Linear Operator Defined on a New Subclass of Multivalent Functions with Negative Coefficients

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Abstract: In the present study, we have introduced linear operator defined on a new subclass of multivalent functions with negative coefficients we derived some properties, like, coefficient inequality, weighted mean, apply of Littlewood theorem integral operator.

Key words: Coefficient inequality, weighted mean, Littlewood theorem integral operator, negative coefficients, integral operator, properties

INTRODUCTION

Let $\omega_p$ be denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=-p}^{\infty} a_k z^k, (p \in \mathbb{C})$$

(1)

Which are analytic and $p$-valent in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{N}_p\omega_p$ denote the subclass of $\omega_p$ of functions of the form:

$$f(z) = z^p + \sum_{k=-p}^{\infty} a_k z^k, (a_k \geq 0, z \in \mathbb{D})$$

(2)

Note that the researchers defined and studied some classes of analytic functions like the form (Eq. 2) by Atshan et al. (2014), Aouf et al. (2016), Breaz and El-Ashwah (2014), El-Qadeem and Mamoun (2018), Yang and Li (2012). Here, we need the linear operator defined by Mahzoon and Latha (2009) such that class can be defined by means of this linear operator. For a function $f \in \mathcal{N}_p\omega_p$, given by Eq. 2, we need the linear operator defined in: Let $\varepsilon, \beta, \gamma, \beta, \gamma, \beta, m, \geq 0, p \in \mathbb{N}$:

$$f(z) = z^p + \sum_{k=-p}^{\infty} a_k z^k$$

Then we define the linear operator:

$$D_{p,m}^{\alpha} f(z) = z^p + \sum_{k=-p}^{\infty} a_k z^k$$

(3)

functions $f$ which satisfy:

$$\left| \frac{(p)(p-1)z^{p-2}-(D_{p,m}^{\alpha} f(z))^p}{\mu(D_{p,m}^{\alpha} f(z))^p + 2\mu(p)(p-1)z^{p-2}} \right| < \gamma$$

(4)

where, $0 < \gamma < 1$, $0 < \mu < 1/2$, $\varepsilon, \beta, \gamma, m, \geq 0, p \in \mathbb{N}$

and $D_{p,m}^{\alpha} f$ is given by Eq. 3. Some of the following properties studied for other class by Atshan et al. (2014) and Li et al. (2017).

MATERIALS AND METHODS

Theorem 1 (Duren, 1983); (Maximum modulus theorem): Suppose that a function $f$ is continuous on boundary of $\mathbb{U}$ ( $\mathbb{U}$ any disk or region). Then, the maximum value of $|f(z)|$ which is always reached, occurs somewhere on the boundary of $\mathbb{U}$ and never in the interior.

Theorem 2: Let the function $f \in \mathcal{N}_p\omega_p$ be defined by Eq. 2. Then $f \in (p, \kappa, \gamma, \beta, m, \varepsilon)$ if and only if:

$$\sum_{k=-p}^{\infty} \left( 1 + \frac{(K-p)\varepsilon}{p+\beta} \right)^n \frac{(k)(k-1)}{\gamma(m+1)} a_k \leq 3\mu\gamma(p)(p-1)$$

(5)

where, $0 < \gamma < 1$, $0 < \mu < 1/2$, $\varepsilon, \beta, \gamma, m, \geq 0, p \in \mathbb{N}$. The result is sharp for the function:

$$f(z) = z^p + \sum_{k=-p}^{\infty} a_k z^k$$

$$\left( 1 + \frac{(K-p)\varepsilon}{p+\beta} \right)^n \frac{(k)(k-1)}{\gamma(m+1)} z^k, k \geq 2$$

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Proof: Assume that the inequality Eq. 5 holds true and let \(|z|=1\) then from Eq. 4, we obtain:

\[
\begin{align*}
\left| (p)(p-1)z^{n-2} - (D_{p,n}^\beta f(z))^m \right| & = \\
\gamma \left| \mu(D_{p,n}^\beta f(z))^m + 2\mu(p)(p-1)z^{n-2} \right| & = \\
\sum_{k=p+1}^{\infty} \left| (k)(k-1) \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2} \right| & \\
-3\mu(p)(p-1)z^{n-2} & \sum_{k=p+1}^{\infty} \mu(k)(k-1) \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2} \leq \\
\sum_{k=p+1}^{\infty} \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2} & \leq \\
\sum_{k=p+1}^{\infty} \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2}. \\
\end{align*}
\]

Letting \(z=1\), through real values:

\[
\begin{align*}
\sum_{k=p+1}^{\infty} \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2} & \leq \\
3\mu(p)(p-1) & \sum_{k=p+1}^{\infty} \mu(k)(k-1) \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2}.
\end{align*}
\]

We obtain inequality Eq. 5. Finally, sharpness follows, if we take:

\[
f(z) = z^{n-2} \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2}, \quad k \geq 2
\]

Corollary 1: Let \(f \in \omega(p, K, \gamma, B, m, e)\). Then:

\[
a_k \leq \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2}, \quad k \geq 2
\]

In 1925, Littlewood (1925) proved the following subordination theorem Duren (1983).

