A Closed U-BG-Filter and Completely Closed U-BG-Filter of a U-BG-BH-Algebra

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Abstract: In this study, we introduce the notion of closed U-BG-filter and completely closed U-BG-filter in U-BG-BH-algebra and observed that every closed and completely closed filter of a U-BG-BH-algebra is a closed and completely closed U-BG-filter. A necessary and sufficient condition is derived for every closed and completely closed U-BG-filter of U-BG-BH-algebra to become a closed or completely closed filter. Some properties of closed and completely closed U-BG-filter are studied with respect to homomorphism, Cartesian products and quotient U-BG-BH-algebra.

Keywords: BH-algebra, U-BG-BH-algebra, filter, U-BG-filter, closed filter, completely closed filter, homomorphism, Cartesian products and quotient U-BG-BH-algebra

INTRODUCTION

Deeba (1980) introduced the notion of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via a filter. Also, Deeba and Thaheem (1990) studied filters in BCK-algebra in 1990. Hoo (1991) was presented the filters in BCI-algebra. Meng (1996) introduced the notion of BCK-filter in BCK-algebra. Abbas and Dahham (2016) discussed the concept of completely closed filter of a BH-algebra and completely closed filter with respect to an element of BH-algebra. The notion of U-BG-BH-algebra was introduced and extensively studied by Abbass and Mahdi (2014). This class of U-BG-BH-algebra was introduced as a combination of the classes of BH-algebra and BG-algebra. Abbass and Hamza (2017) introduced the notion of U-BG-filter of U-BG-BH-algebra. In this study, the notion of closed U-BG-filter and completely closed U-BG-filter of U-BG-BH-algebra are introduced. Some researchers have studied the filters in a practical way different from what we study in our research, for example, by Hameed and Purusothaman. Also, by Jeyachitra and Manickam, researchers proposed and developed a simple and new reconfigurable millimeter-wave photonic transversal filter featuring high quality windowing property.

MATERIALS AND METHODS

In this study, some basic concepts about a BG-algebra, BH-algebra, associative BH-algebra, BH-ideal, regular subset of X, U-BG-BH-algebra, filter, U-BG-filter, subalgebra, normal subset and quotient U-BG-BH-algebra are given.

Definition 1; Kim and Kim (2008): A BG-algebra is a non-empty set X with a constant 0 and a binary operation "∗" satisfying the following axioms:

• x∗x = 0, for all x ∈ X
• x∗0 = x, for all x ∈ X
• (x∗y)∗(0∗y) = x, for all x, y ∈ X

Lemma 1; Kim and Kim (2008): Let (X, ∗, 0) be a BG-algebra. Then:

• The right cancellation law holds in X, i.e., x∗y = z∗y implies x = z,
• 0∗(0∗x) = x, for all x ∈ X
• If x∗y = 0, then x = y, for all x, y ∈ X
• If 0∗x = 0∗y, then x = y for all x, y ∈ X
• (x∗(0∗x))∗x = x for all x ∈ X

Definition 2; Jun et al. (1998): A BH-algebra is a nonempty set X with a constant 0 and a binary operation "∗" satisfying the following conditions:

• x∗x = 0, for all x ∈ X
• x∗y = 0 and y∗x = 0 imply x = y, for all x, y ∈ X
• x∗0 = x, for all x ∈ X

Definition 3; Abbass and Mahdi (2014): A U-BG-BH-algebra is defined to be a BH-algebra X in which there exists a proper subset U of X such that:

• 0 ∈ U, |U| = 2
• U is a BG-algebra

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Definition 4; Baik and Park (2010): A nonempty subset $S$ of a BH-algebra $X$ is called a BH-subalgebra or subalgebra of $X$ if $x^*y \in S$ for all $x, y \in S$.

Definition 5; Abbass and Dahham (2012c): Let, $X$ be a BH-algebra, a non-empty subset $N$ of $X$ is said to be normal of $X$ if $(x^a)(y^b) \in N$ for any $x, y$ and $a, b \in \mathbb{N}$, for all $x, y, a, b \in X$.

Theorem 1; Abbass and Dahham (2012c): Every normal subset $N$ of a BH-algebra $X$ is a subalgebra of $X$.

Definition 6; Abbass and Dahham (2012a): A BH-algebra $X$ is called an associative BH-algebra if $(x^*y)^z = x^*(y^z)$, for all $x, y, z \in X$.

Theorem 2; Abbass and Dahham (2014): Let, $X$ be an associative BH-algebra. Then the following proposition hold:

- $0^*x = x$, for all $x \in X$
- $x^*y = y^*x$, for all $x, y \in X$
- $x^*(x^*y) = y$, for all $x, y \in X$
- $(z^*)(z^*y) = x^*y$, for all $x, y, z \in X$
- $x^*y = 0 \iff x = y$, for all $x, y \in X$
- $(x^*(x^*y))^r = 0$, for all $x, y \in X$
- $(x^*y)^z = (x^*z)^y$, for all $x, y, z \in X$
- $(x^*y)^*(y^*z) = (x^*y)^* (y^*z)$, for all $x, y, z \in X$

Definition 7; Abbass and Mohammed (2013): A subset $R$ of a BH-algebra $X$ is said to be regular if it satisfies: $(\forall x \in X)(\forall y \in X)(x^*y \in R \iff y \in R)$.

