Journal of Engineering and Applied Sciences 14 (16): 5864-5868, 2019
ISSN: 1816-949X

Linear Operator Defined on a New Subclass of Multivalent Functions with Negative Coefficients

Wadhah Abdulelah Hussein and Hamza Barakat Habib
Department of Mathematics, College of Science, University of Diyala, Diyala, Iraq
wadhah.hussein2@gmail.com, halsaadi18@yahoo.com

Abstract: In the present study, we have introduced linear operator defined on a new subclass of multivalent functions with negative coefficients we derived some properties, like, coefficient inequality, weighted mean, apply of Littlewood theorem integral operator.

Key words: Coefficient inequality, weighted mean, Littlewood theorem integral operator, negative coefficients, integral operator, properties

INTRODUCTION

Let ω_p be denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=P+1}^{\infty} a_k z^k, (p \in \mathbb{N}) \quad (1)$$

Which are analytic and p-valent in the open unit disk $\nabla = \{z\in\mathbb{C}:|z|<1\}$. Let $N\omega_p$ denote the subclass of ω_p of functions of the form:

$$f(z) = z^p - \sum_{k=P+1}^{\infty} a_k z^K, (a_k \geq 0, z\in \nabla) \quad (2)$$

Note that the researchers defined and studied some classes of analytic functions like the form (Eq. 2) by Atshan *et al*. (2014), Aouf *et al*. (2016), Breaz and El-Ashwah (2014), El-Qadeem and Mamon (2018), Yang and Li (2012). Here, we need to the linear operator defined by Mahzoon and Latha (2009) such that class can be defined by means of this linear operator. For a function $f\in N\omega_P$ given by Eq. 2, we need the linear operator defined in; Let ε, β, $m\in R$, γ, β, m, ≥ 0, $p\in N$:

$$f(z) = z^p - \sum_{K=p+1}^{\infty} a_K z^K$$

Then we define the linear operator:

$$D^{\varepsilon,\beta}_{P,m}:\omega_p \to \omega_p \text{by}$$

$$D^{\varepsilon,\beta}_{P,m} f(z) = z^p - \sum_{K=p+1}^{\infty}\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_K z^K, z\in\nabla \quad (3)$$

functions f which satisfy:

$$\left|\frac{(p)(p-1)z^{p-2}-(D^{\varepsilon,\beta}_{P,m}f(z))''}{\mu\left(D^{\varepsilon,\beta}_{P,m}f(z)\right)''+2\mu(p)(p-1)z^{p-2}}\right|<\gamma \quad (4)$$

where, $0<\gamma<1$, $0<\mu<1/2$. ε, β, $m\in R$, ε, β, m, ≥ 0, $p\in N$ and $D^{\varepsilon,\beta}_{P,m}f$ is given by Eq. 3. Some of the following properties studied for other class by Atshan *et al*. (2014) and Li *et al*. (2017).

MATERIALS AND METHODS

Theorem 1 (Duren, 1983); (Maximum modulus theorem): Suppose that a function f is continuous on boundary of $\mathbb{U}$ ($\mathbb{U}$ any disk or region). Then, the maximum value of $|f(z)|$ which is always reached, occurs somewhere on the boundary of $\mathbb{U}$ and never in the interior.

Theorem (2): Let the function $f\in N\omega_p$ be defined by Eq. 2. Then $f\in\omega$ (p, K, γ, β, m, ε) if and only if:

$$\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)\, a_k \leq 3\mu\gamma(p)(p-1) \quad (5)$$

where: $0<\gamma<1$, $0<\mu<1/2$.ε, β, $m\in R$, ε, β, $m\geq 0$, $p\in N$. The result is sharp for the function:

$$f(z) = z^p - \frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)} z^k, k\geq 2$$

Corresponding Author: Wadhah Abdulelah Hussein, Department of Mathematics, College of Science, University of Diyala, Diyala, Iraq wadhah.hussein2@gmail.com

Proof: Assume that the inequality Eq. 5 holds true and let $|z| = 1$ then from Eq. 4, we obtain:

$$\left|(p)(p-1)z^{p+n-2}-(D^{\varepsilon,\beta}_{P,m}f(z))''\right|-\gamma\left|\mu\left(D^{\varepsilon,\beta}_{P,m}f(z)\right)''+2\mu(p)(p-1)z^{p+\eta-2}\right|=$$

$$\left|\sum_{k=p+1}^{\infty}(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}\right|-\gamma\left|3\mu(p)(p-1)z^{p-2}-\sum_{k=p+1}^{\infty}\mu(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}\right|\leq$$

$$\sum_{k=p+1}^{\infty}(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k-3\gamma\mu(p)(p-1)+\sum_{k=p+1}^{\infty}\gamma\mu(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k=$$

$$\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)\,a_k-3\gamma\mu(p)(p-1)\leq 0 \qquad (6)$$

