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Deriving the General Laplace Inversion Formula using Complex Integration Results and its Applications in Solving Partial Differential Equations

Maan A. Rasheed and Mustafa A. Sabri
Department of Math, College of Basic Education, Al-Mustansiriyah University, Baghdad, Iraq

Abstract: This study is concerned with Laplace transform and its applications to partial differential equations. We derive the general Laplace inversion formula using some complex analysis results. Furthermore, we apply this formula to find the formal solution of a heat conduction problem which is heat equation with Neumann boundary conditions. We conclude that Laplace transforms with the inversion formula provide a potent technique for solving partial differential equations.

Key words: Laplace transform, inversion formula, heat conduction, Cauchy integral formula, Neumann boundary conditions, equations

INTRODUCTION

The Laplace transform can be helpful in solving ordinary and partial differential equations because it can replace an ODE with on algebraic equation or replace a PDE with an ODE. Another reason that the Laplace transform is useful is that it can help deal with the boundary conditions of a PDE on an infinite domain. Therefore, it has been considered by many researchers, for instance (Atangana and Noutchie, 2014; Blanchard *et al.*, 2012; Coleman, 2013; Gadain and Bachar, 2017; Zhou and Gao, 2017). For its applications science and engineering, we refer to the survey paper (Reddy and Vaithyasubramanian, 2018). In some application of PDEs, situations arise in which the inverse function of Laplace transform cannot be found using the known and classical methods. Therefore, it is important to have a general inversion formula for the Laplace transform. In this study, we will state some results those devoted to find such a formula with involving integration in complex plane. Moreover, we will use the general inversion formula in finding a formal solution of heat equation with nonhomogeneous Neumann boundary conditions.

MATERIALS AND METHODS

Laplace transform: In this study, we will give the general definition of Laplace transform and the sufficient conditions of its existence and uniqueness (Trim, 1996).

Definition 2.1: Let, f be a function of t. The Laplace transform of t is defined as follows:

$$F(S)=\int_0^\infty e^{-st}f(t)dt \tag{1}$$

Provided the improper integer converges. When the Laplace transform exists it is denoted by $L\{f(t)\}$. In fact, this integral might not always converge. The following theorem provides sufficient condition under which the Laplace transform is defined. Before to state the theorem, we need to recall the following definition.

Definition 2.2: A function is said to be of exponential order α if there exist constants T and M>0 in R such that $|f(t)|<Me^{\alpha t}$ for all t>T. This is denoted as $f(t) = 0(e^{\alpha t})$.

Theorem 2.3: Let, f be a function of t that is of exponential order α and is piecewise continuous on $0\le t\le T$ for all $T\in R$. Then, $F(s) = L\{f\}$ exists for all $s\ge\alpha$.

Remark 2.4: We notice that when $L\{f\}$ exists it is unique because it is given by a convergent integral. However, given F(s), there can be countably many functions f_i such that $L\{f_i(t)\} = F(S)$. This follows from the fact that if two function f, g are Lebesgue integrable over a set $A\subseteq R$ and f = g almost everywhere, then:

$$\int_A f=\int_A g \tag{2}$$

Corresponding Author: Maan A. Rasheed, Department of Math, College of Basic Education, Al-Mustansiriyah University, Baghdad, Iraq

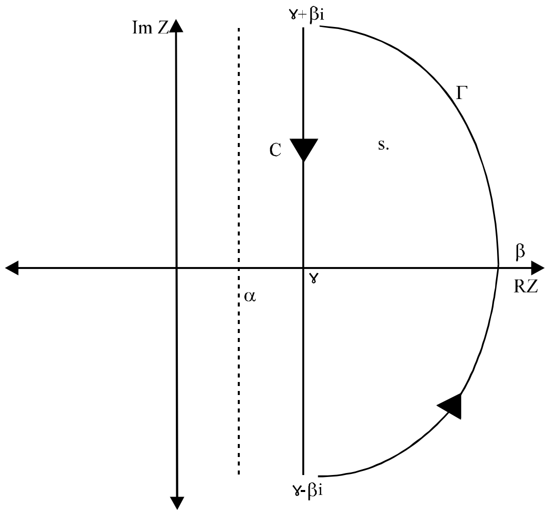

Fig. 1: Right half-plane contour

The inversion formula: As, we have mentioned in section one, finding the inverse of the Laplace transform is the most difficult step in using this technique for solving complicated differential equations. Therefore, we need a general inversion formula when dealing with more complicated expressions. In this study, we state some theorems, to give a way of deriving this formula. In this formulation, we need to use some results from complex analysis which play a key role in the development of an inverse formula (Trim, 1996; Zemainian, 1968) (Fig. 1).

