-filter is not a completely closed U-BG-filter

*	0	1	2	3	4
0	0	1	2	0	0
1	1	0	2	2	1
2	2	1	0	1	2
3	3	2	3	0	2
4	4	1	2	0	0

Proof: straightforward.

Theorem 9: Let, X be a U-BG-BH-algebra and let F be a U-BG-filter of X . Then, F is a completely closed U-BG-filter of X if and only if F is a subalgebra of X contain in U .

Proof: Let, F be a completely closed U-BG-filter of X and let $x, y \in F$, then, $x * y \in F$ (since, F is a completely closed U-BG-filter of X), so, F is a subalgebra of X . Conversely, let F is a subalgebra. Let $x, y \in F$, so we have $x * (x * y) \in F$ and $y * (y * x) \in F$ [by Definition (4)].

Let $x \in F$, $x * y = 0$, $y \in U$, we get $x \in U$ (Since $F \subseteq U$) and applying Lemma (1) (iii), we get $x = y$, hence, $y \in F$, therefore, F is a U-BG-filter of X . Now, let $x, y \in F$, so, $x * y \in F$ for all $x, y \in F$ (since, F is a subalgebra). Then F is completely closed U-BG-filter of X .

Lemma 2: Let, X be a U-BG-BH-algebra and let, N be a normal subset of X contain in U . Then, N is a completely closed U-BG-filter of X .

Proof: Directly from Theorem 1 and 9.

Proposition 11: Let, X be a U-BG-BH-algebra and F be a completely closed U-BG-filter of X . Then, F is a completely closed U-BG-ideal of X .

Proof: Let, F be a completely closed U-BG-filter of X .

- $0 \in F$ (by Proposition 8)
- Let, $x * y \in F$, $y \in F$, $x \in U$. Since, F is a completely closed U-BG-filter of X . Then $(x * y) * y \in F$. Since, F is a U-BG-filter of X , so, we have $y \in U$. Now, take $y = 0$

Since, U is a BG-algebra, then $(x * 0) * 0 \in F$, so, $x \in F$ [by Definition (1) (ii).

- Let $x, y \in F$, so, $x * y \in F$ (Since, F is a completely closed U-BG-filter of X), therefore, F is a completely closed U-BG-ideal of X

Proposition 12: Let, X be U-BG-BH-algebra and F be a closed U-BG-filter such that $x * y \neq 0$, for all $y \notin F$ and $x \in F$. Then, F is a closed filter of X .

Proof: Let, F be a closed U-BG-filter of X . Then, F is a U-BG-filter of X by applying Theorem 3, we get F is a filter of X . Now, let, $x \in F$. Then, $0 * x \in F$ (since, F is a closed U-BG-filter). Therefore, F is a closed filter of X .

Proposition 13: Let, X be U-BG-BH algebra and F be a completely closed U-BG-filter of X such that $x * y \neq 0$, for all $y \in F$ and $x \in F$. Then, F is a completely closed filter of X .

Proof: Let, F be a completely closed U-BG-filter of X . Then, F is a U-BG-filter of X . By applying Theorem 3, we get, F be a filter of X . Now, let $x, y \in F$, so, $x * y \in F$ (since, F is a completely closed U-BG-filter). Therefore, F is a completely closed filter of X .

Theorem 10: Let, X be a normal U-BG-BH algebra and let, R be regular subset of X such that $R \subseteq U$. If R is a U-BG-ideal, then, R is a U-BG-filter of X .

Proof: Let, R be a U-BG-ideal of X .

- Let, $x, y \in R$, since, R is a U-BG-ideal. Then $0 \in R$. Now, $(x * (x * y)) * y \in R$ (By Definition (17) (iii)], so, $x * (x * y) \in R$ [by Definition (13) (ii)]. Similarly, $y * (y * x) \in R$
- Let $x \in R$, $x * y = 0$, $y \in U$, then, $x * y \in R$ and $x \in R$. By Definition 7, we get, $y \in R$, therefore, R is a U-BG-filter of X

Corollary 1: Let, X be a normal U-BG-BH algebra and let, R be regular subset of X which is contain in U . If R is a completely closed U-BG-ideal of X , then, R is a completely closed U-B-filter of X .

