Existence of the Attractor of Recurrent Iterated Function System

Wadia Faid Hassan Al-shameri
Department of Mathematics, Faculty of Science and Arts, Najran University, Najran, Saudi Arabia

Abstract: In this research paper, we present a generalization of Iterated Function System (IFS) called the Recurrent Iterated Function System (RIFS). We explore RIFS, investigate the existence of the fractal attractor of RIFS and present its main properties. Finally, MATLAB program is presented to implement the algorithm for determining the attractor of the RIFS.

Key words: Fractals, iterated function system, fixed points, attractors, existence, algorithm

INTRODUCTION

Iterated Function System (IFS) theory is an important part of fractal theory; it is widely used in fields of image compression due to the pioneering works done by Barnsley (1988a, b), Barnsley et al. (1989). Barnsley (1988a, b) put forward another concept: Recurrent Iterated Function System (RIFS). It reflects the similarities among local regions of a graph. It can generate more complex graphics. That is recurrent modeling is the process of partitioning an object into components and representing each component as a collection of contraction copies of possibly itself and possibly other components. This representation is described by a RIFS which is based on the simpler case of the IFS which represents the entire object as a collection of contraction copies of itself. Barnsley gave a strict proof for the existence and uniqueness of the invariant measure (invariant set) of an RIFS. Based on researches of Barnsley, this study first extends the definition of IFS from the viewpoint of graph theory and product space, and then investigates the proof of the existence and uniqueness of the fractal attractor of the RIFS.

Recurrent modeling by Barnsley (1988a, b) and Barnsley et al. (1989) is the process of partitioning an object into components and representing each component as a collection of contraction copies of possibly itself and possibly other components. This representation is described by a recurrent iterated function system, based on the simpler case of the iterated function system which ultimately represents the entire object as a collection of contraction copies of itself.

This study paper is organized as follows. The second section reviews the concepts that will help us to define several properties of digraphs. Section 3 presents the recurrent Hausdorff distance metric and recurrent Hutchinson operator whereas Section 4 develops the theory of RIFS and investigates the existence of the attractor of RIFS. Section 5 presents the implementation, results conducted and the discussions made. Finally, Section 6 concludes with directions for future research on the relationship between RIFS and recursive fractal interpolation function (RFIF) and provides theoretical basis for their applications.

MATERIALS AND METHODS

Diagraph: Graphs are described here as an ordered pair of sets. The first is the set of vertices, the second is the set of edges. Edges are denoted as ordered pairs of vertices. Now, let us fix the most important notions which furnish the more general setting called diagraph. All notions are well-known and may be found in the literature (Hart, 1996).

A digraph G is a set of N vertices G = {v_i}^{N}_{i=1} and a set of edges G, which are ordered pairs (v_i, v_j), 1 ≤ i, j ≤ N where (v_i, v_j) ∈ G implies G contains a directed edge from v_i to v_j. Since, G_i is a set, the same edge cannot appear twice in G. Thus, the cardinality of G_i is at most the cardinality of G_i squared. The number of edges into a vertex is called the “in-degree” of the vertex. Similarly, the number of edges out of vertex is called “out-degree” of the vertex.

A directed (undirected) path of vertices v_0, v_1, ..., v_k connects vertex v_i to v_j, if and only if for each pair of neighboring vertices v_i, v_{i+1}, the edge (v_i, v_{i+1}) ∈ G. We shall follow (Al-shameri, 2001; Hart, 1996) to review the concepts that will help us to define several properties of digraphs.

A cycle of edges is simply a path from a vertex to itself while digraph G contains a (directed, undirected) cycle if and only if there exists a vertex v ∈ G such that there exists a (directed, undirected) path of vertices in G, connecting vertex v_i to itself. The term cycle will imply directed cycle. A cycle may be as simple as the edge (v_i, v_j). An “acyclic” digraph contains no cycles.
Figure 1a-d are examples of diagram. The graph in Fig. 1a-c is a digraph with the set of vertices $G_v = \{K_1, K_2\}$ and the set of edges $G_e = \{a, b, c, d, e, f\}$. We have, e.g., $e(c) = K_2, e(c) = K_1$, where $e(c)$ and $e(c)$ denoted to the initial and terminal vertex of the edge $e$, respectively where $l: G_v \rightarrow \mathbb{R}^+$; $t: G_v \rightarrow \mathbb{R}^+$ (Al-shameri, 2001; Edgar, 1990).

