Comparing Different Estimators of Reliability Function for Stress-Strength Models with Applications

Inaam Rikan Hassan
University of Information Technology and Communications, Baghdad, Iraq

Abstract: One of the most practical application of reliability as a function of time is a well stress-strength model were this model have several applications like Physics, engineering and components, so here, we introduce the Stress-Strength (S-S) reliability model for system contains one component denoted by \( R = p(y < x) \) where \( y \) is stress random variable and \( x \) is strength random variable were:

\[
R = p(y < x) = \int_{\alpha} f(x) f(y) dy dx
\]

The studied model introduced represents reliability function for stress-strength model, assuming the components of stress and strength are independent and identically distributed as Exponentiated Weibull Distribution (EWD). The model of S-S derived and the reliability of it also found. Then estimating by maximum likelihood and least square methods. The comparison done through simulation using different sets of sample size \((n, m)\) also different sets of initial values of \((\beta, \beta, \theta)\), all the results of comparison explained by tables.

Key words: Stress-strength model, reliability function, maximum likelihood method, least square method, MSE, sets

INTRODUCTION

There are many models for stress-strength reliability but here we explain the S-S system (Ng, 2006; Panahi and Asadi, 2010), i.e., we work on estimating \( R = p(y < x) \) where \( y \) is stress random variable and \( x \) is strength random variable and they are independent random variables follow exponentiated Weibull distribution (Hassan and Basheich, 2012; Salehi and Ahmadi, 2015; Mokhlis, 2006; Singh et al., 2015).

In statistical approach most (stress-strength) models assumed the component of stress and strength are independent and identically distributed, many researchers discuss the reliability of stress-strength model (Al-Zahrani and Basloom, 2016; Amini, 2017; Hussain, 2014; Karam and Sabea, 2017; Salem, 2013) and other like Church and Harris (1970), Bhattacharyya and Johnson (1974) developped model for stress-strength system for \((k)\) components system. Many other researchers studied the model like Pandey and Uddin (1991).

Theoretical aspects: Estimating reliability for stress-strength model (with one component), here we have (stress-strength) model were the stress is \( y \) and strength is \( x \), two independent random variables taking \((x)\) to be random variable follow Exponentiated Weibull Distribution (EWD), \((\beta, \beta, \theta)\) and \((y)\) is independent random variable EWD \((\beta, \theta)\), i.e.:

\[
f(x) = \beta x^{\beta-1} (1-e^{-x})^{\beta-1} e^{-x}
\]

(1)

\[
f(y) = \beta y^{\beta-1} (1-e^{-y})^{\beta-1} e^{-y}
\]

(2)

Where:

- \( \theta = \) Scale
- \( \beta = \) The shape

while the cumulative distribution function is:

\[
F(x, \beta, \theta) = (1-e^{-x})^\beta x, \beta, \theta > 0
\]

(3)

The reliability function is:

\[
R(x, \beta, \theta) = 1-(1-e^{-x})^\beta x, \beta, \theta > 0
\]

(4)

Reliability is the probability that the unit or equipment or machines still work over time \((t)\) when it is necessary to consider the effects of environmental conditions while evaluating reliability, we need here to estimate reliability.
of (stress-strength) model were this model have applications in Physics and Engineering, similar to strength failure and system break down, it have many applications, especially on engineering devices which have more than one component reliability function for stress-strength system (Khan and Jan, 2014; Kizilaslan and Nadar, 2015; Rao, 2012a, b) which defined above, the estimated reliability is:

\[
\hat{R} = \frac{\hat{\beta}_1}{\beta_1 + \hat{\beta}_2}
\]

**MATERIALS AND METHODS**

**Estimation methods**

**Maximum likelihood method:** The estimation of \((\hat{\beta}_1, \hat{\beta}_2)\) through this method as: Let \((x_1, x_2, ..., x_n)\) be a random variable from PDF in Eq. 1 and let \((y_1, y_2, ..., y_m)\) be a random variable from PDF in Eq. 1, then:

\[
L = \prod_{i=1}^{n} f(x_i, \beta_1, \beta_2, \theta) \prod_{j=1}^{m} f(y_j, \beta_2, \theta) = \\
\left[ \beta_1^p \theta^p \prod_{i=1}^{n} \left( 1 - e^{-x_i} \right) \right]^{n-1} \times \\
\left[ \beta_2^m \theta^m \prod_{j=1}^{m} \left( 1 - e^{-y_j} \right) \right]^{m-1}
\]

