Approximation of Sine Series with Coefficient from Class of p-Supremum Bounded Variation Difference Sequences

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Abstract: Recently, the monotone decreasing coefficients of sine series has been generalized by class of p-supremum bounded variation sequences. Further, class of p-supremum bounded variation sequences can be generalized by class of p-supremum bounded variation difference sequences. In this study we compute error about approximation of sine series with coefficient from class p-supremum bounded variation difference sequences.

Key words: Difference sequence, error, p-supremum bounded variation, sine series, coefficients, approximation

INTRODUCTION

Chaundy and Jollife (1917) proved the following classical theorem:

**Theorem 1:** Suppose that \( \{a_n\} \subset [0, \infty) \) is decreasingly tending to zero. A necessary and sufficient conditions for the uniform convergence of the series is:

\[
\sum_{k=1}^{\infty} a_n \sin kx \quad \text{is} \quad \lim_{k \to \infty} k a_n = 0 \tag{1}
\]

The decreasing monotone coefficients (Eq. 1) are said class of Monotone Sequences (MS) and has been generalized by many researchers such as Tikhonov (2008), Zhou et al. (2010) and Korus (2010). These classes are GMS (General Monotone Sequences), NBVS (Non-one sided Bounded Variation Sequences), MVBVS (Mean Value Bounded Variation Sequences) and SBVS (Supremum Bounded Variation Sequences). Zhou et al. (2010) proved that MS \( \subset \) GMS \( \subset \) NBVS and Korus showed that MVBVS \( \subset \) SBVS (Imron and Indradi, 2014).

Furthermore, Liflyand and Tikhonov (2011) generalized GMS to \( g_{M, p} \) p-general monotone sequences, \( 1 \leq p < \infty \). Let \( \alpha = \{a_n\} \) and \( \beta = \{b_n\} \) be two sequences of complex and positive numbers, respectively, a couple \( (\alpha, \beta) \in g_{M, p} \) if there exists \( C > 0 \) such that:

\[
\left( \sum_{k=1}^{\infty} |a_n - a_{n+1}|^p \right)^{1/p} \leq C \beta_n
\]

For \( p, 1 \leq p < \infty \), Imron and Indradi (2013) generalized MVBVS and SBVS to MVBVS \( \Phi \) (p-Mean Value Bounded Variation Sequences) and SBVS \( \Phi \) (Supremum Bounded Variation Sequences). A couple \( (\alpha, \beta) \in \text{MVBVS}_\Phi \) if there exist \( C > 0 \) and \( \lambda > 2 \) such that:

\[
\left( \sum_{k=1}^{\infty} |a_n - a_{n+1}|^p \right)^{1/p} \leq C \sum_{k=1}^{\infty} |b_n|^\lambda
\]

For \( 1 \leq p < \infty \) and \( (\alpha, \beta) \in \text{SBVS}_\Phi \) if there exist \( C > 0 \) and \( \lambda > 1 \) such that:

\[
\left( \sum_{k=1}^{\infty} |a_n - a_{n+1}|^p \right)^{1/p} \leq C \left( \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{n+m} |b_k|^\lambda \right) \right)^{1/\lambda}
\]

For \( 1 \leq p < \infty \). A little modification of definition of SBVS \( \Phi \) gives a class SBVS \( \Phi \). The couple \( (\alpha, \beta) \) is p-supremum bounded variation sequences of second type, written \( (\alpha, \beta) \in \text{SBVS}_\Phi \), if there exist \( C > 0 \) and \( \{b(k)\} \subset [0, \infty) \) tending monotonically to infinity depending only on \( \{a_n\} \) such that:

\[
\left( \sum_{k=1}^{\infty} |a_n - a_{n+1}|^p \right)^{1/p} \leq C \left( \sup_{n \in \mathbb{Z}} \left( \sum_{k=n}^{n+m} |b(k)|^\gamma \right) \right)^{1/\gamma}
\]

holds for \( p, 1 \leq p < \infty \). Imron and Indradi (2013) have shown that MVBVS \( \subset \) SBVS \( \Phi \). Imron and Indradi (2014) generalized of to in the following definition, we consider sequences \( \alpha = \{a_n\} \) and \( \beta = \{b_n\} \) be two sequences of complex and positive numbers, respectively.

**Definition 2:** Let \( \gamma \in \mathbb{N} \), a couple \( (\alpha, \beta) \) is said to be p-Supremum Bounded Variation Sequences order \( n \), written \( (\alpha, \beta) \in \text{SBVS}_{\Phi}^n \), if there exist positive constant \( C \) and \( \gamma > 1 \) such that:

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\[
\left(\sum_{k=m}^{n} |\Delta^k \alpha_k|_p^p \right)^{\frac{1}{p}} \leq C \left( \sup_{|x| \leq m} \sum_{k=1}^{n} |\Delta^k \beta_k| \right)^{1/p}, \quad m \geq n, 1 \leq p \leq \infty,
\]

where \( \Delta^k \alpha_k = \Delta^{k-1} \alpha_k - \Delta^{k-1} \alpha_{k-1} \).