By hypothesis. Hence, by maximum modulus principle, $f\in\omega$ (p, K, γ, β, m, ε). Conversely, let $f\in\omega$ (p, K, γ, β, m, ε). Then:

$$\left|\frac{(p)(p-1)z^{p-2}-(D^{\varepsilon,\beta}_{P,m}f(z))''}{\mu\left(D^{\varepsilon,\beta}_{P,m}f(z)\right)''+2\mu(p)(p-1)z^{p-2}}\right|<\gamma,\ (z\in U)$$

That is:

$$\left|\frac{\sum_{k=p+1}^{\infty}(k)(k-1\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m)a_k z^{k-2}}{3\mu(p)(p-1)z^{p-2}-\sum_{k=p+1}^{\infty}\mu(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}}\right|<\gamma \qquad (7)$$

Since, Re (z)≤|z| for all z (z∈∇), we get:

$$\mathrm{Re}\left\{\frac{\sum_{k=p+1}^{\infty}(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}}{3\mu(p)(p-1)z^{p-2}-\sum_{k=p+1}^{\infty}\mu(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}}\right\}\leq\gamma \qquad (8)$$

We choose the value of z on the real axis, so that, $(D^{\varepsilon,\beta}_{P,m}f(z))''$ is real:

$$\sum_{k=p+1}^{\infty}(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}\leq 3\mu\gamma(p)(p-1)z^{p-2}-\sum_{k=p+1}^{\infty}\gamma\mu(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k z^{k-2}$$

Letting $z\to 1^-$, through real values:

$$\sum_{k=p+1}^{\infty}(k)(k-1)\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m a_k\leq 3\mu\gamma(p)(p-1)-\sum_{k=p+1}^{\infty}\gamma\mu(k)\left(k-1\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m\right)a_k$$

We obtain inequality Eq. 5. Finally, sharpness follows, if we take:

$$f(z)=z^p-\frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)}z^k,\ k\geq 2 \qquad (9)$$

Corollary 1: Let $f\in\omega$ (p, K, γ, β, m, ε). Then:

$$a_k\leq\frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)},\ k\geq 2 \qquad (10)$$

In 1925, Littlewood (1925) proved the following subordination theorem Duren (1983).

Theorem 3; Littlewood (1925): If f and g are analytic in U with $f\prec g$. Then, for $\alpha>0$ and $z = re^{i\vartheta}$ and $(0<r<1)$:

$$\int_0^{2\pi}|f(z)|^\alpha d\vartheta\leq\int_0^{2\pi}|g(z)|^\alpha d\vartheta \qquad (11)$$

We will make use the above theorem to prove.

Theorem (4): Let $f\in\omega$ (p, K, γ, β, m, ε) and suppose that f is defined by:

$$f(z)=z^p-\frac{3\mu\gamma(p)(p-1)}{\left(\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m k\right)(k-1)\right)(\gamma\mu+1)}z^k \qquad (12)$$

$k\geq p+1$

If there exists an analytic function w given by:

$$[w(z)]^k = \frac{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)}{\mu\gamma(p)(p-1)}\sum_{k=p+1}^{\infty} a_k z^k \qquad (13)$$

Then for $z = re^{i\vartheta}$ and $(0<r<1)$:

$$\int_0^{2\pi}\left|f\left(re^{i\vartheta}\right)\right|^a d\vartheta \le \int_0^{2\pi}\left|f\left(re^{i\vartheta}\right)\right|^a d\vartheta, (a>0)$$

Proof: Let f (z) of the form Eq. 2 and f_k (z) defined by Eq. 12 then we must show that:

$$\int_0^{2\pi}\left|1-\sum_{k=p+1}^{\infty} a_k z^k\right|^a d\vartheta \le \int_0^{2\pi}\left|1-\frac{3\mu\gamma(p)(p-1)}{\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m((k)(k-1))(\gamma\mu+1)}z^k\right|^a d\vartheta$$

By applying Littlewood's subordination theorem, it would suffice to show that:

$$1-\sum_{k=p+1}^{\infty} a_k z^k < 1-\frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(K-p)e}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)}z^k$$

By setting:

$$1-\sum_{k=p+1}^{\infty} a_k z^k = 1-\frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m(\gamma\mu+1)\right)(k)(k-1)}[w(z)]^k$$

We find that:

$$[w(z)]^k = \frac{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m(\gamma\mu+1)\right)(k+p+n)(k-1)}{3\mu\gamma(p)(p-1)}\sum_{k=p+1}^{\infty} a_k z^k$$

Which readily yield sw (0) = 0. Furthermore, by using Eq. 5, we obtain:

$$\left|[w(z)]^k\right| = \left|\sum_{k=p+1}^{\infty}\frac{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)}{3\mu\gamma(p)(p-1)}a_k z^k\right| \le$$

$$\sum_{k=p+1}^{\infty}\frac{\left(\left(1+\frac{(K-p)}{p+\beta}\varepsilon\right)^m (k)(k-1)\right)(\gamma\mu+1)}{3\mu\gamma(p)(p-1)}a_k |z|^k \le$$

$$|z|^2\sum_{k=p}^{\infty}\frac{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)}{3\mu\gamma(p)(p-1)}a_k \le$$

$$|z|<1$$

The proof is complete.