Definition 8; Jun et al. (1998): Let, $X$ be a BH-algebra and $I(\ast \in \mathbb{C})$. Then, $I$ is called an ideal of $X$ if it satisfies:

- $0 \in I$
- $x^*y \in I$ and $y \in I \implies x \in I$ for all $x, y \in X$

Definition 9; Saeid et al. (2009): An ideal $I$ of a BCH-algebra $X$ is called a closed ideal of $X$ if for every $x \in I$, we have $0^*x \in I$. We generalize the concept of an ideal to a BH-algebra.

Definition 10: An ideal $I$ of a BH-algebra $X$ is called a closed ideal of $X$ if $0^*x \in I$, for all $x \in I$.

Definition 11; Abbass and Dahham (2012a): An ideal $I$ of a BH-algebra $X$ is called a completely closed ideal of $X$ if $x^*y \in I$, for all $x, y \in I$.

Definition 12: Abbass and Mahdi (2016): Let, $X$ be a BH-algebra and $I$ be a subset of $X$. Then, $I$ is called a BH-ideal of $X$ if it satisfies the following conditions:

- $0 \in I$
- $x^*y \in I$ and $y \in I \implies x \in I$ and $y \in X \implies x^*y \in I$, $x^* \in I$

Definition 13; Abbass and Mhadi (2014): A nonempty subset $I$ of a U-BG-BH-algebra $X$ is called a U-BG-ideal of $X$ related to $U$ if it satisfies:

- $0 \in I$
- $x^*y \in I$, for all $x \in U$, $y \in I$

Definition 14; Abbass and Dahham (2012b): A filter of a BH-algebra $X$ is a non-empty $F$ of $X$ such that:

- $F_0$: if $x \in F$ and $y \in F$, then $(y^*x \in F)$ and $(x^*y) \in F$
- $F_1$: if $x \in F$ and $x^*y = 0$, then $y \in F$. Further $F$ is a closed filter if $0^*x \in F$, for all $x \in F$. In sequel we shall denote $y^*(y^*x)$ by $x^*y$

Definition 15; Abbass and Dahham (2012b): Let, $X$ be a BH-algebra and $F$ is a filter. Then, $F$ is completely closed filter if $x^*y \in F$, for all $x, y \in F$.

Definition 16; Abbass and Hamza (2017): A nonempty subset $F$ of a U-BG-BH-algebra $X$ is called a U-BG-filter of $X$, if it satisfies $(F_0)$ and:

- $F_1$: if $x \in F$ and $x^*y = 0$, then $y \in F$, for all $x \in X$

Theorem 3; Abbass and Hamza (2017): Let, $X$ be a U-BG-BH-algebra and $F$ be a U-BG-filter of $X$ such that $x^*y = 0$, for all $y \in F$ and $x \in F$. Then $F$ is a filter of $X$.

Proposition 1; Abbass and Hamza (2017): Let, $X$ be a U-BG-BH-algebra. Then, every filter of $X$ is a U-BG-filter of $X$.

Proposition 2; Abbass and Hamza (2017): Let, $X$ be a U-BG-BH-algebra and let $\{F_\alpha, \alpha \in \mathbb{A}\}$ be a family of U-BG-filters of $X$. Then, $\bigcup_{\alpha} F_\alpha$ is a U-BG-filter of $X$.

Proposition 3; Abbass and Hamza (2017): Let, $X$ be a U-BG-BH-algebra and let $\{F_\alpha, \alpha \in \mathbb{A}\}$ be a chain of U-BG-filters of $X$. Then, $\bigcup_{\alpha} F_\alpha$ is a U-BG-filter of $X$.

Remark 1: Let $(X, \ast, 0_G)$ and $(Y, \ast, 0_Y)$ be BH-algebra. A mapping $f : X \to Y$ is called a homomorphism if $f(x \ast y_G) = f(x)^* \ast f(y)$ for any $x, y \in X$. A homomorphism $f$ is called a monomorphism (resp., epimorphism) if it injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebra $X$ and $Y$ are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f$. 
X-Y. For any homomorphism f: X→Y, the set \{x∈X: f(x) = 0\} is called the kernel of f, denoted by ker(f), the set \{f(x): x∈X\} is called image of f, denoted by Im(f). Notice that f(0) = 0. By Jun et al. (1998) and the set \{x∈X: f(x) = y, for some y∈Y\} is preimage of f, denoted by f^(-1)(Y) by Abbass and Mhadi (2014).

**Theorem 4; Abbass and Hamza (2017):** Let, \( f(X,∗,0)\rightarrow (Y,∗,0) \) be a U-BG-BH- monomorphism and let \( F \) be a U-BG-filter of \( X \). Then, \( f(F) \) is a (U)-BG-filter of \( Y \).

**Theorem 5; Abbass and Hamza (2017):** Let, \( f(X,∗,0)\rightarrow (Y,∗,0) \) be a U-BG-BH-isomorphism. If \( F \) is a U-BG-filter of \( Y \). Then, \( f'(F) \) is a (U)-BG-filter of \( X \).

**Proposition 4; Abbass and Hamza (2017):** Let, \( X \) and \( Y \) be two U-BG-BH-algebras and \( f(X,∗,0)\rightarrow (Y,∗,0) \) be a BH-homomorphism. Then, \( ker(f) \) is a U-BG-filter of \( X \).