Note that \((\alpha, \beta) \in \text{SBVS}_2(\Delta^1)\) is exactly. This class more general than that one.

**Definition 3:** Let \(n \in \mathbb{N}\), a couple \((\alpha, \beta)\) is said to be \(p\)-Suprema Bound Variation of second type order \(n\), written \((\alpha, \beta) \in \text{SBVS}_2(\Delta^n)\), if there exist \(C > 0\) and \(\{b(k)\} \subset [0, \infty)\) tending monotonically to infinity depending only on \(\{\alpha_k\}\) such that:

\[
\left(\sum_{k=m}^{n} |\Delta^k \alpha_k|_p^p \right)^{\frac{1}{p}} \leq C \left( \sup_{|x| \leq m} \sum_{k=1}^{n} |\Delta^k \beta_k| \right)^{1/p}, \quad m \geq n
\]

For \(1 \leq p < \infty\). Note that \((\alpha, \beta) \in \text{SBVS}_2(\Delta^1)\) is exactly \((\alpha, \beta) \in \text{SBVS}(\Delta^1) = \text{SBVS}_2\). In the present study by definition class of \(p\)-suprema bounded variation of difference sequences, like by Imron (2018) we shall compute the error sine series with coefficient from class \(p\)-suprema bounded variation difference sequences.

**Definition and preliminaries:** In the following definition, we consider sequences \(\alpha = \{\alpha_k\}\) and \(\beta = \{\beta_k\}\) be two sequences of complex and positive numbers, respectively (Korus, 2019).

**Definition 4:** Let a class \(\text{SBVS}_1(\beta, \Delta)\) be given. Class of \(\text{SBVS}_1(\beta, \Delta)\) is defined as \(\{\alpha: (\alpha, \beta) \in \text{SBVS}_2(\Delta^1)\}\).

**Definition 5:** Let a class \(\text{SBVS}_2(\Delta^n)\) be given. Class of \(\text{SBVS}_2(\Delta^n)\) is defined as \(\{\alpha: (\alpha, \beta) \in \text{SBVS}_2(\Delta^n)\}\).

**THEOREM 6:** Let \((\alpha, \beta) \in \text{SBVS}_2(\beta, \Delta^n)\), \(1 \leq p < \infty\), with \(\beta\) real non-negative sequence, if \(\{\alpha_k\}\) decreasing monotone:

\[
\alpha_{m} = m^{\frac{1}{p}} \sup_{|x| \leq m} \sum_{k=1}^{n} |\Delta^k \beta_k|
\]

And:

\[
\left(\sum_{k=m}^{n} |\Delta^k \alpha_k|_p^p \right)^{\frac{1}{p}} \leq C \left( \sup_{|x| \leq m} \sum_{k=1}^{n} |\Delta^k \beta_k| \right)^{1/p}, \quad m \geq n
\]

m for \(m \leq n, m > n\), then:

\[
|g(x) - S_{m-1}(g, x)| \leq C(m + 2M)^{\frac{1}{p}} \sup_{|x| \leq m} \sum_{k=1}^{n} \beta_k
\]

where, \(C\) positive constant only depending on \(\text{SBVS}_2(\beta, \Delta^n)\) with \(m \geq n, x = \pi/M\) and \(x \in (0, \pi]\).

**Proof:** Calculate \(g(x) - S_{m-1}(g, x)\) with:

\[
S_{m}(g, x) = \sum_{j=1}^{m} c_j \sin jx
\]

We have:

\[
g(x) - S_{m-1}(g, x) = \sum_{k=m+1}^{n} c_k \sin kx
\]
Let $x \in (0, \pi]$, for any $M \in \mathbb{N}$ such that $x \in (\pi/M+1, \pi/M]$. Since:

$$\sum_{k=m}^{\infty} c_k \sin nx = \frac{\cos \frac{1}{2} \cdot x - \cos \left( \frac{m+1}{2} \cdot x \right)}{2 \sin \frac{1}{2} x} \leq \frac{1}{\sin \frac{1}{2} x} \leq \frac{\pi}{x}$$

By Abel’s transformation:

$$\sum_{k=m}^{\infty} c_k \sin nx = A = \sum_{k=m}^{\infty} \Delta c_k D_n(x) - c_m D_{n-1}(x) = S + T$$