Theorem 3.1: If f is a function of exponential order α that is $O(e^{\alpha t})$ and f is piecewise continuous on every finite interval $0\leq t\leq T$ then, the Laplace transform $L\{f(t)\} = F(s)$ of f is an analytic function of s in the half-plane $x>\alpha$.

Proof: Trim (1996).

Theorem 3.2: Let, f be piecewise continuous on every finite interval $0\leq t\leq T$ with $f(t) = O(e^{\alpha t})$. Suppose, also that in some half-plane $x>\delta>0$ there exists constants M, R and $k>1$ such that $|F(s)S^K|\leq M$ for $|s|>R$. If the inversion integral of converges then:

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i} e^{st}F(S)dS \qquad (3)$$

Proof: Since, f is of exponential order and it is piece wise continuous on every finite interval $0\leq t\leq T$ by applying. Theorem 3.1, we conclude that F is analytic in the half-plane $x>\alpha$. Next, we apply Cauchy Integral formula, (Churchill, 1960), for F over the following contour. So, we obtain:

$$\int_{c}\frac{F(s)}{\zeta-s}ds+\int_{\Gamma}\frac{F(\zeta)}{\zeta-s}ds = \int_{C\cup\Gamma}\frac{F(\zeta)}{\zeta-s}ds = 2\pi i\,F(s) \qquad (4)$$

By taking the limit of above integral, when B goes to infinity, we obtain:

$$\lim_{B\to\infty}\int_{C}\frac{F(\zeta)}{\zeta-s}ds+\lim_{B\to\infty}\int_{\Gamma}\frac{F(\zeta)}{\zeta-s}ds = 2\pi i\,F(s) \qquad (5)$$

By a complex integral inequality it follows:

$$\left|\int_{\Gamma}\frac{F(\zeta)}{\zeta-s}ds\right| \leq \int_{\Gamma}\frac{|F(\zeta)|}{|\zeta-s|}|ds| \qquad (6)$$

But by our assumption, we have $F(\zeta)\leq M/|\zeta|^k$ for $|\zeta|>R$, So:

$$\lim_{B\to\infty}\left|\int_{\Gamma}\frac{F(\zeta)}{\zeta-s}ds\right| \leq \lim_{B\to\infty}\int_{\Gamma}\frac{M}{|\zeta|^k|\zeta-s|}|ds| \leq \lim_{B\to\infty}\left[\frac{M\pi B}{B^k|B-s|}\right] = 0 \qquad (7)$$

And this due to $|\zeta| = B$ if $\zeta\in\Gamma$. Thus:

$$\lim_{B\to\infty}\int_{\Gamma}\frac{F(\zeta)}{\zeta-s}ds = 0 \qquad (8)$$

From above Eq. 5 be comes:

$$\lim_{B\to\infty}\int_{C}\frac{F(\zeta)}{\zeta-s}d\zeta = \lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i}\frac{-F(\zeta)}{\zeta-s}d\zeta = 2\pi i\,F(s) \qquad (9)$$

Thus:

$$F(s) = \frac{-1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i}\frac{F(\zeta)}{\zeta-s}d\zeta \qquad (10)$$

By taking the inverse Laplace transform for both sides of Eq. 3, we get:

$$f(t) = \frac{-1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i}-F(s)L^{-1}\left\{\frac{1}{s-S}\right\}dS \qquad (11)$$

It is well known that, $L^{-1}\{1/s-s\} = e^{St}$ (Blanchard *et al*., 2012). Thus:

$$f(t) = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i}F(s)e^{st}dS \qquad (12)$$

The next theorem states the sufficient condition for inversion.