Proof: Let, R be a completely closed U-BG-ideal. Then, R is a U-BG-ideal of X by using theorem 10, we get R is a U-BG-filter of X . Now, let $x, y \in R$, hence, $x * y \in R$ (since, R is a completely closed U-BG-ideal). Therefore, R is a completely closed U-B-filter of X .

Theorem 11: Let, X be an associative U-BG-BH-algebra and F be a U-BG-filter contain in U . Then, F is a completely closed U-BG-filter if and only if F is a completely closed U-BG-ideal of X .

Proof: Let, F be a completely closed U-BG- filter: $0 \in F$ (by Proposition 8). Let, $x * y \in F$, $y \in F$, $x \in U$, then $(x * y) * y \in F$ (since, F is a completely closed U-BG-filter). So, $x * (y * y) \in F$ (by Definition (6) by applying Definition (2) (i), we get, $x * 0 \in F$, hence, $x \in F$ (By using Definition (2)(iii)). Therefore, F is a U-BG- ideal of X .

Let, $x, y \in F$. Then, $x * y \in F$ (since, F is a completely closed U-BG-filter), so, F is a completely closed U-BG-ideal. Conversely, let F be a completely closed U-BG-ideal of X .

Table 3: Closed U-BG-ideal is not a closed U-BG- filter

*	0	1	2	3	4
0	0	1	2	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Let, $x, y \in F$, then $x * y \in F$, $y * x \in F$ (since, F is a completely closed U-BG- ideal), then, we have, $x * (x * y) \in F$ and $y * (y * x) \in F$. Let, $x \in F$, $x * y = 0, y \in U$, since, $F \subseteq U$, we have $x \in U$, so, $x = y$ (since, U is a BG-algebra), then, $y \in F$, hence, F is a U-BG-filter of X . Let, $x, y \in F$, then, $x * y \in F$ (since, F is a completely closed U-BG-ideal), so, F is a completely closed U-BG-filter of X .

Theorem 12: Let, X be an associative U-BG- BH-algebra. Then, every closed U-BG-ideal of X is a closed U-BG-filter of X .

Proof: Let, X be an associative BH-algebra and let I be a closed U-BG-ideal of X .

- Let, $x, y \in I$, since, X is an associative, we obtain, $y * (y * x) = (y * y) * x = 0 * x \in I$ (Since I is a closed U-BG-ideal) and $x * (x * y) = (x * x) * y = 0 * y \in I$
- Let, $x \in I$ and $y \in U$ such that $x * y = 0$. Thus, $x = y$ (by Theorem (2) (v)), so, $y \in I$, then we get, I is a U-BG-filter of X
- Let, $x \in I$. By Definition (8) (i), we obtain $0 \in I$, so, $0 * x \in I$ (since, I is a closed U-BG- ideal), therefore, I is a closed U-BG- filter of X .

Remark 8: If X is not associative U-BG-BH-algebra then the Theorem 12 is not correct in general as in the following example:

Example 6: Let, X be a U-BG-BH-algebra, $X = \{0, 1, 2, 3, 4\}$ with binary operation “*” defined as follows Table 3 and $U = \{0, 2\}$ the closed U-BG-ideal $I = \{0, 1\}$ is not a closed U-BG-filter, since, I is not a filter. Since, $1 \in I$, $1 * 2 = 0$, $2 \in U$ but $2 \notin I$.

Proposition 14: Let, X be a U-BG-BH-algebra and the right cancellation law holds in X . Then, every BH-ideal of X is a completely closed U-BH-filter of X .