Theorem 5: Let $\alpha>0$. If $f\in\omega$ (p, K, γ, β, m, ε) and:

$$f(z) = z^p - \frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(K-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)}z^k, k\ge 2$$

Then for $z = re^{i\vartheta}$ and $(0<r<1)$:

$$\int_0^{2\pi}\left|f'\left(re^{i\vartheta}\right)\right|^\alpha d\vartheta \le \int_0^{2\pi}\left|f'_k\left(re^{i\vartheta}\right)\right|^\alpha d\vartheta \qquad (14)$$

Proof:

$$f'(z) = (p)z^{p-1}-\sum_{k=p+1}^{\infty}(k)a_k z^{k-1}$$

$$f'(z) = (p)z^{p-1}-\frac{3\mu\gamma(p)(p-1)(k)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)}z^{k-1}$$

$$k\ge p+1$$

It is sufficient to show that:

$$1-\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)a_k z^k \prec 1-\frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(\gamma\mu+1)\right)(k)(k-1)}\left(\frac{k\eta}{p}\right)z^k$$

By setting:

$$1-\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)a_k z^k = 1-\frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(\gamma\mu+1)\right)(k)(k-1)}\left(\frac{k}{p}\right)[w(z)]^k$$

Hence:

$$[w(z)]^k = \sum_{k=p+1}^{\infty}\frac{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(\gamma\mu+1)\right)(k)(k-1)}{\mu\gamma(p)(p-1)}a_k z^k$$

Which readily yields w (0) = 0. By using theorem (2), we obtain:

$$\left|[w(z)]\right|^k = \left|\sum_{k=p+1}^{\infty}\frac{\left(\left(1+\frac{(k-p)}{p+\beta}\varepsilon\right)^m(k)(k-1)\right)(\gamma\mu+1)}{\mu\gamma(p)(p-1)}a_k z^k\right| \le$$

$$|z|^2\sum_{k=p}^{\infty}\frac{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)}{\mu\gamma(p)(p-1)}a_k \le$$

$$|z|<1$$

The proof is complete. In the following theorem, we obtain weighted mean is in the class ω (p, K, γ, β, m, ε).

Definition (1): Let f_1 and f_2 be in the class ω (p, K, γ, β, m, ε). Then the weighted mean w_j of f_1 and f_2 is given by:

$$w_j(z) = \frac{1}{2}\left[(1-j)f_1(z)+(1+j)f_2(z)\right],\ 0<j<1$$

Theorem (6): Let f_1 and f_2 be in the class ω (p, K, γ, β, m). Then the weighted mean w_j of f_1 and f_2 is also in the class ω (p, K, γ, β, m, ε).

Proof: By definition Eq. 1, we have:

..

$$w_j(z) = \frac{1}{2}\left[(1-j)f_1(z)+(1+j)f_2(z)\right] =$$

$$\frac{1}{2}\left[(1-j)\left(z^p-\sum_{k=p+1}^{\infty}a_{k,1}z^k\right)+(1+j)\left(z^{p+n}-\sum_{k=p+1}^{\infty}a_{k,2}z^k\right)\right] =$$

$$z^p-\sum_{k=p+1}^{\infty}\frac{1}{2}\left[(1-j)a_{k,1}+(1+j)a_{k,2}\right]z^k \tag{15}$$

Since, f_1 and f_2 are in the class ω (p, K, γ, β, m, ε), so, by theorem (1), we get:

$$\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)a_{k,1} \le \mu\gamma(p)(p-1)$$

And:

$$\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(k+p+n)(k+p+\eta-1)\right)(\gamma\mu+1)a_{k,2} \le \mu\gamma(p)(p-1)$$

Hence:

$$\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)]\frac{1}{2}\left[(1-j)a_{k,1}+(1+j)a_{k,2}\right] =$$

$$\frac{1}{2}(1-j)\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)a_{k,1}+$$

$$\frac{1}{2}(1+j)\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m(k)(k-1)\right)(\gamma\mu+1)a_{k,1} \le$$

$$\frac{1}{2}(1-j)\mu\gamma(p)(p-1)+\frac{1}{2}(1+j)\mu\gamma(p)(p-1) = \mu\gamma(p)(p-1)$$

Therefore, w_jєω (p, K, γ, β, m, ε). The proof is complete. In the following theorem, we obtainintegral operator Atshan and Kulkarni (2008) is in the class ω (p, K, γ, β, m, ε).

