Viscous Flow Over a Permeable Stretching/Shrinking Surface in a Nano fluid: A Stability Analysis

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Abstract: This study deals with the boundary layer flow and heat transfer near the stagnation point on a permeable stretching/shrinking surface in a nano fluid. The nanoparticles considered in this study are copper and silver. The governing nonlinear partial differential equations are transformed into a system of nonlinear ordinary differential equations using an appropriate similarity transformation which then solved numerically to study the effect of solid volume fraction or nanoparticle volume fraction parameter \( \phi \) of the nano fluid. Multiple solutions are found for a certain range of shrinking and suction parameters, therefore, a stability analysis is performed to determine which solution is stable and physically realizable. The effects of the governing parameters on the skin friction coefficient, the local Nusselt number and the velocity and temperature profiles were presented and discussed. It was found that the nanoparticle volume fraction substantially affects the fluid flow and heat transfer characteristics.

Key words: Boundary layer, nano fluid, heat transfer, stretching/shrinking surface, dual solutions, stability analysis

INTRODUCTION

Nano fluids is a term proposed by Choi (1995) which defined as liquids that contain suspensions of nanoparticles with the size of 1-100 nm in a base fluid. Nano fluids are expected to have superior heat transfer characteristic due to the presence of the nanoparticles that increase the thermal conductivity. There are several studies on the forced and free convection using nano fluids related with differentially heated enclosures (Khanafer et al., 2003; Tiwari and Das, 2007; Abu-Nada and Oztop, 2009; Muthamaiselvan et al., 2010). An excellent compilation of the published papers on nano fluid can be found in the book written by Das et al. (2008) and the papers by Daungthongsuk and Wongwises (2007), Wang and Mjuumdar (2008) Kakac and Pramujojarenok (2009), Fan and Wang (2011) and Vajjha and Das (2012).

In additional, a few research papers worth mentioning here are the Cheng-Minkowycz problem for natural convective boundary layer flow in a porous medium saturated by a nano fluid by Nield and Kuznetsov (2009) viscous flow due to a permeable stretching/shrinking sheet in a nano fluid by Ariffin et al. (2011) and the mixed convection flow from a horizontal circular cylinder in a nano fluid by Tham et al. (2012). We also mention, here, two of the recent studies done by Bakar et al. (2017) on the rotating flow over a shrinking sheet in nano fluid using Buongiorno model and thermophysical properties of nanoliquids and Uddin et al. (2018) on forced convective slip flow of a nano fluid past a radiating stretching/shrinking sheet.

In this study, we extend the classical problem of stagnation-point flow of a viscous and incompressible (Newtonian) fluid on a stretching/shrinking sheet first considered by Mikkovic and Wang (2006) and Wang (2008) to the case of nano fluids using the model proposed by Tiwari and Das (2007) with two different nanoparticles, namely Copper (Cu) and silver (Ag). We also extend the research by Ariffin et al. (2011) by performing the stability analysis in order to determine the stability of the dual solutions. It is worth mentioning that

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the flow over a continuously stretching/shrinking surface is an important problem in many engineering processes with industrial applications such as wire drawing, hot rolling and glass-fibre production. Later, several other papers on shrinking surfaces were published such as those by Sajid et al. (2008), Noor and Hashim (2009), Nazar et al. (2011), Bhattacharyya et al. (2013) and Rosali et al. (2015) (Table 1).

**MATERIALS AND METHODS**

**Problem formulation:** Consider the steady twodimensional boundary layer flow near the stagnation point over a permeable stretching/shrinking surface in a water based nanofluid containing two types of nanoparticles; Copper (Cu) and silver (Ag). The nanofluid is assumed to be incompressible and the effects of dissipation and radiation are neglected. The base fluid (water) and the nanoparticles are assumed to be in thermal equilibrium and no slip occurs between them. Table 2 displays the thermophysical properties of fluid and nanoparticles. Under these assumptions and following the model equations of nanofluid proposed by Tiwari and Das (2007), the governing boundary layer equations for the problem under consideration can be written as the following:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(1)

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = u_pr + v_pr + w_pr
\]

(2)

\[
\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial z^2}
\]

(3)

