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Homotopy Perturbation Method (HPM) for Solving Nonlinear Volterra Hammerstein Integral Equations

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Abstract: This study discusses the approximate solution of nonlinear Volterra Hammerstein Integral Equations (VHIEs) using Homotopy Perturbation Method (HPM). Based on the previous literature, the use of HPM for solving VHIEs are very limited. Most of them focus toward the solving of Fredholm and Volterra integro-differential equations. In this proposed method, homotopy are constructed in the form Ly+p(Ny+Ny-Ly) = 0 where L, N are linear operator p∈[0, 1] is an embedding parameter to generate approximate series solution converge to the exact solution of VHIEs. We implement our proposed by using two specific examples of VHIEs.

Key words: VHIEs, coverage, parameter, HPM, integro-differential equations, solution

INTRODUCTION

The nonlinear Volterra-Hammerstein integral equation is given in the form of:

$$y(t) = f(t)+\int_a^t k(t,\theta)g(\theta, y(\theta))d\theta \tag{1}$$

where, f, g and k are continuous functions and g (θ, y) is nonlinear in y under assumption that Eq. 1 has a unique solution and y to be determined.

There exists several numerical methods for approximating the solution of Eq. 1 in literature. For example, the traditional successive approximation method (Tricomi, 1985), a variation of Nystrom's method (Lardy, 1981), collocation method (Kumar and Sloan, 1987; Guoqiang, 1993), single term Walsh series method (Sepehrian and Razzaghi, 2005), the hybrid of block-pulse and rationalized Haar functions (Ordokhani, 2009), fixed point method (Maleknejad and Torabi, 2012) and recently alternative Legendre collocation method (Bazm, 2016).

However, based on the previous literature, the use of HPM for solving integral differential equation are very limited. Most of them focus toward, the solving of Fredholm and Volterra integro-differential equations, for example, by Raftari (2010), Afrouzi *et al.* (2011), Aghazadeh and Mohammadi (2012). None of the researchers discuss the HPM method for solving nonlinear Volterra Hammerstein integral equations.

Raftari (2010) solve the special type of linear Volterra integral-differential equations using the Homotopy Perturbation Method (HPM) with finite difference method based on Simpson rule and Trapezoidal rule. The researcher transforms the Volterra integro-differential equation into a matrix equation. By Afrouzi *et al.* (2011) using modified homotopy perturbation method and compared with the exact solution for solving fourth order Volterra integro-differential equations.

Aghazadeh and Mohammadi (2012) present a modification homotopy perturbation method for solving linear and non-linear Volterra and Fredholm integral equations by giving an approximate analytic solution to the equations.

Motivated by previous study using HPM for solving Volterra and Fredholm integral equations, this study will present the use of this method in solving specific case of nonlinear Volterra Hammerstein integral equations.

MATERIALS AND METHODS

Mathematical formulation: This study introduced the formulation of HPM for the approximate solution of Volterra Hammerstein integral Eq. 1 as considered in study. Consider a general non-linear equation in the form Biazar and Ayati (2010):

$$Ly+Ny = 0 \tag{2}$$

where, L and N are linear operator and nonlinear operator, respectively. From Eq. 2, a homotopy can be constructed gamely in the form:

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$$Ly+p(Ny+Ny-Ly) = 0 \quad (3)$$

where, $p\epsilon[0, 1]$ is an embedding parameter to generate approximate series solution converge to the exact solution of Eq. 1. According to Ganji *et al*. (2007) the nonlinear Volterra Hammerstein integral equations of Eq. 1 can be rewritten as:

$$y(t) = f(t)+\int_a^t K(t, \theta)\left[R(y(\theta))+N(y(\theta))\right]d\theta \quad (4)$$

Where:

y(t) = Uunknown function to be determined
K(t, θ) = Kernel of the integral Eq. 1
y(t) = Analytic function
R(y) and N(y) = Linear and nonlinear functions of y, respectively, for all $t\epsilon[a, b]$

In order to present suitable formulation of HPM, we consider the following equation Mirzaei (2010):

$$L(y) = f(t)-y(t)+\int_a^t K(t, \theta)\left[R(y(\theta))+N(y(\theta))\right]d\theta \quad (5)$$