Theorem 3: Littlewood (1925): If \(f\) and \(g\) are analytic in \(U\) with \(f<g\). Then, for \(\alpha>0\) and \(z = re^{i\theta}\) and \((0<\alpha<1)\):

\[
\int_{0}^{2\pi} |f(z)|^\alpha d\theta \leq \int_{0}^{2\pi} |g(z)|^\alpha d\theta
\]

We will make use of the above theorem to prove.

Theorem (4): Let \(f \in \omega(p, K, \gamma, B, m, e)\) and suppose that \(f\) is defined by:

\[
f(z) = z^{n-2} \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2}, \quad k \geq p+1
\]

If there exists an analytic function \(w\) given by:

\[
f(z) = z^{n-2} \left( 1 + \frac{(K-p)e^m}{p+\beta} \right) a_k z^{k-2}
\]
\[
[w(z)]^k = \left( \frac{\binom{n}{m}^{(k-1)}(\gamma+1)}{\sum_{k=p+1}^{n} a_k z^k} \right)^k
\]

Then for \( z = r e^{\theta} \) and \((0 < r < 1)\):
\[
\int_0^{2\pi} f'(r e^{\theta})^m d\theta \leq \int_0^{2\pi} f(r e^{\theta})^m d\theta, (a > 0)
\]

**Proof:** Let \( f(z) \) of the form Eq. 2 and \( f_k(z) \) defined by Eq. 12 then we must show that:
\[
\int_0^{2\pi} \left[ 1 - \sum_{k=p+1}^{n} a_k z^k \right]^m d\theta \leq \int_0^{2\pi} \left[ 1 - \frac{3\mu r (p)(p-1)}{1 + \frac{(K-p)e}{p+\beta}} \right] z^k d\theta
\]

By applying Littlewood's subordination theorem, it would suffice to show that:
\[
1 - \sum_{k=p+1}^{n} a_k z^k < 1 - \frac{3\mu r (p)(p-1)}{1 + \frac{(K-p)e}{p+\beta}} z^k
\]

By setting:
\[
1 - \sum_{k=p+1}^{n} a_k z^k - 1 - \frac{3\mu r (p)(p-1)}{1 + \frac{(K-p)e}{p+\beta}} [w(z)]^k
\]

We find that:
\[
[w(z)]^k = \left( \frac{1 + \frac{(K-p)e}{p+\beta}}{k(m-k)} \right)^m \frac{3\mu r (p)(p-1)}{k(p-k)(\gamma+1)} \sum_{k=p+1}^{n} a_k z^k
\]

Which readily yield \( w(0) = 0 \). Furthermore, by using Eq. 5, we obtain:
\[
[w(z)]^k = \left( \frac{1 + \frac{(K-p)e}{p+\beta}}{k(m-k)} \right)^m \frac{3\mu r (p)(p-1)}{k(p-k)(\gamma+1)} a_k z^k \leq
\]

The proof is complete.

**Theorem 5:** Let \( a > 0 \). If \( f(z) = z^a + \sum_{k=p+1}^{n} \frac{3\mu r (p)(p-1)}{(k)(k-1)(\gamma+1)} z^k, k \geq 2 \)

Then for \( z = r e^{\theta} \) and \((0 < r < 1)\):
\[
\int_0^{2\pi} f'(r e^{\theta})^m d\theta \leq \int_0^{2\pi} f(r e^{\theta})^m d\theta
\]

**Proof:**
\[
f(z) = z^a + \sum_{k=p+1}^{n} \frac{3\mu r (p)(p-1)}{(k)(k-1)(\gamma+1)} z^k
\]

It is sufficient to show that:
\[
1 - \sum_{k=p+1}^{n} \frac{k}{p} a_k z^k < 1 - \frac{3\mu r (p)(p-1)}{1 + \frac{(k-p)e}{p+\beta}} z^k
\]

By setting:
\[
1 - \sum_{k=p+1}^{n} \frac{k}{p} a_k z^k = 1 - \frac{3\mu r (p)(p-1)}{1 + \frac{(k-p)e}{p+\beta}} [w(z)]^k
\]

\[
[w(z)]^k = \left( \frac{1 + \frac{(k-p)e}{p+\beta}}{k(m-k)} \right)^m \frac{3\mu r (p)(p-1)}{k(p-k)(\gamma+1)} a_k z^k
\]