Theorem 3.3; Sufficient condition for inversion: Let, f be a function of complex variable s that is analytic for all

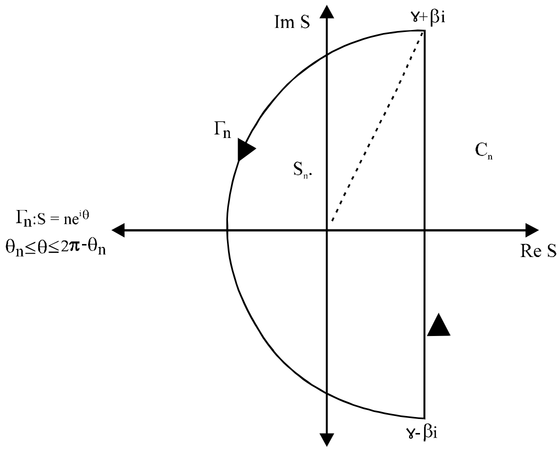

Fig. 2: Left half-plane contour

$s = x+iy$ in the half-plane $\alpha<s$ and $F(s)$ is real-valued when $x>\alpha$. Further, let, $k>1$, M and r be constants such that $|F(s)s^k|<M$ for $|s|>r$ in the half-plane $x>\alpha$. Then, the inversion integral of F along any line $x = \gamma$ defined by:

$$\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i} e^{St}F(s)dS \qquad (13)$$

Converges to a real-valued function of t (Fig. 2).

Proof: It is given that F is analytic function in the half-plane $\alpha<s$ and $e^{St}F(s)$ is a continuous function of y and t where, $z = \gamma+iy$ and $\gamma>\alpha$. Since, $|F(s)S^k|<M$, it follows, there exist $r_0>0$ such that:

$$|F(z)| < \frac{M}{|z^k|} = \frac{M}{(\gamma^2+y^2)^{k/2}} \le \frac{M}{|y|^k} \text{ for } |y| > r_0 \qquad (14)$$

Moreover, F is analytic and real in the half-plane $x>\alpha$. Thus, $F(\bar{S}) = \overline{F(S)}$ and that by the Schwarz reflection principle. Which means $F(\gamma+iy) = F(\gamma-iy)$. So, the inversion integral takes the real form:

$$\frac{e^{\gamma t}}{\pi}\int_0^{\infty}\text{Re}\left[e^{iyt}F(\gamma+iy)\right]dy = \frac{e^{\gamma t}}{\pi}\int_0^{r_0}\text{Re}\left[e^{iyt}F(\gamma+iy)\right]dy + \frac{e^{\gamma t}}{\pi}\int_{r_0}^{\infty}\text{Re}\left[e^{iyt}F(\gamma+iy)\right]dy \qquad (15)$$

Using the fact $|\text{Re}(Z_1Z_2)|\le|\text{Re}Z_1||\text{Re}Z_2|\forall Z_1,\ Z_2\in\mathbb{C}$, we obtain:

$$\left|\text{Re}\left[e^{iyt}F(\gamma+iy)\right]\right| \le \left|\text{Re}\,e^{iyt}\right|\left|\text{Re}\,F(\gamma+iy)\right| = \left|\cos(yt)\right|\left|\text{Re}\,F(\gamma+iy)\right| \le \left|\text{Pe}\,F(\gamma+iy)\right| \le \left|F(\gamma+iy)\right| \qquad (16)$$

By Eq. 14 and above it follows that $|\text{Re}[e^{iyt}F(\gamma+iy)]|\le M/|y|^k\le M/r_0^k$ for $|y|>r_0$. Hence, the improper integral above converges uniformly with respect to t. And since, $\text{Re}[e^{iyt}F(\gamma+iy)]$ is continuous for all t and y it follows that the two integrals above converge to a real-valued function f(t). Another important result for inverting a Laplace transform is given below.

Theorem 3.4:. Let F be a function for which the inversion integral along a line $x = \gamma$ represents the inverse function f and let F be analytic except for isolated singularities $S_n(n = 1, 2, ...,)$ in the half-plane $x<\gamma$. Then the series of residues of $e^{St}F(s)$ at $S = S_n(n = 1, 2, ...,)$ converges to f for each positive t, provided a sequence C_n of Contours can be found that satisfies the following properties:

- C_n Consists of the straight line $x = \gamma$ from γ-$B_n i$ to $\gamma+B_n i$ and some curve Γn beginning at $\gamma+B_n i$ ending at γ-$B_n i$ and lying in $x\le\gamma$
- C_n encloses $S_1, S_2, ..., S_n$
- $\lim_{n\to\infty}\beta_n = \infty$
- $\lim_{n\to\infty}\int_{\Gamma n}e^{St}F(s)ds = 0$

Proof: Trim (1996).