**Remark 2; Kim and Kim (2008):** Let \((X,∗,0)\) be a BH-algebra and let, \( N \) be a normal subalgebra of \( X \). Define a relation \( ∼_N \) on \( X \) by \( x ∼_N y \) if and only if \( x∗y∈N \) where \( x, y∈X \). Then, it is easy to show \( ∼_N \) is an equivalence relation on \( X \). Denote the equivalence class containing \( x \) by \( [x]_N \) i.e., \( [x]_N = \{y∈X: x ∼_N y\} \) and let \( X/N = \{[x]_N: x∈X\} \). If \( * \) denoted on \( X/N \) by \( [x]_N * [y]_N = [xy]_N \). Then \( (X/N,∗,[0]_N) \) is a BH-algebra and it is called quotient BH-algebra of \( X \) by \( N \), the researchers by Abbass and Dahham (2016), generalized this concept to BH-algebra to obtain \((X/N,∗,[0]_N)\) quotient BH-algebra of \( X \) by \( N \).

**Theorem 6; Abbass and Hamza (2017):** Let \((X,∗,0)\) be a U-BG-BH-algebra and \( N \) be a normal subalgebra, if \( F \) is a U-BG-filter in \( X \), then, \( F/N \) is U-N-BG-filter of \((X/N,∗,[0],0)\).

**Proposition 5; Abbass and Mhadi (2016):** Let, \( X \) be a U-BG-BH-algebra. Then, every BH-ideal is a completely closed U-BG-ideal of \( X \).

**Theorem 7; Abbass and Dahham (2012):** Let, \( N \) be a normal subalgebra of a BH-algebra \( X \). Then, \( X/N \) is a BH-algebra.

**Remark 3; Abbass and Hamza (2017):** Let \( \{(X,∗,0): i∈λ\} \) be a family of U-BG-BH-algebra. Define the Cartesian product of all \( X_i \) \( i∈λ \) to be the structure \( [\bigcup]_λ X_i = (\bigcup]_λ X_i, ∗, 0) \) where, \( [\bigcup]_λ X_i \) is the set of tuples \( \{(x_i): i∈λ \) and \( x_i∈X_i\} \) and whose binary operation \( ∗ \) is given by \( (x_i)∗(y_i) = (x_i ∗ y_i) \), for all \( i∈λ \) and \( x_i, y_i∈X_i \). Note that the binary operation \( ∗ \) is componentwise.

**Theorem 8; Abbass and Hamza (2017):** Let \( ([\bigcup]_λ X_i, ∗, 0) \) be a U-BG-BH-algebra. If \( \{(F_i, ∗, 0): i∈λ\} \) be a family of U-BG-filter of \( X \). Then, \( [\bigcup]_λ F_i \) is a U-BG-filter of the product algebra \( [\bigcup]_λ X_i \).

**Definition 17; Zhang et al. (2001):** A BH-algebra \( X \) is said to be normal BH-algebra if it satisfies the following conditions:

- \( 0^*(x^∗y) = 0^*(x)∗0^*(y) \) for all \( x, y∈X \)
- \( (x^∗y)^∗x = 0^*y \), for all \( x, y∈X \)
- \( (x^*(x^∗y))^∗y = 0 \) for all \( x, y∈X \)

**RESULTS AND DISCUSSION**

In this study, the notion of closed and completely closed U-BG-filter of U-BG-BH-algebra are introduced for our discussion, we shall link these notions with the notions which mentioned in preliminaries.

**Definition 18:** A U-BG-filter \( F \) of a U-BG-BH-algebra \( X \) is called a closed U-BG-filter of \( X \) if \( 0^*x∈F \) for all \( x∈X \).

**Example 1:** Consider the U-BG-BH-algebra \( X = \{0, 1, 2, 3, 4\} \) with binary operation \( "*" \) defined as follows Table 1:

where, \( U = \{0, 1, 2\} \), the U-BG-filter \( F = \{1, 3\} \) is a closed U-BG-filter of \( X \). But the U-BG-filter \( F = \{1, 4\} \) is not a closed U-BG-filter, since, \( 4∈F \) and \( 0^*4 = 3∈F \).

**Definition 19:** A U-BG-filter \( F \) of a U-BG-BH-algebra \( X \) is called a completely closed U-BG-filter of \( X \) if \( x^∗y∈F \) for all \( x, y∈F \).

**Example 2:** Consider the U-BG-BH-algebra \( X \) in example 1, the U-BG-filter \( F = \{0, 1, 3\} \) is a completely closed U-BG-filter of \( X \) but the U-BG-filter \( F = \{0, 1, 4\} \) is not a completely closed U-BG-filter of \( X \), since, \( 1, 4∈F \) but \( 1^*4 = 2∉F \).

**Remark 4:** Let, \( X \) be a U-BG-BH-algebra. The filters \( F = \{0\} \) and \( F = X \) are completely closed U-BG-filters of \( X \) which are called a trivial completely closed U-BG-filters of \( X \).

**Proposition 6:** Let, \( X \) be a U-BG-BH-algebra. Then, every closed filter of \( X \) is a closed U-BG-filter of \( X \).

**Proof:** Directly by Proposition 1 and Definition 10.

**Remark 5:** The converse of Proposition 7 is not correct in general as in the following example.

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Example 3: Consider the U-BG-BH-algebra $X$ in Example (1), the U-BG-filter $F = \{0, 1, 3\}$ of $X$ is a closed U-BG-filter of $X$ but it is not a closed filter of $X$ because $F$ is not a filter of $X$, since, $3 \in F$, $3 \times 4 = 0$ but $4 \in F$.