Hence:
\[
[w(z)]^k = \sum_{k=p+1}^{n} \frac{3\mu r (p)(p-1)}{k(p-k)(\gamma+1)} a_k z^k
\]
Which readily yields \( w(0) = 0 \). By using theorem (2), we obtain:

\[
|w(z)|^n = \sum_{k=p^+}^{\infty} \frac{1}{\mu\gamma(p)(p-1)} a_k z^k 
\]

\[
|z|^\sum_{k=p^+}^{\infty} \frac{1}{\mu\gamma(p)(p-1)} a_k z^k \leq \frac{1}{2} \sum_{k=p^+}^{\infty} \frac{1}{\mu\gamma(p)(p-1)} a_k z^k \]

Hence:

\[
\sum_{k=p^+}^{\infty} \frac{1}{\mu\gamma(p)(p-1)} a_k z^k \leq \mu\gamma(p)(p-1)
\]

The proof is complete. In the following theorem, we obtain weighted mean is in the class \( \omega(p, K, \gamma, \beta, m, \varepsilon) \).

**Definition (1):** Let \( f_i \) and \( f_j \) be in the class \( \omega(p, K, \gamma, \beta, m, \varepsilon) \). Then the weighted mean \( w_i \) of \( f_i \) and \( f_j \) is given by:

\[
w_i(z) = \frac{1}{2} [ (1-i) f_i(z) + (1+i) f_j(z) ] \]

**Theorem (6):** Let \( f_i \) and \( f_j \) be in the class \( \omega(p, K, \gamma, \beta, m, \varepsilon) \). Then the weighted mean \( w_i \) of \( f_i \) and \( f_j \) is also in the class \( \omega(p, K, \gamma, \beta, m, \varepsilon) \).

**Proof:** By definition Eq. 1, we have:

\[
w_i(z) = \frac{1}{2} \left[ (1-i) f_i(z) + (1+i) f_j(z) \right] = \frac{1}{2} \left[ (1-i) z^p + \sum_{k=p^+}^{\infty} a_k z^k \right] \left( 1+i \right) \left[ z^p + \sum_{k=p^+}^{\infty} a_k z^k \right] = z^p + \sum_{k=p^+}^{\infty} \frac{1}{2} \left[ (1-i) a_k + (1+i) a_{k+1} \right] z^k
\]

Since, \( f_i \) and \( f_j \) are in the class \( \omega(p, K, \gamma, \beta, m, \varepsilon) \), so, by theorem (1), we get:

\[
\sum_{k=p^+}^{\infty} \frac{1}{\mu\gamma(p)(p-1)} a_k z^k \leq \mu\gamma(p)(p-1)
\]

And:

\[
\sum_{k=p^+}^{\infty} \frac{1}{\mu\gamma(p)(p-1)} a_k z^k \leq \mu\gamma(p)(p-1)
\]

**Theorem 7:** Let \( f(z) \in \omega(p, K, \gamma, \beta, m, \varepsilon) \). Then the integral operator:

\[
F_i(z) = \left( 1+i \right) z^p + \sum_{k=p^+}^{\infty} a_k z^k = \mu\gamma(p)(p-1)
\]

Is also in \( \omega(p, K, \gamma, \beta, m, \varepsilon) \) if \( 0 \leq 1 \leq 2/p \).

**Proof:**

\[
f(z) - z^p - \sum_{k=p^+}^{\infty} a_k z^k
\]

Then:

\[
F_i(z) = \left( 1+i \right) z^p + \sum_{k=p^+}^{\infty} a_k z^k = z^p - \sum_{k=p^+}^{\infty} \frac{1}{p} a_k z^k
\]

\[
za^k = \sum_{k=p^+}^{\infty} g_k z^k
\]

where, \( g_k = \frac{1}{p} k a_k \). But:
\[
\sum_{k=p+1}^{n} \frac{3\mu\gamma(p)(p-1)}{1+(k-p)\epsilon p+\beta} a_k \leq 2 \\
\sum_{k=p+1}^{n} \frac{3\mu\gamma(p)(p-1)}{1+(k-p)\epsilon p+\beta} a_k \leq \frac{t(p)}{2} \leq 1 \\
\leq \sum_{k=p+1}^{n} \frac{\mu\gamma(p)(p-1)}{1+(k-p)\epsilon p+\beta} a_k \\
\leq 3\mu\gamma(p+\eta)(p+\eta-1)
\]

By 3.26:

\[ F(\epsilon) = (p, K, \gamma, \beta, m, \epsilon). \] So, the proof is complete.

RESULTS AND DISCUSSION

The scope of this study has caused several limitations which however, provide basis for future research along the path to geometric function theory of a complex variable in several areas. These areas include, application of multivalent functions with negative coefficients and development of solving problems in physics is an important area for future research.

CONCLUSION

The main objective of this study is to study linear operator defined on a new subclass of multivalent functions with negative coefficients. Main techniques used to derive our results are mainlycoefficient inequality, weighted mean, apply of Littlewood theorem integral operator. It is observed that some of these results are the best possible and by giving different values to the parameters involved.

REFERENCES