As with any geometric model, there are certain properties that recurrent models may satisfy. For example, the partitioning may satisfy the open set property and the directed graph that describes how components combine to form components may be weakly or strongly connected. The following concepts are needed for digraph connectedness. We shall follow (Deo, 1974).

A digraph $G$ is (strongly, weakly)-connected if and only if every pair of vertices $v_i, v_j \in G$, is connected by a (directed, undirected) path of vertices (Al-shameri, 2001; Edgar, 1990). We follow the convention, implied from this definition, that strongly-connected implies weakly-connected which differs form (Deo, 1974). Hence, a weakly-connected digraph may or may not be strongly-connected too (Deo, 1974; Edgar, 1990; Hart, 1996). If digraph $G$ is weakly connected, then the cardinality of $G_v$ is at least one less than the cardinality of $G_v$.

A strongly-connected digraph necessarily contains a cycle. A strictly weakly-connected digraph necessarily contains no cycle. An Iterated Function System (IFS) is a couple $(X, \mathcal{D})$ of a complete metric space together with a finite set of contraction mappings $\mathcal{D}: X \times X, n = 1, 2, \ldots, N$ where the metric $d$ is a distance function between elements of $X$, i.e., $d:X \times X \rightarrow \mathbb{R}^+$. Note that, $\mathcal{D}$ transforms a subset of the complete metric space $A \subseteq X$ onto smaller subsets $\mathcal{D}(A)$, i.e., an IFS models an object by constructing it out of smaller copies of itself (Barnsley and Demko, 1985). Symbolically, an IFS $\{X; \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n\}$ models the set $A \subseteq X$ is called the attractor of the IFS as the union of attractorlets $A_n = \bigcup_{i=1}^{N} A_n$ as the solution $A$ of:

$$A = \bigcup_{i=1}^{N} A_n$$

It is often convenient to write an IFS somewhat more briefly as IFS $\{X; \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n\}$. Let, $q(X) = \{A \subseteq X, A$ is compact $\}$ be the collection or space of all compact subsets of the complete metric space $X$. Then, the metric $d$ is now used to define a metric $h(d)$ denotes the Hausdorff metric (distance) between elements $A$ and $B$ of $q(X)$ as follows. Let, $h: q(X) \times q(X) \rightarrow \mathbb{R}$ be a mapping such that:

$$h(A, B) = \max\{d(A, B), d(B, A)\}$$

is the Hausdorff metric between $A$ and $B$ of $q(X)$ (Al-Shameri, 2015; Barnsley and Demko, 1985; Falconer, 1999). The recurrent versions of the standard IFS analysis tools operate on $N$-tuples of sets. These will be denoted by:

$$S = (S_1, S_2, \ldots, S_N)$$

A set $N$-tuple $S$ of $n$-dimensional sets $S_i$ belongs to the $n$-dimensional space $(\mathbb{R}^n)^N$. Moreover, functions on these sets in this space are denoted with a superscripted $N$ (i.e., $f^{(N)}$).

One can think of $(\mathbb{R}^n)^N$ as $N$ copies of $\mathbb{R}^n$ overlayed on top of each other. A set $S \subseteq \mathbb{R}^n$ is better understood by drawing each of its $N$ parts $S_i$ on a separate clear sheet. Then, the set $S$ can be visualized in the space by overlaying all $N$ transparent sheets (Barnsley et al., 1989). The subset relation in $(\mathbb{R}^n)^N$ is defined as the logical “and” of the subset relation of its components. That is, if $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ then, $A \subseteq B$ if and only if, $A_i \subseteq B_i \quad \forall i = 1, \ldots, N$. Notice that, if $A \subseteq B$ then $\cup A_i \subseteq \cup B_i$.

The recurrent Hausdorff metric and recurrent Hutchinson operator: The extension of the Hausdorff distance (metric) as originally defined by Barnsley et al. (1989), combines the individual Hausdorff distances of the component sets. One possible space is $P(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$. A subspace of $P(\mathbb{R}^n)$ is $q$ the collection of non-empty compact subsets of $\mathbb{R}^n$. Then the Hausdorff distance between the points $A$ and $B$ in $q$ is defined by $h(A, B) = d(A, B) \vee d(B, A)$ where $(A, B) = \max\{d(x, B), x \in A\}$.
and \(x/y\) means the maximum of \(x\) and \(y\). We also call \(h\) the Hausdorff metric on \(\mathcal{C}\). Now, \((\mathcal{C}, h)\) is a complete metric space (Hart, 1996).