Remark: If the assumptions of theorem 3.2 and 3.3 are satisfied, then, $f(t) = L^{-1}\{F(s)\}$ is given by the Laplace inversion Eq. 13. Moreover, by theorem 3.4, we can get:

$$f(t) = L^{-1}\{F(S)\} = \sum_{n=1}^{m}\text{Res}(e^{St}F(s)S_n) \qquad (17)$$

where, m is the number of singular points, S_n of $e^{st}F(s)$ inside the contour $C_n\cup\Gamma_n$.

RESULTS AND DISCUSSION

Applications of Laplace inversion formula: In this study, we consider the solution of the problem of heat conduction, introduced by Newcomb (1958). In the process, we use the inversion formula derived in last section, to obtain the final solution.

Heat conduction problem: Consider the linear flow of heat in a solid initially at zero temperature and bounded by a pair of infinite parallel planes at $x = 0$ and $x = d$ such that at $x = 0$ there is no flow of heat perpendicular to the plane and at $x = d$ there is a uniform thermal flux $N(1\text{-}Mt)$ into the solid N and M are constants determined from the rate of evolution of heat and t represents time (sec). If u, k and α

denote the temperature, thermal conductivity and diffusivity, respectively, the solution of the differential equation:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{\alpha}\frac{\partial u}{\partial t}=0, \quad 0<x<d \tag{18}$$

Is required with the boundary conditions:

$$\frac{\partial u}{\partial t}=0 \text{ at } x=0 \tag{19}$$

$$k\frac{\partial u}{\partial t}=N(1\text{-}Mt) \text{ at } x=d \tag{20}$$

The solution of heat conduction problem: The problem of Heat conduction Eq. 18 with Eq. 19 and Eq. 20 is difficult to be solved by using some known techniques such as separation of variables and Fourier transform and that due to the complexity of the Boundary conditions. Therefore, we aim to find a formal solution to this problem using Laplace inversion formula derived in last section.

Theorem 4.1: If we use Laplace transform technique with Laplace inversion Eq. 13 to solve the heat conduction problem Eq. 18-20. Then, the formal solution takes the form:

$$u(x,t)=\frac{N\sqrt{\alpha}}{k}\left(\left(\sum_{n=0}^{\infty}\left[\frac{\left(2(n+1)d\text{-}x\right)^2}{2\alpha}+\frac{\left(2(n+1)d+x\right)^2}{2\alpha}+2t\right]\text{-} M\sum_{n=0}^{\infty}\left[\frac{\left(2(n+1)d\text{-}x\right)^{4t}}{4!\alpha^2}+\frac{\left(2(n+1)d+x\right)^2}{4!\alpha^2}+t^2\right]\right)\right) \tag{21}$$

Proof: The proof of this theorem is divided into five steps:

Step one: Taking the Laplace transform. First of all, we introduce the Laplace transform of u:

$$\bar{u}=\int_0^\infty e^{-st}u(x,t)dt \tag{22}$$

By using integration by part, we obtain:

$$\bar{u}=\frac{1}{s}\int_0^\infty e^{-st}u_t dt \tag{23}$$

Take the second derivative of Eq. 22 with respect to x:

$$\bar{u}_{xx}=\int_0^\infty e^{-st}u_{xx}dt \tag{24}$$

From Eq. 24 and 25, we get:

$$\bar{u}_{xx}=-\frac{s}{\alpha}\bar{u}\int_0^\infty e^{-st}\left[u_{xx}-\frac{1}{\alpha}u_t\right]dt \tag{25}$$

Thus:

$$\bar{u}_{xx}=-\frac{s}{\alpha}\bar{u}=0, \quad 0<x<d \tag{26}$$

Next, we find the new initial boundary conditions. Since, $u_x(0, t) = 0$, it follows that:

$$\bar{u}_x(0,t)=\int_0^\infty e^{-st}u_x(0,t)dt=0 \tag{27}$$

Thus:

$$\bar{u}_x(0,t)=0\forall t \tag{28}$$

And:

$$\bar{u}_x(d,t)=\int_0^\infty e^{-st}u_x(d,t)dt \tag{29}$$

From Eq. 21, we obtain:

$$\bar{u}_x(d,t)=\int_0^\infty e^{-st}\left[\frac{N}{K}(1\text{-}Mt)\right]dt \tag{30}$$