\[
\log L = n \log \beta_1 + m \log \beta_2 + n \log \theta + \\
(\theta-1) \sum_{i=1}^{n} \log x_i + (\beta_1 - 1) \sum_{i=1}^{n} \log \left( 1 - e^{-x_i} \right) + \\
(\beta_2 - 1) \sum_{j=1}^{m} \log y_j + (\beta_2 - 1) \sum_{j=1}^{m} \log \left( 1 - e^{-y_j} \right)
\]

\[
\frac{\partial \log L}{\partial \theta} = \frac{n+m}{\theta} + \left[ \sum_{i=1}^{n} \log x_i + \sum_{j=1}^{m} \log y_j \right] - \sum_{i=1}^{n} x_i \left( 1 - e^{-x_i} \right) - \sum_{j=1}^{m} y_j \left( 1 - e^{-y_j} \right)
\]

\[
\frac{\partial \log L}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n} \log \left( 1 - e^{-x_i} \right)
\]

\[
\frac{\partial \log L}{\partial \beta_2} = \frac{m}{\beta_2} + \sum_{j=1}^{m} \log \left( 1 - e^{-y_j} \right)
\]

From these two Eq. 9 and 10, we obtain MLE’s for two shape parameters \(\hat{\beta}_1, \hat{\beta}_2\):

\[
\hat{\beta}_1 = \frac{n}{\sum_{i=1}^{n} \log \left( 1 - e^{-x_i} \right)}
\]

\[
\hat{\beta}_2 = \frac{m}{\sum_{j=1}^{m} \log \left( 1 - e^{-y_j} \right)}
\]

The maximum likelihood for scale parameter \((\theta)\) is:

\[
\frac{\partial \log L}{\partial \theta} = \frac{n+m}{\theta} + \left[ \sum_{i=1}^{n} \log x_i + \sum_{j=1}^{m} \log y_j \right] - \sum_{i=1}^{n} x_i \left( 1 - e^{-x_i} \right) - \sum_{j=1}^{m} y_j \left( 1 - e^{-y_j} \right)
\]

\[
\frac{\partial \log L}{\partial \beta_2} = \frac{m}{\beta_2} + \sum_{j=1}^{m} \log \left( 1 - e^{-y_j} \right)
\]

**Estimation of (S-S) Model by least square method:** The estimator by this method obtained by minimizing sum square for the difference between \([F(x_i)]\) and some non parametric estimators of \([F(x_i)]\) which may be:

\[
\hat{F}(x_i) = \frac{i}{n+1} = p_i
\]

Let:

\[
F_i(x_i, \alpha, \theta) = \left( 1 - e^{-x_i} \right)^\alpha
\]

\[
\hat{\beta}_i \ln \left( 1 - e^{-x_i} \right) = \ln p_i
\]

\[
T = \sum_{i=1}^{n} \left[ \hat{\beta}_i \ln \left( 1 - e^{-x_i} \right) - \ln p_i \right]^2
\]

\[
\frac{\partial T}{\partial \beta_i} = 2 \sum_{i=1}^{n} \left[ \hat{\beta}_i \ln \left( 1 - e^{-x_i} \right) - \ln p_i \right] \ln \left( 1 - e^{-x_i} \right) = 0
\]

\[
\hat{\beta}_{LS} = \frac{\sum_{i=1}^{n} \left[ \ln p_i \ln \left( 1 - e^{-x_i} \right) \right]}{\sum_{i=1}^{n} \left[ \ln \left( 1 - e^{-x_i} \right) \right]^2}
\]

Using same method, we find the least square estimator of \((\hat{\beta}_2)\) for \((y)\) were \((y)\) is stress-random EWD \((\hat{\beta}_2, \theta)\).
\[
\hat{\beta}_{\text{MLE}} = \frac{\sum_{i=1}^{n} \left[ n_{i} \ln \left( 1 - e^{-\beta_{i} x_{i}} \right) \right]}{\sum_{i=1}^{n} \left[ \ln \left( 1 - e^{-\beta_{i} x_{i}} \right) \right]^2}
\]

Then from Eq. 14 and 15:

\[
\tilde{R}_{\text{MLE}} = \frac{\hat{\beta}_{\text{MLE}}}{\hat{\beta}_{\text{MLE}} + \hat{\beta}_{\text{MLE}}}
\] (16)

**RESULTS AND DISCUSSION**

We apply simulation procedure to find numerical results for estimated reliability of stress-strength model using different sample size and different set of initial values of \((\beta_1, \beta_2, \theta)\) were:

**Step 1:**

\[
F(x) = \left( 1 - e^{-x^q} \right)^{\frac{1}{q}}
\]

\[
\bar{r}_q = \left\{ 1 - e^{-x^q} \right\}^{\frac{1}{q}}
\]

\[
\tilde{r}_q = \left[ \ln \left( 1 - \bar{r}_q \right) \right]^{\frac{1}{q}}
\]