Subject to the boundary conditions:

\[
u = u(x) = ax, \quad v = 0, \quad w = w_w, \quad T = T_w \text{ at } z = 0
\]

\[
u - u(x) - cx, \quad T - T_w \text{ as } z \to \infty
\]

where, \(x\) and \(y\) are the Cartesian coordinates with \(x\) and \(y\) in the plane of the stretching/shrinking sheet \((z = 0)\) while the \(z\)-coordinate being measured normal to the stretching/shrinking sheet. \(u, v\) and \(w\) are the velocity components along the \(x, y\) and \(z\), respectively, \(w_w\) is the mass flux velocity with \(w_w > 0\) for suction and \(w_w < 0\) for injection, \(T\) is the non-dimensional temperature of the nanofluid, \(T_w\) is the constant surface temperature distribution, \(T_s\) is the uniform temperature of the ambient nanofluid, \(u_w(x)\) is the velocity of the stretching/shrinking sheet, \(u(x)\) is the velocity of the external flow potential flow of the nanofluid with \(c\) being a constant where, \(c > 0\) for a stretching sheet and \(c < 0\) for a shrinking sheet and \(a\) is a positive constant. Further, \(\mu_f\) is the effective viscosity of the nanofluid and \(\alpha_f\) is the thermal diffusivity of the nanofluid which are given in Table 2 and are defined as:

\[
\mu_f = \frac{\mu_f}{(1-\phi)^{1.5}}, \quad \alpha_f = \frac{k_f}{(\rho C_p)^{\frac{1}{3}}} k_f = \frac{k_f + 2k_f - 2\phi(k_f - k_p)}{k_f + 2k_f + \phi(k_f - k_p)}
\]

(5)

\[
\rho C_p = (1-\phi)\rho C_p f + \phi \rho C_p p, \quad \rho_{eff} = (1-\phi)\rho t + \phi \rho t
\]

where, \(\phi\) is the nanoparticle volume fraction, \(\rho_{eff}\) is the effective density of the nanofluid, \((\rho C_p)_{eff}\) is the heat capacity of the nanofluid, \(k_f\) is the effective thermal conductivity of the nanofluid, \(\rho t\) is the reference density of the fluid fraction, \(\rho t\) is the reference velocity of the solid fraction, \(\mu f\) is the viscosity of the fluid fraction, \(k f\) is the thermal conductivity of the fluid, \(k s\) is the thermal conductivity of the solid, \((\rho C_p)_f\) is the heat capacity of the fluid and \((\rho C_p)_s\) is the heat capacity of the solid. Following Miklavcic and Wang (2006) and Arifin et al. (2011), the similarity solutions of Eq. 1-3 are expressed in terms of the following variables:

\[
u - \xi \tau(\eta), \quad v = \xi (m-1) \gamma(\eta), \quad w = \sqrt{\xi} \sqrt{\frac{m f}{\xi}} \xi(\eta), \quad \theta(\eta) = \frac{T - T_w}{T_s - T_w}, \quad \eta = \sqrt{\frac{\xi}{v}}
\]

(6)

where, \(\xi\) denotes differentiation with respect to \(\eta\). Here, we have \(m = 1\) when the sheet shrinks in the \(x\)-direction only and \(m = 2\) when the sheet shrinks axisymmetrically. Equation 1 is automatically satisfied.
while substituting 6 into Eq. 2 and 3 reduce the basic equations to the following ordinary differential equations:

\[ \kappa_s f'' + m f' + 1 + f'' = 0 \]  

(7)

\[ \kappa_t \theta' + Prmf' = 0 \]  

(8)

while the boundary conditions Eq. 4 become:

\[ f(0) = s, \quad f'(0) = \lambda, \quad \theta(0) = 1 \]

\[ f'(\eta) \rightarrow 1, \quad \theta(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \]  

(9)

Here, Pr = \nu/\alpha_s is the Prandtl number, s = -w_d/(\omega \nu) is the suction (s > 0) parameter, \lambda = \alpha/c is the stretching (\lambda > 0) or shrinking (\lambda < 0) parameter and \kappa_s and \kappa_t are two constants relating to the properties of the nanofluid which are defined as:

\[ \kappa_s = \frac{1}{(1 - \phi)^2 (1 - \phi + \phi_p/\rho_p)}, \quad \kappa_t = \frac{\kappa_{ns}/\kappa_t}{(1 - \phi) + (\phi (\rho C_p)/(\rho C_p))} \]  

(10)

The physical quantities of interest are the skin friction coefficient \( C_f \) and the local Nusselt number Nu which are given by:

\[ Re_v^{1/2} C_f = -\frac{1}{(1 - \phi)^2} f'(0), \quad Nu = -\frac{k_s}{\kappa_t} \theta'(0) \]  

(11)

where, \( Re_v = u(\nu) x/\nu \) is the local Reynolds number.