With:

$$D_n(x) = \sum_{j=1}^{n} \sin jx$$

And:

$$S = \sum_{k=m}^{\infty} \Delta c_k D_n(x), \quad T = \sum_{k=m}^{\infty} c_k D_{n-1}(x)$$

$$|S| \leq \frac{\pi}{X} |C_m| \quad \text{and} \quad |T| \leq \frac{\pi}{X} |C_m|$$

So:

$$|A| \leq \frac{2\pi}{X} |C_m| \leq 2(M+1)|C_m|$$

$$\leq (m+2M) \sum_{k=m}^{\infty} |\Delta C_i|$$

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Because of:

$$\Delta a_i = \Delta a_{i+1} + \Delta^2 a_{i+1} + \ldots + \Delta^m a_{i+1} + \Delta^m a_i \quad \text{(4)}$$

We get:

$$|A| \leq (m+2M) \sum_{i=m}^{\infty} |\Delta a_{i+1} + \ldots + \Delta^m a_{i+1}|$$

$$\leq (m+2M) \sum_{i=m}^{\infty} |\Delta^m a_i| = I_1 + I_2$$

By Holder inequality, we get:

$$I_1 \leq (m+2M) \sum_{i=m}^{\infty} \left( \sum_{\nu \geq 2m} |\Delta^\nu a_{i+1} + \ldots + \Delta^\nu a_{i+1}| \right)^{\frac{1}{p}} \left( \left(2^m m\right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

$$\leq (m+2M) \sum_{i=m}^{\infty} \left( \sum_{\nu \geq 2m} |\Delta^\nu a_{i+1} + \ldots + \Delta^\nu a_{i+1}| \right)^{\frac{1}{p}} \left( \left(2^m m\right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

Further:

$$I_2 = (m+2M) \sum_{i=m}^{\infty} |\Delta^m a_i|$$

$$\leq (m+2M) \sum_{i=m}^{\infty} \left( \sum_{\nu \geq m} |\Delta^\nu a_{i+1} + \ldots + \Delta^\nu a_{i+1}| \right)^{\frac{1}{p}} \left( \left(2^m m\right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

We defined:

$$\alpha_m = \frac{2\pi}{X} \sup_{i \geq m} \beta_i$$

And by Holder inequality we get:

$$I_2 \leq (m+2M) \left( \sum_{i=m}^{\infty} \left( \sum_{\nu \geq m} |\Delta^\nu a_{i+1} + \ldots + \Delta^\nu a_{i+1}| \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}$$

$$\leq (m+2M) \sum_{i=m}^{\infty} \left( \sum_{\nu \geq m} |\Delta^\nu a_{i+1} + \ldots + \Delta^\nu a_{i+1}| \right)^{\frac{1}{p}} \left( \left(2^m m\right)^{\frac{1}{q}} \right)^{\frac{1}{q}}$$

$$\leq 2C(m+2M) m^{\frac{1}{q}} \sup_{i \geq m} \beta_i$$

So, we have:

$$\left| g(x) - S_{n+1}(g, x) \right| \leq 6C(m+2M) m^{\frac{1}{q}} \sup_{i \geq m} \beta_i$$

With:

$$m \geq n, \quad x = \frac{\pi}{M} \quad \text{and} \quad x \in (0, \pi]$$
Theorem 7: Let \( \alpha \in SBVS_{\lambda}(\beta, \Delta^\nu) \) 1 < \( p \leq \infty \) with \( \beta \) real non-negative sequence, if:

\[
m 
\left( \sum_{k=n}^{m} | \Delta \sigma_{k+1} + \ldots + \Delta^{\nu} \sigma_{k+1} | \right)^{\frac{1}{p}} \leq \frac{C}{m} \left( \sum_{k=m+1}^{2m} \beta_k \right), \ n \geq m
\]

And:

\[
\frac{1}{m} \sum_{k=m}^{2m} \beta_k
\]

Decreasing monotone, then:

\[
E_n \left( f \right) \leq 2 \max_{v=[n, m]} v | C_{v+n} | + 6C \frac{1}{m} \sum_{k=n+1}^{2m} \beta_k + 6C \frac{m+2M}{2m} \frac{1}{m} \sum_{k=n+1}^{2m} \beta_k
\]

With \( C \) positive constant depending class of on SBVS\(_{\lambda}(\beta, \Delta^\nu) \) and \( M = \pi/\xi \) for \( \xi \in (0, \pi) \), \( m \geq n \).