Proof: Let, I be a BH-ideal of X . By Proposition 5, we get I is a completely closed U-BG-ideal of X :

- Let, $x, y \in I$, so, we have $x * (x * y) \in F$, $y * (y * x) \in F$ (since, I is a completely closed U-BG-ideal of X
- Let, $x \in I$, $x * y = 0$, $y \in U$, then, $x * y = y * y$, so, by Lemma (1) (i), we obtain $x = y$, imply that $y \in F$. Therefore, I is a U-BG-filter of X

- Let $x, y \in I$. By Definition (12) (iii), we get $x * y \in I$. Then, I is a completely closed U-BG-filter of X

Remark 9: The converse of Proposition 14 is not correct in general as in the following example:

Example 7: Consider U-BG-BH-algebra X in Example 1, $F = \{0, 3\}$ is a completely closed U-BG-filter of X but it is not a BH-ideal of X , since, $3 \in F$, $2 \in X$ but $3 * 2 = 1 \notin F$.

Proposition 15: Let $\{F_i, i \in \lambda\}$ be a family of closed U-BG-filters of a U-BG-BH-algebra X . Then, $\bigcap_{i \in \lambda} F_i$ is a closed U-BG-filters of X .

Proof: Since, F_i is a closed U-BG-filters, $\forall i \in \lambda$, then, F_i is a U-BG-filter, $\forall i \in \lambda$ (by Definition 18), so, $\bigcap_{i \in \lambda} F_i$ is a U-BG-filter (by Proposition 2).

Now, let $x \in \bigcap_{i \in \lambda} F_i$, hence, $x \in F_i, \forall i \in \lambda$, so, $0 * x \in F_i, \forall i \in \lambda$

(Since, F is a closed U-BG-filter of X . By Definition 18), then $0 * x \in \bigcap_{i \in \lambda} F_i$. So, we get, $\bigcap_{i \in \lambda} F_i$ is a closed U-BG-filter of X .

Proposition 16: Let $\{F_i, i \in \lambda\}$ be a family of completely closed U-BG-filters of a U-BG-BH-algebra X . Then, $\bigcap_{i \in \lambda} F_i$ is a completely closed U-BG-filter of X .

Proof: Since, F_i is a completely closed U-BG-filter of $X, \forall i \in \lambda$, then, F_i is a U-BG-filter of $X, \forall i \in \lambda$ (by Definition 19). By proposition 2, we obtain $\bigcap_{i \in \lambda} F_i$ is a U-BG-filter of X .

Now, let $x, y \in \bigcap_{i \in \lambda} F_i$, so, $x, y \in F_i, \forall i \in \lambda$, then, $x * y \in F_i, \forall i \in \lambda$ (since, F is a completely closed U-BG-filter), hence, $x * y \in \bigcap_{i \in \lambda} F_i$. Therefore, $\bigcap_{i \in \lambda} F_i$ is a completely closed U-BG-filter of X .

Proposition 17: Let $\{F_i, i \in \lambda\}$ be a chain of closed U-BG-filters of a U-BG-BH-algebra X . Then, $\bigcup_{i \in \lambda} F_i$ is a closed U-BG-filter of X .

Proof: Since, F_i is a closed U-BG-filter of $X, \forall i \in \lambda$, then, F_i is a U-BG-filter of $X, \forall i \in \lambda$ (by Definition 18), so, $x \in \bigcup_{i \in \lambda} F_i$ is a U-BG-filter of X (by Proposition 3). Now, let $x \in \bigcup_{i \in \lambda} F_i$,

Then, there exist $F_i, F_k \in \{F_i, i \in \lambda\}$ such that $x \in F_i, y \in F_k$ and either $F_i \subseteq F_k$ or $F_k \subseteq F_i$ (since, $\{F_i, i \in \lambda\}$ is a chain). If $F_i \subseteq F_k$, then, $x \in F_k$, hence, $0 * x \in F_k$ (since, F_k is a closed U-BG-filter). Similarly, if $F_k \subseteq F_i \Rightarrow 0 * y \in F_i$, therefore, $\bigcup_{i \in \lambda} F_i$ is a closed U-BG-filter of X .