Stability analysis: In the earlier study, we have mentioned the existence of dual solutions. In order to determine which of these solutions are physically realizable in the real world applications, we have to perform a stability analysis. This analysis has been performed by many researchers such as Weidman et al. (2006), Harris et al. (2009), Weidman and Sprague (2011) and most recently by Akbar et al. (2017) and Najib et al. (2018). First, we have to consider the unsteady problem. Equation 1 holds while Eq. 2 and 3 become:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \]  

(12)

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_d \frac{\partial^2 T}{\partial z^2} \]  

(13)

where, \( t \) denotes the time. Based on Eq. 6 and following Weidman et al. (2006), we now introduce the following new dimensionless variable:

\[ u = -\frac{\partial f}{\partial \eta}, \quad v = \alpha/(\omega l_y) \frac{\partial f}{\partial \eta}, \quad w = \alpha/(\omega l_y^2) \frac{\partial f}{\partial \eta} \]

\[ \theta = (T - T_\infty)/(T_\infty - T_\infty), \quad \eta = (\omega l_y^2) \]  

(14)

where \( \tau \) is a dimensionless time variable. Substituting Eq. 14 into 12 and 13 yield the following:

\[ \kappa_s \frac{\partial^2 f}{\partial \eta^2} + m \frac{\partial f}{\partial \eta} + 1 + \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \eta^2} \]  

(15)

\[ \kappa_t \frac{1}{Pr} \frac{\partial \theta}{\partial \eta} + m \frac{\partial \theta}{\partial \eta} - \frac{\partial \theta}{\partial \eta} = 0 \]  

(16)

which is subject to the boundary conditions:

\[ f(0, \tau) = s, \quad \frac{\partial f}{\partial \eta}(0, \tau) = \lambda, \quad \theta(0, \tau) = 1 \]

\[ \frac{\partial f}{\partial \eta}(\eta, \tau) \rightarrow 1, \quad \theta(\eta, \tau) \rightarrow 0 \text{ as } \eta \rightarrow \infty \]  

(17)

To determine the stability of the solution \( f = f_0(\eta), \theta = \theta_0(\eta) \) satisfying the boundary value problem Eq. 7-9, we write:

\[ f(\eta, \tau) = f_0(\eta) + e^{-\gamma \tau} f_1(\eta, \tau), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma \tau} \theta_1(\eta, \tau) \]  

(18)

where, \( \gamma \) is an unknown eigenvalue and \( F(\eta, \tau) \) and \( \Theta(\eta, \tau) \) are small relative to \( f_0(\eta) \) and \( \theta_0(\eta) \). Solutions of the eigenvalue problem Eq. 15-17 give an infinite set of eigenvalues \( \gamma_1, \gamma_2, ..., \); if the smallest eigenvalue \( \gamma_1 \) is positive, there is an initial decay which indicates that the flow is stable, however, if \( \gamma_1 \) is negative there is an initial growth of disturbances which indicates that the flow is unstable (Weidman and Sprague (2011)). Substituting Eq. 18 into 15 and 16, we obtain the following linearized problem:

\[ \kappa_s \frac{\partial^2 F}{\partial \eta^2} + m \left( \int f_0 \frac{\partial^2 f}{\partial \eta^2} + f_1^* \frac{\partial^2 f}{\partial \eta^2} \right) \left( 2f_0^* \gamma \right) \frac{\partial F}{\partial \eta} + \frac{\partial^2 F}{\partial \eta^2} = 0 \]  

(19)

\[ \kappa_t \frac{1}{Pr} \frac{\partial G}{\partial \eta} + m \left( \int f_0 \frac{\partial G}{\partial \eta} + F_0 \frac{\partial G}{\partial \eta} \right) + \gamma G \frac{\partial G}{\partial \eta} = 0 \]  

(20)
Subject to the following boundary conditions:

\[ F(0, \tau) = 0, \quad \frac{\partial}{\partial \eta} F(0, \tau) = 0, \quad G(0, \tau) = 0 \]
\[ \frac{\partial}{\partial \eta} F(\eta, \tau) \rightarrow 0, \quad G(\eta, \tau) \rightarrow 0 \text{ as } \eta \rightarrow \infty \] (21)

By setting \( \tau = 0 \), we obtain the solutions \( f(\eta) = f_0(\eta) \) and \( \theta(\eta) = g_0(\eta) \). Hence, \( F = F_0(\eta) \) and \( G = G_0(\eta) \) in Eq. 19 and 20 identify initial growth or decay of the solution Eq. 18. To test our numerical procedure, we have to solve the following linear eigenvalue problem:

\[ \kappa F_0'' + m \left( f_0 F_0' + f_0 G_0' + G_0 F_0' \right) \left( 2f_0' - \eta \right) F_0' = 0 \] (22)

\[ \kappa_2 G_0'' + Pr \left( f_0 G_0' + f_0 G_0' + G_0 G_0' \right) \gamma G_0 = 0 \] (23)