**Step 2:** Let \((y_i)\) be values of the random variable \((x_i)\):

\[
y_i = \left[ \ln \left( 1 - \bar{r}_q \right) \right]^{\frac{1}{q}}
\]

Use:

\[
R = \frac{\hat{\beta}_1}{\hat{\beta}_1 + \hat{\beta}_2}
\]

From Eq. 16 and:

\[
\tilde{R}_{\text{MLE}} = \frac{\hat{\beta}_{\text{MLE}}}{\hat{\beta}_{\text{MLE}} + \hat{\beta}_{\text{MLE}}}
\]

The comparison done using statistical measure mean square error. Using random sample for \((x_i)\) and \((y_i)\) with sizes \((n, m)\) as:

<table>
<thead>
<tr>
<th>((n, m))</th>
<th>((\beta_1, \beta_2, \theta))</th>
<th>(R)</th>
<th>(\tilde{R}_{\text{MLE}})</th>
<th>(\text{MSE}_{\text{MLE}})</th>
<th>(\bar{R}_u)</th>
<th>(\text{MSE}_{\text{U}})</th>
<th>Best</th>
</tr>
</thead>
<tbody>
<tr>
<td>(15, 10)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
<td>(15, 30)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
<td>(15, 50)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
<td>(30, 10)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
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<td>0.83856</td>
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<td>0.03847</td>
<td>0.80144</td>
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<td>LS</td>
</tr>
<tr>
<td>(50, 30)</td>
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<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
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<td>LS</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
<td>(75, 10)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
<td>(75, 30)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
<tr>
<td>(75, 50)</td>
<td>([2.5, 3, 1.5, 3, 4, 4])</td>
<td>0.8155</td>
<td>0.83856</td>
<td>0.03847</td>
<td>0.80144</td>
<td>0.0052</td>
<td>LS</td>
</tr>
</tbody>
</table>

While the shape parameters:

\[
x_i = [15, 30, 50]
\]

\[
y_j = [30, 50, 100]
\]

From estimated values of reliability function for stress-strength model for different sets of initial values \((\beta_1, \beta_2, \theta) = [2.5, 3, 1.5, 3, 4, 4]\) and different sets of sample size \((n, m)\) in Table 1, we find \(\tilde{R}_{\text{MLE}}\) is best with percentage \([\{11/16\} \times 100\% = 69\%]\) while \(\bar{R}_{\text{MLE}}\) is best with \([\{5/16\} \times 100\% = 31\%]\). For the second set chosen of initial values (Table 2), also, we find that \(\tilde{R}_{\text{MLE}}\) is best with percentage \([\{11/16\} \times 100\% = 69\%]\) while \(\bar{R}_{\text{MLE}}\) is best with \([\{5/16\} \times 100\% = 31\%]\). For the third table, we find that \(\tilde{R}_{\text{MLE}}\) and \(\bar{R}_{\text{MLE}}\) with equal percentage \([\{8/16\} \times 100\% = 50\%]\) while \(\bar{R}_{\text{MLE}}\) is best with \([\{8/16\} \times 100\% = 50\%]\) (Table 3). The fourth table, we find that all values of \(\tilde{R}_{\text{MLE}}\) are the best (Table 4).
Table 3: MSE for $(\hat{\gamma}, \hat{\beta}, \hat{\delta})$ with $(\alpha = 1.5, \beta = 35, \theta = 4)$