By integration by part, we get:

$$\bar{u}_x(d,t)=\frac{N}{K}\left[\frac{1}{s}-\frac{M}{s^2}\right] \tag{31}$$

Step two: Solving the new problem. To solve the ordinary differential Eq. 26 with the boundary conditions Eq. 27 and 28, we will use Euler method assuming that $\bar{u}=e^{qx}$. By direct calculations it follows that the general solution of Eq. 27 takes the form:

$$\bar{u}=A_1e^{qx}+A_2e^{-qx} \quad \text{where } q=\sqrt{\frac{s}{\alpha}} \tag{32}$$

So:

$$\bar{u}=\frac{A_1}{2}\left(e^{qx}+e^{-qx}\right)+\frac{A_1}{2}\left(e^{qx}-e^{-qx}\right)+\frac{A_2}{2}\left(e^{qx}+e^{-qx}\right)-\frac{A_2}{2}\left(e^{qx}-e^{-qx}\right), \tag{33}$$

Thus:

$$\bar{u}_x=A\cosh qx+B\sinh qx \tag{34}$$

Where:

$$A=\left(\frac{A_1}{2}+\frac{A_2}{2}\right),\ B=\left(\frac{A_1}{2}-\frac{A_2}{2}\right) \tag{35}$$

To find the constant A, B we use the boundary conditions:

$$\bar{u}_x = Aq\sinh qx + Bq\cosh qx \qquad (36)$$

So, $\bar{u}_x = (0) = 2Bq = 0$, thus, B = 0. This leads to $\bar{u} = A\cos qx,\ q = \sqrt{s/\alpha}$ by Eq. 31, we get:

$$\bar{u}_x(d) = Aq\sinh qd = \frac{N}{K}\left(\frac{1}{s}-\frac{M}{s^2}\right) \qquad (37)$$

It follows that:

$$A = \frac{N}{K}\left(\frac{1}{s}-\frac{M}{s^2}\right)\frac{1}{q\sinh(qd)} \qquad (38)$$

Thus, the solution of problem Eq. 29 with Eq. 30 and 31 becomes:

$$\bar{u} = \frac{N}{K}\left(\frac{1}{s}-\frac{M}{s^2}\right)\frac{\cosh(qx)}{q\sinh(qd)} \qquad (39)$$

Step three: Write the solution Eq. 33 in an infinite series form. To simplify the last Eq. 33, we need to prove the following proposition.

Proposition 4.2: For any x∈R we have:

$$\frac{1}{e^x - e^{-x}} = \sum_{n=0}^{\infty} e^{-(2n+1)x} \qquad (40)$$

Proof: Let us start with the left hand side $1/e^x - e^{-x} = 1/e^x(1-e^{-2x})$ and by using the geometric series, it follows:

$$\frac{1}{e^x - e^{-x}} = \frac{1}{e^x}\sum_{n=0}^{\infty}(e^{2x}) = \sum_{n=0}^{\infty} e^{-(2n+1)x} \qquad (41)$$

Now, by Eq. 34, the solution Eq. 33 becomes as follows:

$$\bar{u} = \frac{N}{K}\left(\frac{1}{sq}-\frac{M}{s^2 q}\right)\left[\sum_{n=0}^{\infty} e^{-q[(2n+1)d-x]} + \sum_{n=0}^{\infty} e^{-q[(2n+1)d+x]}\right] \qquad (42)$$

Or it can be rewritten as follows:

$$\frac{K}{N\sqrt{\alpha}}\bar{u} = \frac{1}{s^{3/2}}\sum_{n=0}^{\infty}[e^{-a_n\sqrt{s}} + e^{-b_n\sqrt{s}}] - \frac{M}{s^{5/2}}\sum_{n=0}^{\infty}[e^{-a_n\sqrt{s}} + e^{-b_n\sqrt{s}}] \qquad (43)$$

Where:

$$a_n = \frac{2(n+1)d-x}{\sqrt{\alpha}},\ b_n = \frac{2(n+1)d-x}{\sqrt{\alpha}},\ n = 0, 1, 2, \ldots, \qquad (44)$$

Step four: Taking Laplace inverse to Eq. 43. By taking the Laplace inverse to the two side of Eq. 43, we get:

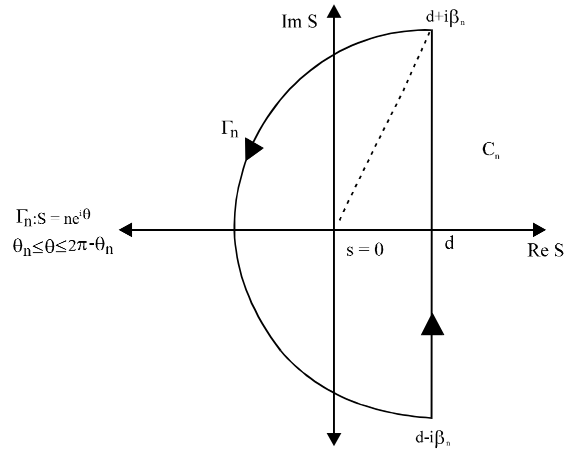

Fig. 3: Left half-plane contour for problem Eq. 18

$$\frac{K}{N\sqrt{\alpha}}u = \sum_{n=0}^{\infty}\left[L^{-1}\left\{\frac{e^{-a_n\sqrt{s}}}{s^{3/2}}\right\} + L^{-1}\left\{\frac{e^{-b_n\sqrt{s}}}{s^{3/2}}\right\}\right] - M\sum_{n=0}^{\infty}\left[L^{-1}\left\{\frac{e^{-a_n\sqrt{s}}}{s^{5/2}}\right\} + L^{-1}\left\{\frac{e^{-b_n\sqrt{s}}}{s^{5/2}}\right\}\right] \qquad (45)$$

Set:

$$F_1(s) = \frac{e^{-a_n\sqrt{s}}}{s^{3/2}},\ F_2(s) = \frac{e^{-a_n\sqrt{s}}}{s^{5/2}},\ n\in N \qquad (46)$$

It is clear that, we can find a constant M such that:

$$|F_1(s)| \le \frac{M}{s^{3/2}}, |F_2(s)| \le \frac{M}{s^{5/2}},\ \text{for } s>0 \qquad (47)$$

Moreover, F_1, F_2 are analytic in half-plane s>0. Therefore, to find the Laplace inverse to F_1 and F_2, we can apply theorem (3.2) and (3.2) and by the inversion Eq. 3, we obtain (Fig. 3):

$$L^{-1}\{F_1(s)\} = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i} e^{st}F_1(s)dS \qquad (48)$$

$$L^{-1}\{F_2(s)\} = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-\beta i}^{\gamma+\beta i} e^{st}F_2(s)dS \qquad (49)$$

For γ>0. Next, in order to apply theorem (3.4) which can help us to find the integrals in the right hand side of each of Eq. 48 and 49, we need to consider the contour below. It is clear that this contour satisfies the conditions 1-3 of theorem 3.4. Moreover, the Functions F_1, F_2 are analytic except for the isolated singular point S = 0. Thus, to apply theorem (3.4) it is only left to show that for each of F_1, F_2 the condition 4 of this theorem is satisfied.

Proposition 4.3: Let, $F(s) = e^{-\alpha\sqrt{s}}/s^{k/2}, k\in N, k>2, \alpha>0$. Then:

$$\lim_{n\to\infty}\int_{\Gamma_n} e^{st}F(s)ds = 0,\quad t>0 \tag{50}$$

Where:

$$\Gamma_n : s = ne^{i\theta}, \theta_n \le \theta \le 2\pi-\theta_n \tag{51}$$

Proof: By substituting $s = ne^{i\theta}$ in the function $|e^{st}F|$, we obtain:

$$\left|e^{st}F(s)\right| = \frac{\left|e^{st-\alpha\sqrt{s}}\right|}{\left|s^{k/2}\right|} = \frac{\left|e^{nt(\cos\theta+i\sin\theta)-\alpha\sqrt{n}(\cos(\theta/2)+i\sin(\theta/2))}\right|}{n^{k/2}} \le \frac{\left|e^{nt\cos\theta-\alpha\sqrt{n}\cos(\theta/2)+i\sin(\theta/2)}\right|.\left|e^{i(nt\sin\theta-\alpha\sqrt{n}\sin(\theta/2))}\right|}{n^{k/2}} \le \frac{e^{nt\cos\theta-\alpha\sqrt{n}\cos(\theta/2)}}{n^{k/2}} \tag{52}$$

Thus:

$$\left|\int_{\Gamma_n} e^{st}F(s)ds\right| \le \int_{\Gamma_n}\left|e^{st}F(s)\right||ds| \le \int_{\theta=\theta_n}^{2\pi-\theta_n}\frac{e^{nt\cos\theta-\alpha\sqrt{n}\cos(\theta/2)}}{n^{k/2}}(n)d\theta \tag{53}$$