Proposition 18: Let, $\{F_i, i \in \lambda\}$ be a chain of completely closed U-BG-filters of a U-BG-BH-algebra X . Then, $\bigcup_{i \in \lambda} F_i$ is a completely closed U-BG-filter of X .

Proof: Since, F_i is a completely closed U-BG-filter of $X, \forall i \in \lambda$, we get F_i is a U-BG-filter of $X, \forall i \in \lambda$. By definition (19), Therefore, $\bigcup_{i \in \lambda} F_i$ is a U-BG-filter of X (by Proposition 3). Now, let $x, y \in \bigcup_{i \in \lambda} F_i$, so, there exist $F_i, F_k \in \{F_i, i \in \lambda\}$ such that $x \in F_i, y \in F_k$, then, either $F_i \subseteq F_k$ or $F_k \subseteq F_i$ (since, $\{F_i, i \in \lambda\}$ is a chain). If $F_i \subseteq F_k$, hence, $x, y \in F_k$, so, $x * y \in F_k$ (since, F_k is a completely closed U-BG filter]. Similarly, if $F_k \subseteq F_i \Rightarrow x * y \in F_i$. Therefore, $\bigcup_{i \in \lambda} F_i$ is a completely closed U-BG-filter of X .

Proposition 19: Let, $f: (X, *, 0) \rightarrow (Y, *,' 0')$ be a U-BG-BH-monomorphism and let F be a closed U-BG-filter of X . Then, $f(F)$ is a closed $f(U)$ -BG-filter of Y .

Proof: Let, F be a closed U-BG-filter of X , then, F is a U-BG-filter of X (by Definition (18)), so, by Theorem (4), we obtain $f(F)$ is a $f(U)$ -BG-filter of Y . Now, let $y \in f(F)$. Then, there exist $a \in F$ such that $y = f(a)$, hence, $0' / *' y = 0' / *' f(a) = f(0) *' f(a) = f(0 * a)$.

Since, $0 * a \in F$ (by Definition 18), we get $f(0 * a) \in f(F)$, so, $0' / *' y \in f(F)$. Therefore, $f(F)$ is a closed $f(U)$ -BG-filter of Y .

Proposition 20: Let, $f: (X, *, 0) \rightarrow (Y, *,' 0')$ be a U-BG-BH-monomorphism and let F be a completely closed U-BG-filter of X . Then, $f(F)$ is a completely closed $f(U)$ -BG-filter of Y .

Proof: Let, F be a completely closed U-BG-filter of X , so, F is a U-BG-filter of X (by Definition 19), then, $f(F)$ is a $f(U)$ -BG-filter of Y (Theorem 4).

Now, let, $x, y \in f(F)$, so, there exist $a, b \in F$ such that $x = f(a), y = f(b)$, then $x *' y = f(a) *' f(b) = f(a * b)$. Since, F is a completely closed U-BG-filter of X , then, we obtain $a * b \in F$, so, $f(a * b) \in f(F)$, hence, $x *' y \in f(F)$. Therefore, $f(F)$ is a completely closed $f(U)$ -BG-filter of X .

Theorem 13: Let, $f: (X, *, 0) \rightarrow (Y, *,' 0')$ be a U-BG-BH-isomorphism. If F is a closed U-BG-filter of Y . Then, $f^{-1}(F)$ is a closed $f^{-1}(U)$ -BG-filter of X .

Proof: Let, F be a closed U-BG-filter of Y . Then, F is a U-BG-filter of Y (by Definition 18), so, $f^{-1}(F)$ is a $f^{-1}(U)$ -BG-filter of X (by Theorem 5). Now, let, $y \in f^{-1}(F)$, hence, we have, $f(y) \in F$ and $0' / *' f(y) \in F$ (since, F is a closed U-BG-filter of Y), then, $f(0 * y) \in F$, so, we get, $0 * y \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a closed $f^{-1}(U)$ -BG-filter of X .

Theorem 14: Let, $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a U-BG-BH-isomorphism. If F is a completely closed U-BG-filter of Y . Then, $f^{-1}(F)$ is a completely closed $f^{-1}(U)$ -BG-filter of X .