Proposition 7: Let $X$ be a U-BG-BH-algebra. Then every completely closed filter of $X$ is a completely closed U-BG-filter of $X$.

Proof: Its directly by Definition (19) and Proposition (1).

Remark 6: The converse of Proposition (7) is not correct in general as in the following example.

Example 4: Consider the U-BG-BH-algebra $X$ in example 1, the U-BG-filter $F = \{0, 1, 3\}$ of $X$ is a completely closed U-BG-filter of $X$ but it is not a completely closed filter of $X$ because $F$ is not a filter, since, $1 \notin F$, $1 \times 4 = 0$ but $4 \in F$.

Proposition 8: Let, $X$ be a U-BG-BH-algebra and $F$ be a completely closed U-BG-filter of $X$. Then, $0 \in F$.

Proof: Let, $F$ be a completely closed U-BG-filter of $X$ and let $x \in F$, then $x^* \in F$ (since, $F$ is a completely closed U-BG-filter of $X$). So, by using Definition (2) (1), $0 \in F$ (since, $x^*x = 0$).

Proposition 9: Let, $X$ be a U-BG-BH-algebra. Then, every completely closed U-BG-filter of $X$ is a closed U-BG-filter of $X$.

Proof: Its directly from Proposition (8) and by applying Definition (19), we get $0 \in x \in F$, so, $I$ is a closed U-BG-filter of $X$.

Remark 7: The converse of Proposition (9) is not correct in general as in the following example.

Example 5: Consider a U-BG-BH-algebra, $X = \{0, 1, 2, 3, 4\}$ with binary operation `$*$` defined as follows Table 2: where, $U = \{0, 1, 2\}$. The U-BG-filter $F = \{0, 3, 4\}$ a closed U-BG-filter of $X$ but it is not a completely closed U-BG-filter, since, $3 \in F$ and $3 \times 4 = 2 \in F$.

Proposition 10: Let, $X$ be a U-BG-BH-algebra and $F$ be a completely closed U-BG-filter of $X$. Then, $F$ is BH-algebra with the same binary operation on $X$ and the constant 0.

Proof: straightforward.

Theorem 9: Let, $X$ be a U-BG-BH-algebra and let $F$ be a U-BG-filter of $X$. Then, $F$ is a completely closed U-BG-filter of $X$ if and only if $F$ is a subalgebra of $X$ contain in $U$.

Proof: Let, $F$ be a completely closed U-BG-filter of $X$ and let $x \in F$, then, $x^*y \in F$ (since, $F$ is a completely closed U-BG-filter of $X$). So, $x^*y \in F$. Conversely, let $F$ be a subalgebra of $X$. Now, let $x \in F$, so, $x^*y \in F$ for all $x \in F$ (since, $F$ is a subalgebra). Then $F$ is completely closed U-BG-filter of $X$.

Lemma 2: Let, $X$ be a U-BG-BH-algebra and let, $N$ be a normal subset of $X$ contain in $U$. Then, $N$ is a completely closed U-BG-filter of $X$.

Proof: Directly from Theorem 1 and 9.

Proposition 11: Let, $X$ be a U-BG-BH-algebra and $F$ be a completely closed U-BG-filter of $X$. Then, $F$ is a completely closed U-BG-ideal of $X$.

Proof: Let, $F$ be a completely closed U-BG-filter of $X$.

- $0 \in F$ (by Proposition 8).
- Let, $x^*y \in F$, $y \in U$. Since, $F$ is a completely closed U-BG-filter of $X$. Then, $(x^*y)^* \in F$. Since, $F$ is a U-BG-filter of $X$, so, we have $y \in F$. Now, take $y = 0$.

Since, $U$ is a B-H-algebra, then $(x^*0)^* \in F$, so, $x \in F$ (by Definition (1) (ii)).

- Let $x \in F$, so, $x^*y \in F$ (Since, $F$ is a completely closed U-BG-filter of $X$), therefore, $F$ is a completely closed U-BG-ideal of $X$.

Proposition 12: Let, $X$ be a U-BG-BH-algebra and $F$ be a closed U-BG-filter such that $x^*y \neq 0$, for all $y \in F$ and $x \in F$. Then, $F$ is a closed filter of $X$.
Proof: Let, F be a closed U-BG-filter of X. Then, F is a U-BG-filter of X by applying Theorem 3, we get F is a filter of X. Now, let, xεF. Then, 0*εF (since, F is a closed U-BG-filter). Therefore, F is a completely closed filter of X.

Proposition 13: Let, X be a U-BG-BH algebra and F be a completely closed U-BG-filter of X such that x*y = 0, for all yεF and xεF. Then, F is a completely closed filter of X.

Proof: Let, F be a completely closed U-BG-filter of X. Then, F is a U-BG-filter of X by applying Theorem 3, we get F is a filter of X. Now, let x, yεF, so, x*yεF (since, F is a completely closed U-BG-filter). Therefore, F is a completely closed filter of X.

Theorem 10: Let, X be a normal U-BG-BH algebra and let, R be regular subset of X such that RεU. If R is a U-BG-ideal, then, R is a U-BG-filter of X.

Proof: Let, R be a U-BG-ideal of X.