Since, $\cos\theta$ is asymmetric function and $\cos(\theta/2)\ge 0$, for $\pi/2\le\theta\le\pi$ it follows that it is easy to show that, the linear equation which passes through the two points $(\pi/2, 0)$, $(\pi, -1)$ takes the form $h(\theta) = -2/\pi\theta+1$. Thus:

$$\left|\int_{\Gamma_n} e^{st}F(s)ds\right| \le \frac{4}{n^{(k/2-1)}}\int_{\pi/2}^{\pi} e^{nt(1-2/\pi\theta)}d\theta = \frac{-2\pi}{n^{k/2}t}e^{nt(1-2/\pi\theta)}\Big|_{\pi/2}^{\pi} = \frac{-2\pi}{n^{k/2}}\left[e^{-nt}-1\right] = \frac{2\pi}{n^{k/2}t}\left[1-e^{-nt}\right] \tag{54}$$

By taking the limit when $n\to\infty$, we obtain:

$$\lim_{n\to\infty}\left|\int_{\Gamma_n} e^{st}F(s)dS\right| \le \lim_{n\to\infty}\frac{2\pi}{n^{k/2}t}\left[1-e^{-nt}\right] = 0 \tag{55}$$

Which leads to:

$$\lim_{n\to\infty}\int_{\Gamma_n} e^{st}F(s)ds = 0 \tag{56}$$

By proposition 4.3, we see that the condition 4 of theorem (3.4) is satisfied for each of F_1, F_2. Therefore, by applying theorem (3.4) it follows that Laplace inverse for F_1, F_2 can be found by taking the residues of $e^{st}F_1(s)$, $e^{st}F_2(s)$, respectively, at the singular point (s = 0). Thus, Eq. 48 and 48 become:

$$L^{-1}\{F_1(s)\} = \text{Res}\left(e^{st}F_1(s), 0\right) \tag{57}$$

$$L^{-1}\{F_2(s)\} = \text{Res}\left(e^{st}F_2(s), 0\right) \tag{58}$$

Before to move on step five, we need to state the following definition, (Yang, 2011).

Definition 4.4: Suppose that z_0 is an isolated singular point of F(z). Then, there is a local fractional Laurent series:

$$F(z) = \sum_{k=-\infty}^{\infty}\alpha_k\left(z-z_0\right)^{k\alpha} \tag{59}$$

valid for $|z-z_0|^\alpha \le R^\alpha$, $\alpha\in Q$, $0<\alpha<1$. The coefficient α_{-1} of $(z-z_0)^{-\alpha}$ is called the residue of F(z) at $z = z_0$ and is frequently written as $\text{Res}(F(z), z_0)$. Moreover, if $F(z) = \phi(z)/(z-z_0)^{n\alpha}$ where, ϕ is analytic function, $\alpha\in Q$, $0<\alpha<1$ and z_0 is a pole of order $n\alpha$, $n\in N$ then, $\text{Res}(F(z), z_0) = \alpha_{-1}$.

Step five: Finding the residues. In this step, we prove that:

$$\text{Res}\left(e^{st}F_1(s), 0\right) = t+\frac{a_n^2}{2} \tag{60}$$

$$\text{Res}\left(e^{st}F_2(s), 0\right) = \frac{t^2}{2}+\frac{a_n^2 t}{2}+\frac{a_n^4}{4!} \tag{61}$$

To prove this, let us expand $e^{st}F_1(s)$, $e^{st}F_2(s)$ in a Laurent expansion around s = 0 and by definition 4.4, the residue for each of these functions will be the coeffient of $s^{-1/2}$ in each of these series. Let us start with $e^{st}F_1(s)$:

$$e^{st}F_1(s) = \frac{1}{s^{3/2}}\sum_{n=0}^{\infty}\frac{\left(st-a_n\sqrt{s}\right)^n}{n!} = \frac{1}{s^{3/2}}\left[1+\left(st-a_n\sqrt{s}\right)+\frac{\left(st-a_n\sqrt{s}\right)^2}{2!}, \ldots,\right] = \frac{1}{s^{3/2}}+\frac{t}{\sqrt{s}}-\frac{a_n}{s}+\frac{\sqrt{s}t^2}{2}-a_nt+\frac{a_n^2}{2\sqrt{s}}+, \ldots, = \frac{1}{s^{3/2}}+\left(t+\frac{a_n^2}{2}\frac{1}{\sqrt{s}}\right)-\frac{a_n}{s}+\frac{\sqrt{s}t^2}{2}-a_nt, \ldots, \tag{62}$$