Proof: Let, F be completely a closed U-BG-filter of Y . So, F be a U-BG-filter of Y (by Definition 19), then, $f^{-1}(F)$ is a $f^{-1}(U)$ -BG-filter of X (by Theorem 5). Now, let, $x, y \in f^{-1}(F)$, hence, $f(x) \in F, f(y) \in F$. Since, F is a completely closed U-BG-filter of Y , we obtain $f(x) *' f(y) \in F$, then, $f(x * y) \in F$, so, we get $x * y \in f^{-1}(F)$, therefore, $f^{-1}(F)$ is a completely closed $f^{-1}(U)$ -BG-filter of X .

Proposition 21: Let, $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a U-BG-BH-homomorphism. Then $\ker(f)$ is a completely closed filter of X .

Proof: Let, $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a U-BG-BH-homomorphism. Then $\ker(f)$ is a U-BG-filter (by Proposition 4). Now, let, $x, y \in \ker(f)$, we have $f(x) = f(y) = 0'$ and then, $f(x * y) = f(x) *' f(y) = 0'$ by Remark (1), we get $x * y \in \ker(f)$, therefore, $\ker(f)$ is a completely closed U-BG-filter of X .

Proposition 22: Let, $f: (X, *, 0) \rightarrow (Y, *, 0')$ be a U-BG-BH-homomorphism. Then, $\ker(f)$ is a closed filter of X .

Proof: Directly by Proposition 9 and 21.

Proposition 23: Let, X be a U-BG-BH-algebra, N be a normal subalgebra of X and F be a closed U-BG-filter of X . Then, F/N is a closed U/N -BG-filter of X/N .

Proof: Let, F be a closed U-BG-filter of X , so, F is a U-BG-filter of X , (by Definition 18), then, F/N is a U/N -BG-filter of X/N (by Theorem 6). Now, let $(0)_N, (y)_N \in F/N$, since, $0 * y \in F$ (by F is closed U-BG-filter), so $[0]_N * [y]_N = [0 * y]_N \in F/N$, therefore, F/N is a closed U/N -BG-filter in X/N .

Proposition 24: Let, X be a U-BG-BH-algebra, N be a normal subalgebra of X and F is a completely closed U-BG-filter of X . Then, F/N is a completely closed U/N -BG-filter of X/N .

Proof: Let, F be a completely closed U-BG-filter of X by Definition (19), we obtain F is a U-BG-filter of X , then, F/N is a U/N -BG-filter of X/N [by Theorem (6)].

Now, let $[x]_N, [y]_N \in F/N$, so, $[x]_N * [y]_N = [x * y]_N \in F/N$ [since $x * y \in F$ by F is completely closed U-BG-filter], then, we get F/N is a completely closed U/N -BG-filter in X/N .

Corollary 2: Let, X be a U-BG-BH-algebra, N be a normal subalgebra of X and F is a completely closed U-BG-filter in X . Then, F/N is a closed U/N -BG-filter of X/N .

Proof: Let, F be a completely closed U-BG-filter of X . By Proposition (9), we get F is a closed U-BG-filter of X , then, F/N is a closed U/N -BG-filter in X/N . By Proposition (24).

Theorem 15: Let $(\prod_{i \in \lambda} X_i, \otimes, (0_i))$ be a $\prod_{i \in \lambda} U_i$ -BG-BH-algebra. If $\{F_i: i \in \lambda\}$ be a family of a closed U_i -BG-filter of X_i . Then $\prod_{i \in \lambda} F_i$ is a closed $\prod_{i \in \lambda} U_i$ -BG-filter of the product algebra $\prod_{i \in \lambda} X_i$.