The recurrent version of Hausdorff metric denoted by \(h^R\) measures the distance between two subsets \(A = (A_1, A_2, ..., A_N)\) and \(B = (B_1, B_2, ..., B_N)\) of \((\mathbb{R}^n)^N\) (or \(\mathcal{C}^N\)) as:

\[
h^R(A, B) = \max_{i=1,...,N} h(A_i, B_i)
\]

Now, \((\mathcal{C}, h)\) is a complete metric space (Hart, 1996). Here, a set \(\mathcal{A} \subseteq (\mathbb{R}^n)^N\) is compact if and only if every one of its parts \(A_i\) is compact in \(\mathbb{R}^n\). A Recurrent Iterated Function System (RIFS) consists of a finite set of strictly contractive maps \(\{\psi_i\}_{i=1}^N\) from \(\mathbb{R}^n\) into itself, along with an \(N\)-vertex weakly-connected digraph \(G = (G_i, G_e)\) containing some directed cycle from every vertex \(v_i \in G_i\) back to itself. The digraph \(G\) is used to restrict map compositions. The iteration sequence \(f^k\) is allowed if and only if a directed edge from vertex \(v_i\) to \(v_j\) exists in \(G\). Symbolically, we write \(((\{\psi_i\}_{i=1}^N, G)\) for RIFS.

According to connectedness of the digraph, many different names for IFS enhancement are given (Barnsley et al., 1989; Prusinkiewicz and Hammel, 1991; Prusinkiewicz et al., 1990; Reuter, 1987). In this study, we mention that \((f^k, G)\) is a RIFS where \(f^k = \{\psi_i\}_{i=1}^N\) and \(G\) is weakly-connected digraph. If each partition is the union of images of every partition including itself, then the graph \(G\) is complete and the RIFS is simply an IFS. Hence, every IFS is an RIFS and we will focus in the upcoming section on existence of the attractor of RIFS representation.

The degree of overlap of the partitioning is dictated by the open set property. An RIFS \(((\{\psi_i\}_{i=1}^N, G)\) satisfies the open set property, if and only if there exists a set-vector \(U = (U_1, U_2, U_3, ..., U_N)\) of open sets \(U_i \subset \mathbb{R}^n\) such that \(f(U) = \cup U_i\) and \(\phi\forall i = j\). An attractor is "just touching" (Barnsley, 1988a, b), if and only if it is connected and its RIFS satisfies the open set property. The recurrent Hutchinson operator as first developed by Barnsley et al. (1989) is a generalization of the standard Hutchinson operator. Let \((\{\psi_i\}_{i=1}^N, G)\) be an RIFS. Then, the recurrent Hutchinson operator \(f^R : (\mathbb{R}^n)^N \rightarrow (\mathbb{R}^n)^N\) is defined (Hutchinson, 1981):

\[
f^R(A) = (f_1(A), f_2(A), ..., f_N(A))
\]

where, \(A = (A_1, A_2, ..., A_N) \in (\mathbb{R}^n)^N\) and:

\[
f_i(A) = \cup_{(i, v)\in G_e} f(A_i)
\]

The recurrent Hutchinson operator is a contraction. However, this proof will need the following and its proof based on the version that appears in Barnsley et al. (1989). Lemma (Hart, 1996) for collections \(A = (A_i)_{i=1}^N\) and \(B = (B_i)_{i=1}^N\) where \(A_i, B_i\) are subset of metric space \((X, d)\).

\[
h^R\left(\bigcup_{i=1, ..., N} A_i, \bigcup_{i=1, ..., N} B_i\right) \leq \max_{i=1, ..., N} h(A_i, B_i)
\]

Proof: Let:

\[
\epsilon = h^R\left(\bigcup_{i=1, ..., N} A_i, \bigcup_{i=1, ..., N} B_i\right)
\]

Then:

\[
\bigcup_{i=1, ..., N} A_i \cup \bigcup_{i=1, ..., N} B_i + \epsilon
\]

For each \(1 \leq i \leq N\) we have:

\[
A_i \subset \bigcup_{i=1, ..., N} B_i + \epsilon
\]

and specifically:

\[
A_i \subset B_i + \epsilon_{A_i}
\]

where, \(\epsilon_{A_i}\) is minimal still, \(\epsilon_{A_i} \geq \epsilon\). Likewise, for each \(i\) there is a corresponding \(\epsilon_{B_i} \geq \epsilon\). For each \(i\) the maximum thickening radius matches or exceeds the original thickening radius:

\[
\max_{i}\{\epsilon_{A_i} \cap \epsilon_{B_i}\} \geq \epsilon
\]

as does its maximum over \(i\). The proof is complete once the reader realizes that the left-hand side of this inequality is the Hausdorff distance between \(A_i\) and \(B_i\) and the right hand side is the Hausdorff distance between the union of \(A_i\) and the union of \(B_i\). Now, we are ready to prove that the recurrent Hutchinson operator is a contraction.