Thus:

$$\text{Res}\left(F_1(s), 0\right) = t+\frac{a_n^2}{2} \tag{63}$$

Next, we consider the second function:

$$e^{st}F_2(s) = \frac{1}{s^{5/2}}\sum_{n=0}^{\infty}\frac{\left(st-a_n\sqrt{s}\right)^n}{n!} = \frac{1}{s^{5/2}}\left[1+\left(st-a_n\sqrt{s}\right)+\frac{\left(st-a_n\sqrt{s}\right)^2}{2}+\frac{\left(st-a_n\sqrt{s}\right)^3}{6}+\frac{\left(st-a_n\sqrt{s}\right)^4}{4!}, \ldots,\right] \tag{64}$$

Thus:

$$e^{st}F_2(s) = \frac{1}{s^{5/2}}+\frac{t}{s^{3/2}}-\frac{a_n}{s^2}+\frac{t^2}{2\sqrt{s}}-\frac{a_n t}{s}+\frac{a_n^2}{2s^{3/2}}+\frac{\sqrt{s}t^3}{6}-\frac{a_n t^2}{3}+\frac{a_n^2 t}{2\sqrt{s}}-\frac{a_n t^2}{6}+\frac{a_n^3}{6s}+\frac{s^{3/2}t^4}{4!}-\frac{2a_n t^3 s}{4!}+\frac{a_n^2 t^2\sqrt{s}}{4!}-\frac{a_n t^3 s}{4!}+\frac{2a_n^2 t^2\sqrt{s}}{4!}-\frac{a_n^3 t}{4!}-\frac{a_n s t^3}{4!}+\frac{2a_n^2 t^2\sqrt{s}}{4!}-\frac{a_n^3 t}{4!}+\frac{a_n^2 t^2\sqrt{s}}{4!}-\frac{2a_n^3 t^2 s}{4!}+\frac{a_n^4}{4!\sqrt{s}}+, \ldots, \tag{65}$$

So,

$$\text{Res}\left(e^{st}F_2(s), 0\right) = \frac{t^2}{2}+\frac{a_n^2 t}{2}+\frac{a_n^4}{4!} \tag{66}$$

Thus, by Eq. 63-65 and 40, we obtain:

$$L^{-1}\left\{\frac{e^{-a_n\sqrt{s}}}{s^{3/2}}\right\} = t+\frac{a_n^2}{2} \tag{67}$$

$$L^{-1}\left\{\frac{e^{-a_n\sqrt{s}}}{s^{5/2}}\right\} = \frac{t^2}{2}+\frac{a_n^2 t}{2}+\frac{a_n^4}{4!} \tag{68}$$

By using a similar way, we can show that:

$$L^{-1}\left\{\frac{e^{-b_n\sqrt{s}}}{s^{3/2}}\right\} = t+\frac{b_n^2}{2} \tag{69}$$

$$L^{-1}\left\{\frac{e^{-b_n\sqrt{s}}}{s^{5/2}}\right\} = \frac{t^2}{2}+\frac{b_n^2 t}{2}+\frac{b_n^4}{4!} \tag{70}$$

Thus, Eq. 45 becomes:

$$u(x,t) = \frac{N\sqrt{\alpha}}{k}\left(\sum_{n=0}^{\infty}\left[\frac{\left(2(n+1)d-x\right)^2}{2\alpha}+\frac{\left(2(n+1)d+x\right)^2}{2\alpha}+2t\right]-M\sum_{n=0}^{\infty}\left[\frac{\left(2(n+1)d-x\right)^4 t}{4!\alpha^2}+\frac{\left(2(n+1)d+x\right)^2}{4!\alpha^2}+t^2\right]\right) \tag{71}$$

Which is the formal solution of the heat conduction problem Eq. 18-20.

CONCLUSION

We see that, complex analysis has an important role in finding the inverse Laplace transform for general types of functions. Moreover, Laplace transform with using the inversion formula derived in section 3 can be considered a robust technique for solving complicated boundary problems of partial differential equations such as the problem of heat conduction which has been considered in this study.

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