Proof: Let $\{F_i: i \in \lambda\}$ be a family of a closed U_i -BG-filter of X_i . By Definition (18), we obtain $\{F_i: i \in \lambda\}$ be a family of a U_i -BG-filter of X_i , then, $\prod_{i \in \lambda} F_i$ is a $\prod_{i \in \lambda} U_i$ -BG-filter of $\prod_{i \in \lambda} X_i$ (by Theorem 8). Now, let $y = (y_i) \in \prod_{i \in \lambda} F_i$ for all $y_i \in F_i$ and $i \in \lambda$, so, $(0_i) \otimes (y_i) = (0_i * y_i)$, since, F_i is a closed U_i -BG-filter of X_i , then $0 * y_i \in F_i$ (By definition 18), hence, $(0) \otimes (y) \in \prod_{i \in \lambda} F_i$, then, $\prod_{i \in \lambda} F_i$ is a closed $\prod_{i \in \lambda} U_i$ -BG-filter of $\prod_{i \in \lambda} X_i$.

Theorem 16: Let $(\prod_{i \in \lambda} X_i, \otimes, (0_i))$ be a $\prod_{i \in \lambda} U_i$ -BG-BH-algebra. If $\{(F_i, *, 0_i): i \in \lambda\}$ be a family of a completely closed U_i -BG-filter of X_i . Then, $\prod_{i \in \lambda} F_i$ is a completely closed $\prod_{i \in \lambda} U_i$ -BG-filter of the product algebra $\prod_{i \in \lambda} X_i$.

Proof: Let $\{F_i: i \in \lambda\}$ be a family of a completely closed U_i -BG-filter of X_i . Then $\{F_i: i \in \lambda\}$ be a family of a U_i -BG-filter of X_i (by Definition 19), so, $\prod_{i \in \lambda} F_i$ is a $\prod_{i \in \lambda} U_i$ -BG-filter of $\prod_{i \in \lambda} X_i$ (by Theorem 8). Now, let $x = (x_i), y = (y_i) \in \prod_{i \in \lambda} F_i$ for all $x_i, y_i \in F_i, x_i \in F_i$ and $i \in \lambda$, then $x \otimes y = (x_i) \otimes (y_i) = (x_i * y_i) \in \prod_{i \in \lambda} F_i$, since, F_i is a completely closed U_i -BG-filter of X_i , then $x_i * y_i \in F_i$, hence, $(x_i) \otimes (y_i) \in \prod_{i \in \lambda} F_i$. Therefore, $\prod_{i \in \lambda} F_i$ is a completely closed $\prod_{i \in \lambda} U_i$ -BG-filter of $\prod_{i \in \lambda} X_i$.

Proposition 25; (Extension property for closed U-BG-filter in U-BG-BH-algebra): Let X be a normal U-BG-BH-algebra F is a completely closed U-BG-filter of X , G is a closed ideal of X such that $G \subseteq U$. Then G is a completely closed U-BG-filter of X .

Proof: Let $x, y \in G$, since, F be a completely closed U-BG-filter of X , then, $0 \in F$ (by Proposition 8), so, $0 \in G$ (since, $F \subseteq G$) by Definition (17), we get $(x * (x * y)) * y \in G$. So, by Definition (8), we obtain $x * (x * y) \in G$. Similarly, $y * (y * x) \in G$.

Let, $x \in G, x * y = 0, y \in U$, so, $x \in U$ (since, $G \subseteq U$). Then, $x * y = y * y$, imply that $x = y$ (By Lemma (1)), so, we obtain $y \in G$. Therefore, G is U-BG-filter of X . Now, since, G is a closed ideal of X , thus, $0 * y \in G$ (by Definition 10). By Definition (17), we obtain $(x * y) * x \in G$. So, we have $(x * y) * x \in G, x \in G$, then $x * y \in G$ (Since, G is an ideal of X), therefore, G is a completely closed U-BG-filter of X .

CONCLUSION

In this study, the notions of closed and completely closed U-BG-filter of U-BG-BH-algebra are introduced. Furthermore, the results are examined in terms of the relationship between closed and completely closed U-BG-filters. In addition, the relationship between the closed and completely closed U-BG-filters with the other filters as well as some special ideals are also presented the important characteristics of closed and completely closed U-BG-filters are analyzed.

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