Define the homotopy function H(y, p) such that:

$$\begin{cases}\mathcal{H}(y, 0): F(y) = 0\\ \mathcal{H}(y, 1): L(y) = 0\end{cases} \quad (6)$$

where, F(y) is an integral operator with known solution y which can be obtained easily. Typically, According to Mirzaei (2010), we can define the homotopy formula by:

$$H(y, p) = (1-p)F(y)+p\,L(y) \quad (7)$$

We assume that the solution of Eq. 2 as given by the following power series:

$$y(t) = \sum_{i=0}^{\infty} p^i y_i(t) \quad (8)$$

If we set p = 1, the approximate solution of Eq. 2 is:

$$y(t) = \lim_{p\to 1}\sum_{i=0}^{\infty} p^i y_i(t) = y_0(t)+y_1(t)+y_2(t)+, \dots \quad (9)$$

RESULTS AND DISCUSSION

Numerical experiment: This study we show the implementation of HPM for the approximate solution of Volterra Hammerstein integral equations:

Example 3.1: Consider the following Volterra Hammerstein integral equation (Maleknejad and Torabi, 2012):

$$Y(t) = \frac{2t^6}{15}-\frac{t^4}{3}+t^2-1+\int_a^b (t-2\theta)y(\theta)^2\,d\theta,\ t\in[0, 1] \quad (10)$$

From Maleknejad and Torabi (2012) the exact solution of Eq. 10 is given by:

$$Y(t) = t^2-1 \quad (11)$$

Using Eq. 5, we define the following operators:

$$L(y) = y(t)-\frac{2t^6}{15}+\frac{t^4}{3}-t^2+1+\int_a^b (t-2\theta)y(\theta)^2\,d\theta$$

$$F(y) = y(t)-y_0(t)$$

where, $y_0(t)$ is the initial function or guess and by following the homotopy theory (Afrouzi *et al*., 2011), we have freedom to choose the proper function in order to obtain the best solution conversance to the exact solation of Eq. 10. Substituting F(u) and L(u) in Eq. 7 and equating the terms whit identical power of, we obtain the following equations:

$$p^0 = y_0(t) = -1$$

$$p^1 = y_1(t) = \frac{2t^6}{15}-\frac{t^4}{3}+t^2+\int_0^t (t-2\theta)y_0(\theta)^2\,d\theta$$

$$p^2 = y_2(t) = \frac{2t^6}{15}-\frac{t^4}{3}+t^2+\int_0^t (t-2\theta)\left[2y_0(\theta)y_1(\theta)\right]d\theta$$

$$\vdots$$

$$p^{n+1} = y_{n+1}(t) = \frac{2t^6}{15}-\frac{t^4}{3}+t^2+\int_0^t (t-2\theta)\left[\sum_{n=0}^{k-1} y_n(\theta)y_{k-1-n}(\theta)\right]d\theta$$

Such that $n\geq 0$. By using Eq. 9, we have the following results (Fig. 1 and 2). In order to detect the accuracy of

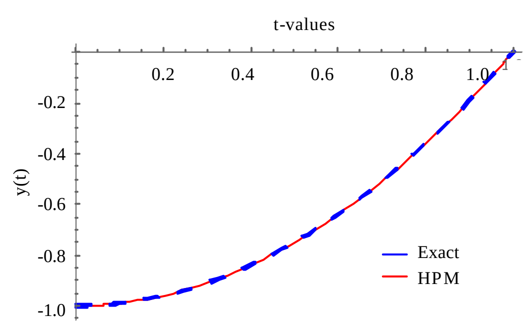

Fig. 1: Exact and tenth order HPM solution of Eq. 10 for all $t\epsilon[0, 1]$ (HPM and exact)

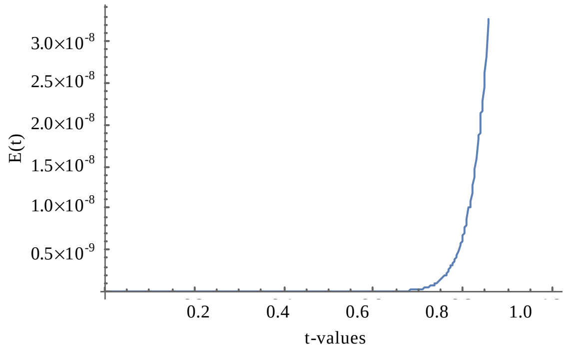

Fig. 2: Accuracy tenth order HPM solution of Eq. 10 for all $t\in[0, 1]$ (Absolute error)