- Let, x, yεR, since, R is a U-BG-ideal. Then 0εR. Now, (x*(y*x))εR (by Definition (17) (iii)), so, x*(y*x)εR (by Definition (13) (ii)). Similarly, y*(y*x)εR.
- Let xεR, y*x = 0, yεU, then, x*yεR and xεF. By Definition 7, we get, yεR, therefore, R is a U-BG-filter of X.

Corollary 1: Let, X be a normal U-BG-BH algebra and let, R be regular subset of X which is contained in U. If R is a completely closed U-BG-ideal of X, then, R is a completely closed U-BG-filter of X.

Proof: Let, R be a completely closed U-BG-ideal. Then, R is a U-BG-ideal of X by using theorem 10, we get R is a U-BG-filter of X. Now, let x, yεR, hence, x*yεR (since, R is a completely closed U-BG-ideal). Therefore, R is a completely closed U-BG-filter of X.

Theorem 11: Let, X be an associative U-BG-BH algebra and F be a U-BG-filter contain in U. Then, F is a completely closed U-BG-filter if and only if F is a completely closed U-BG-ideal of X.

Proof: Let, F be a completely closed U-BG-filter: 0εF (by Proposition 8). Let, x*yεF, yεF, xεU, then (x*y)*yεF (since, F is a completely closed U-BG-filter). So, x*(y*y)εF (by Definition (6) by applying Definition (2) (i)), we get, x*0εF, hence, xεF (by using Definition (2) (iii)). Therefore, F is a U-BG-ideal of X.

Let, x, yεF. Then, x*yεF (since, F is a completely closed U-BG-filter), so, F is a completely closed U-BG-ideal. Conversely, let F be a completely closed U-BG-ideal of X.

Table 3: Closed U-BG-ideal is not a closed U-BG-filter

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Let, x, yεF, then x*yεF, y*xεF (since, F is a closed U-BG-ideal), then, we have, x*(y*x)εF and y*(y*x)εF. Let, xεF, x*y = 0, yεU, since, FεU, we have x=x, so, x=x (since, U is a BG-algebra), then, yεF, hence, F is a U-BG-filter of X. Let, x, yεF, then, x*yεF (since, F is a completely closed U-BG-ideal), so, F is a completely closed U-BG-filter of X.

Theorem 12: Let, X be an associative U-BG-BH algebra. Then, every closed U-BG-ideal of X is a closed U-BG-filter of X.

Proof: Let, X be an associative BH-algebra and let I be a closed U-BG-ideal of X.

- Let, x, yεI, since, X is an associative, we obtain, y*(y*x) = (y*y)*x = 0*εF (since, I is a closed U-BG-ideal) and x*(x*y) = (x*x)*y = 0*εF.
- Let, xεI and yεU such that x*y = 0. Thus, x = y (by Theorem (2) (v)), so, yεI, then, we get, I is a U-BG-filter of X.
- Let, xεI. By Definition (8) (i), we obtain 0εI, so, 0*εI (since, I is a closed U-BG-ideal), therefore, I is a closed U-BG-filter of X.

Remark 8: If X is not associative U-BG-BH algebra then the Theorem 12 is not correct in general as in the following example:

Example 6: Let, X be a U-BG-BH algebra, X = {0, 1, 2, 3, 4} with binary operation "*" defined as follows Table 3 and U = {0, 2} the closed U-BG-ideal I = {0, 1} is not a closed U-BG-filter, since, I is not a filter. Since, 1εI, 2*2 = 0, 2εU but 2εF.

Proposition 14: Let, X be a U-BG-BH algebra and the right cancellation law holds in X. Then, every BH-ideal of X is a completely closed U-BF-filter of X.

Proof: Let, I be a BH-ideal of X. By Proposition 5, we get I is a completely closed U-BG-ideal of X.

- Let, x, yεI, so, we have x*(x*y)εF, y*(y*x)εF (since, I is a completely closed U-BG-ideal of X.
- Let, xεI, x*y = 0, yεU, then, x*y = y*y, so, by Lemma (1) (i), we obtain x = y, imply that yεF. Therefore, I is a U-BG-filter of X.
Remark 9: The converse of Proposition 14 is not correct in general as in the following example:

Example 7: Consider U-BG-BH-algebra X in Example 1, F = {0, 3} is a completely closed U-BG-filter of X but it is not a BH-ideal of X, since, 3 ∈ F, 2 ∈ X but 3 ∗ 2 = 1 ∉ F.

Proposition 15: Let \( \{F_i, i ∈ I\} \) be a family of closed U-BG-filters of a U-BG-BH-algebra X. Then, \( \bigcap_{i ∈ I} F_i \) is a closed U-BG-filter of X.

Proof: Since, \( F_i \) is a closed U-BG-filter, \( \forall i ∈ I \), then, \( F_i \) is a U-BG-filter, \( \forall i ∈ I \) (by Definition 18), so, \( \bigcap_{i ∈ I} F_i \) is a U-BG-filter (by Proposition 2).

Now, let \( x ∈ \bigcap_{i ∈ I} F_i \), hence, \( x ∈ F_i, \forall i ∈ I \), so, \( 0^* x ∈ F_i, \forall i ∈ I \) (since, F is a closed U-BG-filter of X). By Definition 18, then \( 0^* x ∈ \bigcap_{i ∈ I} F_i \). So, we get, \( \bigcap_{i ∈ I} F_i \) is a closed U-BG-filter of X.