Proof:

\[
E_n \left( g \right) = \inf_{p \in \mathbb{P}_{\lambda}} \| g - p \| \leq \| g - p^0 \|
\]

With:

\[
p^0(x) = \sum_{v=n}^{m} C_v \sin v x + \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x
\]

We get:

\[
\left\| g(x) - \left( \sum_{v=n}^{m} C_v \sin v x + \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x \right) \right\| \leq \sum_{v=n+1}^{m} C_v \sin v x - \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x = \sum_{v=n+1}^{m} C_v \sin v x + \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x = \sum_{v=n+1}^{m} C_v \sin v x - \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x = \sum_{v=n+1}^{m} C_v \sin v x - \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x \right\|
\]

\[
\left\| g(x) - \left( \sum_{v=n}^{m} C_v \sin v x + \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x \right) \right\| \leq 2 \sum_{v=n}^{m} C_v \sin v x + \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x = A + B
\]

With:

\[
A = \sum_{v=n}^{m} C_v \sin v x
\]

And:

\[
B = \sum_{v=n+1}^{m} C_{v+n} \sin (v-n) x
\]

For any \( x \in (0, \pi) \) and \( j = [\pi/2x] \) by \( [x] \) is integer part of \( x \). If \( j \geq m \), then:

\[
A = \sum_{v=n}^{m} C_{v+n} \sin v x \leq \pi \sum_{v=n}^{m} v | \sigma_{v+n} | \leq \frac{\pi}{2} \max_{v=[n, m]} v | C_{v+n} |
\]

If \( j < m \), then sum:

\[
\sum_{v=n}^{m} C_{v+n} \sin v x
\]

Broken into:

\[
\sum_{v=n}^{m} C_{v+n} \sin v x = \sum_{v=n}^{j} C_{v+n} \sin v x + \sum_{v=j+1}^{m} C_{v+n} \sin v x = I_1 + I_2
\]

By Eq. 5, we get:

\[
\left\| \sum_{v=n}^{j} C_{v+n} D_v (x) \right\| \leq \frac{\pi}{2} \sum_{v=n}^{j} | \sigma_{v+n} | \left\| I_1 \right\| \leq \frac{\pi}{2} \max_{v=[n, m]} v | C_{v+n} |
\]

By Abel’s transformation:

\[
\left\| I_1 \right\| \leq \sum_{v=n}^{j} | \sigma_{v+n} | \left\| D_v (x) \right\| \leq \left( \sum_{v=n}^{j} | \sigma_{v+n} | \right) \left\| D_v (x) \right\|
\]

With:

\[
D_v (x) = \sin v x
\]

For \( x \in (0, \pi) \) we find \( |D_v (x)| < \pi/\xi \), so that, from Eq. 4, we get:

\[
\sum_{v=n}^{j} | \sigma_{v+n} | \leq \frac{\pi}{2} \sum_{v=n}^{2m-1} | \sigma_{v+n} | + | \sigma_{v+n} | \leq \frac{\pi}{2} \sum_{v=n}^{2m-1} | \sigma_{v+n} | + | \sigma_{v+n} | \leq \frac{\pi}{2} \sum_{v=n}^{2m-1} | \sigma_{v+n} | + | \sigma_{v+n} | \leq \frac{\pi}{2} \sum_{v=n}^{2m-1} | \sigma_{v+n} | + | \sigma_{v+n} |
\]

where, \( t \) non-negative integer and \( 2^j \leq m \leq 2^n \). Further we defined:

\[
\alpha_m = \frac{1}{m} \sum_{k=m+1}^{2m} \beta_k
\]
And we get:

\[
\pi \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \Delta \alpha_{n,i} + \cdots + \Delta^{-1} \alpha_{n,i} \right) + \Delta \alpha_{n,i} = I_n + I_0
\]

By Holder inequality, we get:

\[
I_0 = \pi \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \left| \Delta \alpha_{n,i} + \cdots + \Delta^{-1} \alpha_{n,i} \right| \right)^{1/p} \leq \left( \sum_{j=0}^n \left( \sum_{i=0}^n C \sup_{i=0}^{n} \beta_i \right) \right)^{1/p}
\]

\[
a(m)^{1/p} \left( \sum_{i=0}^{n} \beta_i \right)^{1/p} \leq \left( \sum_{i=0}^{n} \beta_i \right)^{1/p}
\]

where, C positive constant depending class of SBVS2\(_p\), \(m > n\) and \(M = \pi/\alpha\) for \(x \in (0, \pi]\).

**CONCLUSION**

In this study we have introduced the class SBVS2\(_p\)(\(\Delta^n\)). We have investigated that error of sine series with coefficient from class \(p\)-supremum bounded variation difference sequences is:

\[
E_n(f) \leq 2 \max_{v \in [1, m]} \|C_{v+m}\|^p \|C_{v+m}\|_{p}^p \sup_{i=0}^{n} \beta_i + 6C\(2m+2M)(2m)^{1/p} \sup_{i=0}^{n} \beta_i
\]

where, C positive constant depending class of SBVS2\(_p\), \(m > n\) and \(M = \pi/\alpha\) for \(x \in (0, \pi]\).

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