Theorem 7: Let f (z)єω (p, K, γ, β, m). Then the integral operator:

$$F_\iota(z) = (1-\iota)z^p+\iota(p)\int_0^z\frac{f(s)}{s}ds\ (\iota\ge 0,\ z\in U)$$

Is also in ω (p, K, γ, β, m, ε) if 0≤l≤2/p.

Proof: If

$$f(z) = z^p-\sum_{k=p+1}^{\infty}a_k z^k$$

Then:

$$F_\iota(z) = (1-\iota)z^p+\iota(p)\int_0^z\left(\frac{s^p-\sum_{k=2}^{\infty}a_k s^k}{s}\right)ds =$$

$$(1-\iota)z^p+\iota(p)\left[\frac{z^p}{p}-\sum_{k=p+1}^{\infty}\frac{a_k z^k}{k}\right] =$$

$$z^p-\sum_{k=p+1}^{\infty}\frac{\iota(p)}{k}a_k z^k =$$

$$z^p-\sum_{k=p+1}^{\infty}g_k z^k$$

where, g_k = l (p)/ka_k. But:

$$\sum_{k=p+1}^{\infty} \frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)} g_k =$$

$$\sum_{k=p+1}^{\infty} \frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)} \frac{\iota(p)}{k} a_k \leq$$

$$\sum_{k=p+1}^{\infty} \frac{3\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)} \frac{\iota(p)}{2} a_k$$

Where:

$$\frac{\iota(p)}{2} \leq 1$$

$$\leq \sum_{k=p+1}^{\infty} \frac{\mu\gamma(p)(p-1)}{\left(\left(1+\frac{(k-p)\varepsilon}{p+\beta}\right)^m (k)(k-1)\right)(\gamma\mu+1)} a_k$$

By 3.26:

$$\leq 3\mu\gamma(p+\eta)(p+\eta-1)$$

Fl $(z)\in\omega(p, K, \gamma, \beta, m, \varepsilon)$. So, the proof is complete.

RESULTS AND DISCUSSION

The scope of this study has caused several limitations which however, provide basis for future research along the path to geometric function theory of a complex variable in several areas. These areas include, application of multivalent functions with negative coefficients and development of solving problems in physics is an important area for future research.

CONCLUSION

The main objective of this study is to study linear operator defined on a new subclass of multivalent functions with negative coefficients. Main techniques used to derive our results are mainlycoefficient inequality, weighted mean, apply of Littlewood theorem integral operator. It is observed that some of these results are the best possible and by giving different values to the parameters involved.

REFERENCES

Aouf, M.K., A.O. Mostafa and W.K. Elyamany, 2016. Certain subclass of multivalent functions with higher order derivatives and negative coefficients. Intl. J. Open Prob. Complex Anal., 8: 1-17.

Atshan, W.G. and S.R. Kulkarni, 2008. A generalized ruscheweyh derivatives involving a general fractional derivative operator defined on a class of multivalent functions II. Intl. J. Math. Anal., 2: 97-109.

Atshan, W.G., A.H. Battor and A.M. Dereush, 2014. On a new class of multivalent functions with negative coefficient defined by hadamard Product involving a linear operator. Am. J. Math. Stat., 4: 147-155.

Breaz, N. and R.M. El-Ashwah, 2014. Quasi-hadamard product of some uniformly analytic and P-valent functions with negative coefficients. Carpathian J. Math., 30: 39-45.

Duren, P.L., 1983. [Univalent Functions (Basic Theories of Mathematical Sciences 259)]. Springer, Berlin, Germany, ISBN-13:978-0387907956, Pages: 384 (In German).

El-Qadeem, A.H. and M.A. Mamon, 2018. Comprehensive subclasses of multivalent functions with negative coefficients defined by using a Q-difference operator. Trans. A. Razmadze Math. Inst., 172: 510-526.

Li, X., D. Ding, L. Xu, C. Qin and S. Hu, 2017. Certain subclasses of multivalent functions defined by higher-order derivative. J. Funct. Spaces, 2017: 1-6.

Littlewood, J.E., 1925. On inequalities in the theory of functions. Proc. London Math. Soc., 2: 481-519.

Mahzoon, H. and S. Latha, 2009. Neighborhoods of multivalent functions. Intl. J. Math. Anal., 3: 1501-1507.

Yang, J. and S. Li, 2012. Properties of certain subclass of multivalent functions with negative coefficients. Intl. J. Math. Sci., 2012: 1-21.