Proposition 16: Let \( \{F_i, i ∈ I\} \) be a family of completely closed U-BG-filters of a U-BG-BH-algebra X. Then, \( \bigcap_{i ∈ I} F_i \) is a completely closed U-BG-filter of X.

Proof: Since, \( F_i \) is a completely closed U-BG-filter of X, \( \forall i ∈ I \), then, \( F_i \) is a U-BG-filter of X, \( \forall i ∈ I \) (by Definition 19). By Proposition 2, we obtain \( \bigcap_{i ∈ I} F_i \) is a U-BG-filter of X.

Now, let \( x, y ∈ \bigcap_{i ∈ I} F_i \), so, \( x, y ∈ F_i, \forall i ∈ I \), then, \( x^* y ∈ F_i, \forall i ∈ I \) (since, F is a completely closed U-BG-filter), hence, \( x^* y ∈ \bigcap_{i ∈ I} F_i \). Therefore, \( \bigcap_{i ∈ I} F_i \) is a completely closed U-BG-filter of X.

Proposition 17: Let \( \{F_i, i ∈ I\} \) be a chain of closed U-BG-filters of a U-BG-BH-algebra X. Then, \( \bigcup_{i ∈ I} F_i \) is a closed U-BG-filter of X.

Proof: Since, \( F_i \) is a closed U-BG-filter of X, \( \forall i ∈ I \), then, \( F_i \) is a U-BG-filter of X, \( \forall i ∈ I \) (by Definition 18), so, \( x ∈ \bigcup_{i ∈ I} F_i \) is a U-BG-filter of X (by Proposition 3). Now, let \( x ∈ \bigcup_{i ∈ I} F_i \), then, there exist \( F_{i_0}, F_{i_1} ∈ \{F_i, i ∈ I\} \) such that \( x ∈ F_{i_0}, F_{i_1} \), and either \( F_{i_0} ⊆ F_{i_1} \) or \( F_{i_1} ⊆ F_{i_0} \) (since, \( \{F_i, i ∈ I\} \) is a chain). If \( F_{i_0} ⊆ F_{i_1} \), then \( x ∈ F_{i_0} \) hence, \( 0^* x ∈ F_{i_0} \) (since, \( F_{i_0} \) is a closed U-BG-filter). Similarly, if \( F_{i_1} ⊆ F_{i_0} \), then \( 0^* y ∈ F_{i_0} \), therefore, \( \bigcup_{i ∈ I} F_i \) is a closed U-BG-filter of X.

Proposition 18: Let \( \{F_i, i ∈ I\} \) be a chain of completely closed U-BG-filters of a U-BG-BH-algebra X. Then, \( \bigcup_{i ∈ I} F_i \) is a completely closed U-BG-filter of X.

Proof: Since, \( F_i \) is a completely closed U-BG-filter of X, \( \forall i ∈ I \), we get \( F_i \) is a U-BG-filter of X, \( \forall i ∈ I \) (by Definition 19). By Proposition 3, \( \bigcup_{i ∈ I} F_i \) is a U-BG-filter of X (by Proposition 3). Now, let \( x, y ∈ \bigcup_{i ∈ I} F_i \), so, there exist \( F_j, F_k ∈ \{F_i, i ∈ I\} \) such that \( x ∈ F_j, y ∈ F_k \). Then, either \( F_j ⊆ F_k \), or \( F_k ⊆ F_j \) (since, \( \{F_i, i ∈ I\} \) is a chain). If \( F_j ⊆ F_k \), hence, \( x ∈ F_j \), so, \( x^* y ∈ F_j \) (since, \( F_j \) is a completely closed U-BG-filter). Similarly, if \( F_k ⊆ F_j \), then \( 0^* y ∈ F_k \), therefore, \( \bigcup_{i ∈ I} F_i \) is a completely closed U-BG-filter of X.

Proposition 19: Let, \( f(X, *, 0) - (Y, *, 0') \) be a U-BG-BH-monomorphism and let \( F \) be a closed U-BG-filter of X. Then, \( f(F) \) is a closed \( f(U)-\)BG-filter of Y.

Proof: Let, \( F \) be a closed U-BG-filter of X, then \( F \) is a U-BG-filter of X (by Definition 18), so, by Theorem 4, we obtain \( f(F) \) is a \( f(U)-\)BG-filter of Y. Now, let \( x ∈ f(F) \), then, there exist \( a ∈ F \) such that \( y = f(a) \), hence, \( 0^* y = 0^* f(a) = f(0^* a) \).

Since, \( 0^* a = f(0^* a) \), we get \( f(0^* a) = f(F) \), so, \( 0^* y = f(F) \), therefore, \( f(F) \) is a closed \( f(U)-\)BG-filter of Y.

Proposition 20: Let, \( f(X, *, 0) - (Y, *, 0') \) be a U-BG-BH-monomorphism and let \( F \) be a completely closed U-BG-filter of X. Then, \( f(F) \) is a completely closed \( f(U)-\)BG-filter of Y.

Proof: Let, \( F \) be a completely closed U-BG-filter of X, so, \( F \) is a U-BG-filter of X (by Definition 19), then, \( f(F) \) is a \( f(U)-\)BG-filter of Y (Theorem 4).