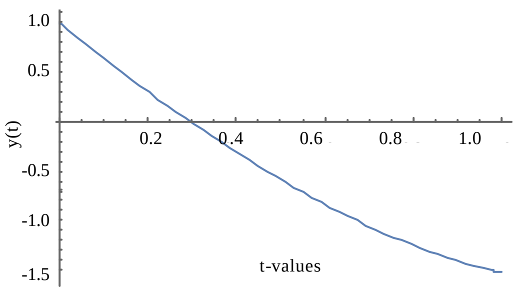

Fig. 3: Tenth order HPM solution of Eq. 13 for all $t\in[0, 1]$

tenth order HPM approximate solution of Eq. 10, we define the absolute error between the exact solution Y(t) and HPM solution y(t) as follows:

$$E(t) = |Y(t)-y(t)| \tag{12}$$

Example 3.2: Consider the following Volterra Hammerstein integral equation:

$$Y(t) = t^2-\frac{56t}{15}+1+\int_0^t (t-\theta)y(\theta)^2\, d\theta,\ t\in[0, 1] \tag{13}$$

Using Eq. 5, we define the following operators:

$$L(y) = y(t)-t^2+\frac{56t}{15}-1+\int_a^b (t-\theta)y(\theta)^2\, d\theta$$

where, $y_0(t)$ is the initial function or guess and by following the homotopy as in example 1, we have freedom to choose the proper function. Substituting F(u) and L(u) in Eq. 7 and equating the terms whit identical power of, we obtain the following equations (Fig. 3 and 4):

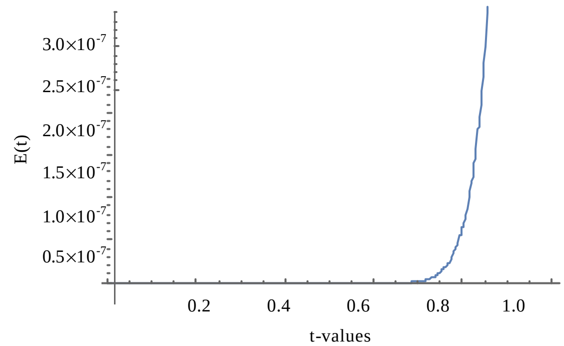

Fig. 4: Accuracy tenth order HPM solution of Eq. 13 for all and $t\in[0, 1]$ (Residual error)

$$p^0 = y_0(t) = 1$$

$$p^1 = y_1(t) = t^2-\frac{56t}{15}+\int_0^t (t-\theta)y_0(\theta)^2\, d\theta$$

$$p^2 = y_2(t) = t^2-\frac{56t}{15}+\int_0^t (t-\theta)\left[2y_0(\theta)y_1(\theta)\right] d\theta$$

$$\vdots$$

$$p^{n+1} = y_{n+1}(t) = t^2-\frac{56t}{15}+\int_0^t (t-\theta)\left[\sum_{n=0}^{k-1} y_n(\theta)y_{k-1-n}(\theta)\right] d\theta$$

Such that $n\geq 0$. By using Eq. 9 we have the following results. Moreover, in order to detect the accuracy of HPM $y(t)_{HPM}$ solution without need to compare with the exact solution of Eq. 13 we define the following residual error (Jameel and Ismail, 2015) as follows:

$$E(t) = \left|t^2-\frac{56t}{15}+1+\int_0^t (t-\theta)y(\theta)_{HPM}{}^2\, d\theta\right| \tag{14}$$

CONCLUSION

We have successfully developed an approximate-analytical method based on HPM to obtain an approximate solution of nonlinear VHIEs. From the numerical examples, the accuracy of HPM are determined through comparison with exact solution (in example 3.1) and using residual error (as in example 3.2 because there is no exact analytical solution for the given VHIE).

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