**Theorem (Hart, 1996):** Let \(f^R\) be the recurrent version of Hutchinson operator of RIFS \(((\{\psi_i\}_{i=1}^N, G)\). Then \(f^R\) is a contraction on the metric space \((\mathbb{C}, h^R)\).

**Proof:** Let, \(A = (A_1, A_2, ..., A_N), B = (B_1, B_2, ..., B_N) \in (\mathbb{C})^N\).

Consider the following chain of inequalities:

\[
h^R\left(f^R(A), f^R(B)\right) = \max_{j=1, ..., N} h(f_j(A), f_j(B))
\]

\[
\leq \max_{j=1, ..., N} \sup_{i=1, ..., N} h(f_i(A_j), f_i(B_j))
\]

\[
\leq \max_{j=1, ..., N} s h(A_j, B_j)
\]

\[
\leq s \max_{i=1, ..., N} h(f_i(A_i), f_i(B_i))
\]

where:

\[
s = \max_{i=1, ..., N} \text{Lip } f_i
\]
Since, \( s < 1 \), \( f^n \) is a contraction on the complete metric space \( (\mathbb{R}^n, \rho^n) \).

Existence and uniqueness of the attractor of recurrent iterated function system: Now, we consider fundamental result that a unique set be associated with an RIFS as a consequence of the following theorem and proof are based on (Barnsley et al., 1989).

**Theorem 1 (Hart, 1996):** For any RIFS \( \{ f_t \} \) on \( (\mathbb{R}^n, G) \) there exists a unique \( N \)-tuple, \( A \in (\mathbb{R}^n)^N \) of non-empty compact sets such that:

\[
A = f^n(A)
\]

**Proof:** Since, \( (\mathbb{R}^n, \rho^n) \) is complete metric space (the finite product of complete spaces (the finite product of complete: Theorem 76 of (Kaplansky, 1977)) and \( f^n \) is a contraction on \( (\mathbb{R}^n, \rho^n) \) the Contraction Mapping Principle implies that \( f^n \) possesses a unique fixed point in \( (\mathbb{R}^n, \rho^n) \).

The attractor of a RIFS is not an \( N \)-tuple because we want to deal in an \( n \)-dimensional space not an \( n \times N \)-dimensional space. Thus, we have from (Barnsley et al., 1989), the attractor of a RIFS.

Let, \( A = (A_1, A_2, \ldots, A_n) \) be a set such that it is invariant under the recurrent Hutchinson operator of a RIFS \( f^n \) (G):

\[
A = f^n(A)
\]

Then, the attractor \( A \) of the RIFS is given by:

\[
A = \bigcap_{i=1}^{N} A_i
\]

We need the following result.

**Corollary 1; (Hart, 1996):** Let \( (X, d) \) be a complete metric space and let: \( f: X \to X \) be a contraction on \( X \). Then all points in \( X \) converge to the same fixed point under iteration of \( f \). Finally, we have the useful result that any initial set \( N \)-tuple will iterate to the attractor of a RIFS.

**Corollary 2; (Hart, 1996):** Consider the RIFS \( f^n \) on \( (\mathbb{R}^n, G) \) \( A \in (\mathbb{R}^n)^N \) of non-empty compact sets such that \( A = f^n(A) \). Let \( B \) be an \( N \)-tuple of nonempty bounded sets. Then:

\[
\lim_{n \to \infty} (f^n)(B) = A
\]

**Proof:** Since, each component of \( B \) is bounded, there exists a compact set \( B' \supset B \) for all \( B \in B \). Let:

\[
B' = (B', B', \ldots, B')
\]

Since, each component of \( B \) is non-empty, there exists a compact set \( B' \subset B \). Let:

\[
B' = (B', B', \ldots, B')
\]

Then, by Corollary 1 and Theorem 1:

\[
A \supset \lim_{n \to \infty} (f^n)(B') \supset \lim_{n \to \infty} (f^n)(B) \supset \lim_{n \to \infty} (f^n)(B') \subset A
\]

they are all equal.