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>MSE</th>
<th>MSE</th>
<th>MSE</th>
<th>MSE</th>
<th>Best</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 10)</td>
<td>0.624</td>
<td>0.84752</td>
<td>0.05310</td>
<td>0.7582</td>
<td>0.1925</td>
</tr>
<tr>
<td>(15, 10)</td>
<td>0.624</td>
<td>0.8036</td>
<td>0.0644</td>
<td>0.8013</td>
<td>0.0356</td>
</tr>
<tr>
<td>(15, 30)</td>
<td>0.624</td>
<td>0.8044</td>
<td>0.0632</td>
<td>0.8082</td>
<td>0.0352</td>
</tr>
<tr>
<td>(10, 10)</td>
<td>0.624</td>
<td>0.8052</td>
<td>0.0644</td>
<td>0.7922</td>
<td>0.0343</td>
</tr>
<tr>
<td>(15, 10)</td>
<td>0.624</td>
<td>0.8172</td>
<td>0.0601</td>
<td>0.7761</td>
<td>0.0441</td>
</tr>
<tr>
<td>(10, 30)</td>
<td>0.624</td>
<td>0.8394</td>
<td>0.0552</td>
<td>0.7863</td>
<td>0.0473</td>
</tr>
<tr>
<td>(30, 50)</td>
<td>0.624</td>
<td>0.8421</td>
<td>0.0535</td>
<td>0.7861</td>
<td>0.0855</td>
</tr>
<tr>
<td>(30, 30)</td>
<td>0.624</td>
<td>0.8351</td>
<td>0.0572</td>
<td>0.7626</td>
<td>0.0672</td>
</tr>
<tr>
<td>(50, 10)</td>
<td>0.624</td>
<td>0.8353</td>
<td>0.0641</td>
<td>0.7331</td>
<td>0.0663</td>
</tr>
<tr>
<td>(50, 30)</td>
<td>0.624</td>
<td>0.8541</td>
<td>0.0441</td>
<td>0.7412</td>
<td>0.0656</td>
</tr>
<tr>
<td>(30, 50)</td>
<td>0.624</td>
<td>0.8561</td>
<td>0.0443</td>
<td>0.7435</td>
<td>0.0453</td>
</tr>
<tr>
<td>(50, 100)</td>
<td>0.624</td>
<td>0.8572</td>
<td>0.0440</td>
<td>0.7343</td>
<td>0.0443</td>
</tr>
<tr>
<td>(75, 100)</td>
<td>0.624</td>
<td>0.8606</td>
<td>0.0440</td>
<td>0.7424</td>
<td>0.032</td>
</tr>
<tr>
<td>(75, 50)</td>
<td>0.624</td>
<td>0.8612</td>
<td>0.0336</td>
<td>0.7431</td>
<td>0.031</td>
</tr>
<tr>
<td>(75, 50)</td>
<td>0.624</td>
<td>0.8622</td>
<td>0.0335</td>
<td>0.7422</td>
<td>0.022</td>
</tr>
<tr>
<td>(75, 100)</td>
<td>0.624</td>
<td>0.8634</td>
<td>0.0333</td>
<td>0.7566</td>
<td>0.021</td>
</tr>
</tbody>
</table>

Table 4: MSE for $(\hat{\gamma}, \hat{\beta}, \hat{\delta})$ with $(\alpha = 4, \beta = 4, \theta = 4)$

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>MSE</th>
<th>MSE</th>
<th>MSE</th>
<th>MSE</th>
<th>Best</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 10)</td>
<td>0.634</td>
<td>0.6203</td>
<td>0.01666</td>
<td>0.5162</td>
<td>0.05691</td>
</tr>
<tr>
<td>(15, 10)</td>
<td>0.634</td>
<td>0.6208</td>
<td>0.0176</td>
<td>0.5098</td>
<td>0.0639</td>
</tr>
<tr>
<td>(15, 50)</td>
<td>0.634</td>
<td>0.6244</td>
<td>0.0246</td>
<td>0.5086</td>
<td>0.0538</td>
</tr>
<tr>
<td>(10, 100)</td>
<td>0.634</td>
<td>0.6255</td>
<td>0.02098</td>
<td>0.4992</td>
<td>0.10463</td>
</tr>
<tr>
<td>(30, 10)</td>
<td>0.634</td>
<td>0.6261</td>
<td>0.02099</td>
<td>0.4998</td>
<td>0.10572</td>
</tr>
<tr>
<td>(30, 30)</td>
<td>0.634</td>
<td>0.6265</td>
<td>0.0328</td>
<td>0.5022</td>
<td>0.04709</td>
</tr>
<tr>
<td>(30, 100)</td>
<td>0.634</td>
<td>0.6268</td>
<td>0.07015</td>
<td>0.4998</td>
<td>0.06159</td>
</tr>
<tr>
<td>(30, 100)</td>
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<td>0.6093</td>
<td>0.02089</td>
<td>0.4975</td>
<td>0.10408</td>
</tr>
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<td>0.6234</td>
<td>0.02306</td>
<td>0.4999</td>
<td>0.14073</td>
</tr>
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<td>0.6271</td>
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<td>0.14662</td>
</tr>
<tr>
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<td>0.6225</td>
<td>0.02195</td>
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<td>0.0435</td>
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<tr>
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<td>0.6268</td>
<td>0.00182</td>
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<td>0.0167</td>
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<td>0.0032</td>
</tr>
<tr>
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<tr>
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<td>0.6225</td>
<td>0.0003</td>
<td>0.5448</td>
<td>0.00502</td>
</tr>
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</table>

CONCLUSION

We can conclude that the MLE estimators for reliability function of the studied stress-strength model is best as compared with least square estimators.

REFERENCES


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