Now, let \( x = f(a) \), so, there exist \( a, b ∈ F \) such that \( x = f(a), y = f(b) \), then \( x^* y = f(a)^* f(b) = f(a^* b) \). Since, \( F \) is a completely closed U-BG-filter of X, then, we obtain \( a^* b ∈ F \), hence, \( x^* y ∈ f(F) \), therefore, \( f(F) \) is a completely closed \( f(U)-\)BG-filter of X.

Theorem 13: Let, \( f(X, *, 0) - (Y, *, 0') \) be a U-BG-BH-isomorphism. If \( F \) is a closed U-BG-filter of X, then, \( f(F) \) is a closed \( f(U)-\)BG-filter of X.

Proof: Let, \( F \) be a closed U-BG-filter of X. Then, \( F \) is a U-BG-filter of X (by Definition 18), so, \( f(F) \) is a \( f(U)-\)BG-filter of X (by Theorem 5). Now, let, \( y ∈ f(F) \), hence, we have, \( f(y) ∈ F \) and \( 0^* y = f(F) \) (since, \( F \) is a closed U-BG-filter of Y), then, \( f(0^* y) = f(F) \), therefore, \( f(F) \) is a closed \( f(U)-\)BG-filter of X.
Theorem 14: Let, \( f(X, *, 0) = (Y, *, Y) \) be a U-BG-BH-isomorphism. If \( F \) is a completely closed U-BG-filter of \( Y \). Then, \( f'(F) \) is a completely closed \( f'(U) \)-BG-filter of \( X \).

Proof: Let \( F \) be completely closed U-BG-filter of \( Y \). So, \( F \) is a U-BG-filter of \( Y \) (by Definition 19), then, \( f'(F) \) is a \( f'(U) \)-BG-filter of \( X \) (by Theorem 5). Now, let, \( x, \ y \in f'(F) \), hence, \( f(x)eF, f(y)eF \). Since, \( f \) is a completely closed U-BG-filter of \( Y \), we obtain \( f(x)eF, f(y)eF \), then, \( f(x)\cdot y \in f(F) \), so, we get \( x\cdot y = f'(F) \), therefore, \( f'(F) \) is a completely closed \( f'(U) \)-BG-filter of \( X \).

Proposition 21: Let, \( f(X, *, 0) = (Y, *, 0) \) be a U-BG-BH-homomorphism. Then \( Ker(f) \) is a completely closed filter of \( X \).

Proof: Let, \( f(X, *, 0) = (Y, *, 0) \) be a U-BG-BH-homomorphism. Then \( Ker(f) \) is a U-BG-filter (by Proposition 4). Now, let, \( x, y \in Ker(f) \), we have \( f(x) = f(y) = 0 \) (and then, \( f(x)\cdot y = f(x) = 0 \) by Remark 1), we get \( x, y \in Ker(f) \), therefore, \( Ker(f) \) is a completely closed U-BG-filter of \( X \).

Proposition 22: Let, \( f(X, *, 0) = (Y, *, 0) \) be a U-BG-BH-homomorphism. Then, \( Ker(f) \) is a closed filter of \( X \).

Proof: Directly by Proposition 9 and 21.

Proposition 23: Let, \( X \) be a U-BG-BH-algebra, \( N \) be a normal subalgebra of \( X \) and \( F \) be a closed U-BG-filter of \( X \). Then, \( F/N \) is a closed U/N-BG-filter of \( X/N \).

Proof: Let, \( F \) be a closed U-BG-filter of \( X \), so, \( F \) is a U-BG-filter of \( X \) (by Definition 18), then, \( F/N \) is a U/N-BG-filter of \( X/N \) (by Theorem 6). Now, let, \( (0)_N, (y)_N \in F/N \), since, \( 0 \cdot y \in F \) (by \( F \) is closed U-BG-filter), so \( (0)_N, (y)_N \), \( (0\cdot y)_N \in F/N \), therefore, \( F/N \) is a closed U/N-BG-filter of \( X/N \).

Proposition 24: Let, \( X \) be a U-BG-BH-algebra, \( N \) be a normal subalgebra of \( X \) and \( F \) is a completely closed U-BG-filter of \( X \). Then, \( F/N \) is a completely closed U/N-BG-filter of \( X/N \).

Proof: Let, \( F \) be a completely closed U-BG-filter of \( X \) (by Definition 19), we obtain \( F \) is a U-BG-filter of \( X \), then, \( F/N \) is a U/N-BG-filter of \( X/N \) (by Remark 6).

Now, let, \( (x)_N, (y)_N \in F/N \), so, \( (x\cdot y)_N \in F/N \) (since \( x\cdot y \in F \) by \( F \) is closed U-BG-filter), then, we get \( F/N \) is a completely closed U/N-BG-filter of \( X/N \).

Corollary 2: Let, \( X \) be a U-BG-BH-algebra, \( N \) be a normal subalgebra of \( X \) and \( F \) is a completely closed U-BG-filter in \( X \). Then, \( F/N \) is a closed U/N-BG-filter of \( X/N \).

Proof: Let, \( F \) be a completely closed U-BG-filter of \( X \). By Proposition (9), we get \( F \) is a closed U-BG-filter of \( X \), then, \( F/N \) is a closed U/N-BG-filter in \( X/N \). By Proposition (24).