**RESULTS AND DISCUSSION**

The attractor of RIFS is determined by the digraph \( G \) or by the \( n \times n \) transition matrix (adjacency matrix) of the digraph representing the probabilities of the choice of the transformations (affine maps) in the space \( \mathbb{R}^n \) in which:

\[
w_i \left( \begin{array}{c} x \\ y \\ \vdots \\ y_i \\ \vdots \\ y_n \end{array} \right) = \left( \begin{array}{c} a_{i1} \cdot x + b_{i1} \\ a_{i2} \cdot y + b_{i2} \\ \vdots \\ a_{in} \cdot y_i + b_{in} \end{array} \right), \quad i = 1, 2, \ldots, n
\]

whereas \( n \) is the number of the transformations \( w_i \) defined in the space \( \mathbb{R}^n \). The digraph corresponding to the transition matrix for the RIFS:

\[
P_i = (P_i) = \begin{pmatrix}
0.3 & 0.6 & 0.1 \\
0.1 & 0.5 & 0.4 \\
0.4 & 0.4 & 0.2
\end{pmatrix}
\]

given in Fig. 2. The code for RIFS is presented in Table 1 and the attractor of RIFS is shown in Fig. 3 while the digraph corresponding to the transition matrix:

\[
P_j = (P_j) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\]

Fig. 2: Digraph for RIFS code given in Table 1 whose RIFS attractor is a Version of Sierpinski Gasket
given in Fig. 4. The code for RIFS is presented in Table 2 and the attractor of RIFS is shown in Fig. 5.

Both RIFS fractal attractors shown in Fig. 3 and 5 were constructed by using MATLAB program listed in the algorithm 1. This program is using the random iteration algorithm (Al-Shameri, 2015; Al-Shameri, 2001; Barnsley, 1988a, b) for generation these fractal attractors of the RIFS in the space $\mathbb{R}^2$. The program inputs are $p_0$, $w$, $p$, $a$, $k$ and $r$ where $p_0$ is the initial point represented by a $1 \times 2$ matrix, $p$ is the transition

<table>
<thead>
<tr>
<th>$w_i$</th>
<th>$a_i$</th>
<th>$b_i$</th>
<th>$c_i$</th>
<th>$d_i$</th>
<th>$e_i$</th>
<th>$f_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>128</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>128</td>
<td>128</td>
</tr>
</tbody>
</table>

Table 1: Recurrent IFS code for sierpinski gasket

<table>
<thead>
<tr>
<th>$w_i$</th>
<th>$a_i$</th>
<th>$b_i$</th>
<th>$c_i$</th>
<th>$d_i$</th>
<th>$e_i$</th>
<th>$f_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>128</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>128</td>
<td>128</td>
</tr>
</tbody>
</table>

Table 2: Recurrent IFS code for quadtree fractal

Algorithm 1:

% MATLAB program (Hahn and Valentine, 2007) plots the fractal attractor of RIFS of $\mathbb{R}^2$
% clear all, clc
% The version of Sierpinski Gasket
% $w = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}$
% p0 = [0 0 0 0 0 0]
% P = [0.1 0.1 0.1 0.1 0.1 0.1]
% A = [0.0 0.1 0.1 0.0 0.1 0.1]
% % The quadtree fractal
% % w0 = [0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]
% % p0 = [0 0 0 0 0 0 0 0]
% % P0 = [0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1]

Fig. 3: Recurrent IFS attractor (Version of Sierpinski Gasket) corresponding to the diagraph represented in Fig. 2 with RIFS code in Table 1 (adjacency) matrix $n \times n$ of the directed graph (diagraph) representing the probabilities (transition of the graph).

Fig. 4: Diagraph for RIFS code given in Table 2 whose RIFS attractor is a quadtree fractal.
CONCLUSION

The mathematical principles behind RIFS have been introduced by Barnsley (1988a, b) and Barnsley et al., (1989). In this study we explored the RIFS and investigated the proof of the existence and uniqueness of its attractor. We observe that the RIFS from the viewpoint of graph theory reflects the similarities among local regions of a diagraph, it generates more complex fractal attractors using random iteration algorithm. Future research may focus on recursive fractal interpolation function which is an extension of fractal interpolation function. The latter is the attractor of IFS while recursive fractal interpolation function is the attractor of RIFS.

ACKNOWLEDGEMENT

This research is supported by the Scientific Research Deanship at Najran University, Kingdom of Saudi Arabia under research code: NU/ESCI/14/031.

REFERENCES