Theorem 15: Let, \( (\bigwedge_{\lambda} X, \wedge, (0)) \) be a \( \bigwedge_{\lambda} \) U-BG-BH-algebra. If \( \{F_{\lambda} : \lambda \in \Lambda \} \) be a family of a closed U-BG-filter of \( X \). Then \( \bigwedge_{\lambda} F_{\lambda} \) is a closed \( \bigwedge_{\lambda} \) U-BG-filter of the product algebra \( \bigwedge_{\lambda} X \).

Proof: Let \( \{F_{\lambda} : \lambda \in \Lambda \} \) be a family of a closed U-BG-filter of \( X \). By Definition (18), we obtain \( \{F_{\lambda} : \lambda \in \Lambda \} \) be a family of a U-BG-filter of \( X \), then \( \bigwedge_{\lambda} F_{\lambda} \) is a \( \bigwedge_{\lambda} \) U-BG-filter of \( \bigwedge_{\lambda} X \) (by Theorem 8). Now, let, \( y = (y) \in \bigwedge_{\lambda} F_{\lambda} \), for all \( y \in F_{\lambda} \), and \( \lambda \in \Lambda \), so, \( (0) \in \bigwedge_{\lambda} (y) \) = \( (0) \cdot y \), since, \( F_{\lambda} \) is a closed U-BG-filter of \( X_{\lambda} \), then \( 0 \cdot y \in F_{\lambda} \) (by Definition 18), hence, \( (0) \in \bigwedge_{\lambda} (y) \in \bigwedge_{\lambda} F_{\lambda} \), then, \( \bigwedge_{\lambda} F_{\lambda} \) is a closed \( \bigwedge_{\lambda} \) U-BG-filter of \( \bigwedge_{\lambda} X \).

Theorem 16: Let \( (\bigwedge_{\lambda} X, \wedge, (0)) \) be a \( \bigwedge_{\lambda} \) U-BG-BH-algebra. If \( \{F_{\lambda} : \lambda \in \Lambda \} \) be a family of a completely closed U-BG-filter of \( X \). Then, \( \bigwedge_{\lambda} F_{\lambda} \) is a completely closed \( \bigwedge_{\lambda} \) U-BG-filter of the product algebra \( \bigwedge_{\lambda} X \).

Proof: Let \( \{F_{\lambda} : \lambda \in \Lambda \} \) be a family of a completely closed U-BG-filter of \( X \). Then \( \{F_{\lambda} : \lambda \in \Lambda \} \) be a family of a U-BG-filter of \( X \) (by Definition 19), so \( \bigwedge_{\lambda} F_{\lambda} \) is a \( \bigwedge_{\lambda} \) U-BG-filter of \( \bigwedge_{\lambda} X \) (by Theorem 8). Now, let, \( x = (x) \), \( y = (y) \in \bigwedge_{\lambda} F_{\lambda} \), for all \( x \in F_{\lambda} \), \( y \in F_{\lambda} \), and \( \lambda \in \Lambda \), then \( x\cdot y = (x)\cdot (y) = (x\cdot y) \in \bigwedge_{\lambda} F_{\lambda} \), since, \( F_{\lambda} \) is a completely closed U-BG-filter of \( X_{\lambda} \), then \( x\cdot y \in F_{\lambda} \) (by Definition 18), hence, \( (x)\cdot (y) \in \bigwedge_{\lambda} F_{\lambda} \), therefore, \( \bigwedge_{\lambda} F_{\lambda} \) is a completely closed \( \bigwedge_{\lambda} \) U-BG-filter of \( \bigwedge_{\lambda} X \).

Proposition 25: (Extension property for closed U-BG-filter in U-BG-BH-algebra): Let \( X \) be a normal U-BG-BH-algebra \( F \) is a completely closed U-BG-filter of \( X \), \( G \) is a closed ideal of \( X \) such that \( G \subseteq U \). Then \( G \) is a completely closed U-BG-filter of \( X \).

Proof: Let \( x, y \in G \), since, \( F \) be a completely closed U-BG-filter of \( X \), then, \( 0 \in F \) (by Proposition 8), so, \( 0 \in G \) (since, \( F \subseteq G \) by Definition (17)), we get \( (x\cdot y) \in G \). So, by Definition (8), we obtain \( x\cdot y \cdot 0 \cdot y \in G \). Similarly, \( y \cdot 0 \cdot x \cdot y \in G \).

Let, \( x \in G \), \( x\cdot y = 0 \cdot y \cdot x \in G \), so, \( x \in G \) (since, \( G \subseteq U \)). Then, \( x\cdot y = y\cdot x \), imply that \( x = y \) (by Lemma (1)), so, we obtain \( y \in G \). Therefore, \( G \) is U-BG-filter of \( X \). Now, since, \( G \) is a closed ideal of \( X \), thus, \( 0 \in G \) (by Definition 10). By Definition (17), we obtain \( (x\cdot y) \cdot x \cdot G \). So, we have \( (x\cdot y) \cdot x \in G \), \( x \in G \), then \( x\cdot y \in G \) (Since, \( G \) is an ideal of \( X \)), therefore, \( G \) is a completely closed U-BG-filter of \( X \).
CONCLUSION

In this study, the notions of closed and completely closed U-BG-filter of U-BG-BH-algebra are introduced. Furthermore, the results are examined in terms of the relationship between closed and completely closed U-BG-filters. In addition, the relationship between the closed and completely closed U-BG-filters with the other filters as well as some special ideals are also presented. The important characteristics of closed and completely closed U-BG-filters are analyzed.

REFERENCES