Along with the boundary conditions:

\[ F_0(0) = 0, \quad F_0'(0) = 0, \quad G_0(0) = 0 \]
\[ F_0'(\eta) \rightarrow 0, \quad G_0(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \] (24)

It should be stated that for particular cases of \( \lambda \), \( s \), \( m \) and \( Pr \), the stability of the corresponding steady laminar flow solutions \( f_0(\eta) \) and \( g_0(\eta) \) are determined by the smallest eigenvalue \( \gamma \). Harris et al. (2009) suggested that the range of possible eigenvalues can be determined by relaxing a boundary condition on \( F_0(\eta) \) or \( G_0(\eta) \). Therefore, for the present problem, we relax the condition that \( G_0'(\eta) \rightarrow 0 \) as \( \eta \rightarrow \infty \) for a fixed value of \( \gamma \) we solve the system Eq. 22-23 along with the new boundary condition \( F_0'(0) = 1 \).

**RESULTS AND DISCUSSION**

Numerical solutions to the nonlinear ordinary differential Eq. 7 and 8 along with the boundary conditions Eq. 9 were obtained using the “bvp4c” function in MATLAB Kierzenka and Shampine (2001) for different values of the nanoparticle volume fraction \( \Phi \). Following Khanafar et al. (2003); Tiwari and Das (2007); Abu-Nada and Öztop (2009), we have considered the range of nanoparticle volume fraction as 0 \( \leq \Phi \leq 0.2 \). The Prandtl number \( Pr \) of the base fluid (water) is kept constant at 6.7850 throughout the study. The thermophysical properties of fluid and nanoparticles used in this study are given in Table 2. To verify the accuracy of the present method, the present numerical results for \( \Gamma'(0) \) are compared by Wang (2008) for various values of the stretching/shrinking parameter \( \lambda \) when \( \Phi = 0 \) (regular Newtonian fluid). The comparisons which presented in Table 3 are found to be in very good agreement and thus, we are confident the present results are accurate.

Variation of the reduced skin friction coefficient \( \Gamma'(0) \) and reduced local Nusselt number \( -\Theta'(0) \) with \( \lambda \) for different types of nanoparticles when \( m = 2 \), \( s = 0.5 \) and \( \Phi = 0.1 \) are shown in Fig. 1 and 2, respectively. It seems that there are regions of unique solutions for \( \lambda > 1 \), dual solutions for \( \lambda < \lambda_c < 1 \) and no solution for \( \lambda < \lambda_c \), where, \( \lambda_c \) is the critical value of \( \lambda \) beyond which the boundary layer separates from the surface and the solutions based upon the boundary-layer approximations are not possible. It is important to mention that in this study, the second

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Fig. 1: Variation of \( \Gamma'(0) \) with \( \lambda \) for different types of nanoparticles.

Table 3: Values of \( \Gamma'(0) \) for \( \Phi = 0 \) for Cu nanoparticles with different \( m \) and \( s = 0 \)

Table 4: Values of \( \lambda \) for Cu nanoparticles for several \( m \) and \( s \) when \( \Phi = 0 \)

Table 5: Smallest eigenvalues \( \lambda \) for Cu nanoparticles when \( \Phi = 0.1 \) and different values of \( \lambda \) and \( s \)
Fig. 2: Variation of $-\theta'(0)$ with $\lambda$ for different types of nanoparticles.

Fig. 3: Velocity profiles $\Gamma(\eta)$ for different types of nanoparticles when $m = 1$ and $2$, $s = 0.5$, $\lambda = -1.5$, and $\phi = 0.1$.

Fig. 4: Temperature profiles $\theta(\eta)$ for different types of nanoparticles when $m = 1$ and $2$, $s = 0.5$, $\lambda = -1.5$, and $\phi = 0.1$.

solutions only occur for the shrinking ($\lambda < 0$) case. Several values of $\lambda_c$ for Cu nanoparticles for different $m$ and $s$ when $\phi = 0.1$ are displayed in Table 4 and 5. From Fig. 1, it can be seen that the values of the reduced skin friction coefficient $\Gamma'(0)$ for nanoparticle Ag are higher than Cu while opposite behavior can be observed in Fig. 2, specifically when $\lambda > 1$. From Table 4, we found that the values of $|\lambda|$ increase with the increase of $s$. Hence, the suction parameter widen the range of $\lambda$ for which the solutions exist.

Figure 3 and 4 display the velocity $\Gamma(\eta)$ and temperature profiles $\theta(\eta)$ for both Cu and Ag nanoparticles for $m = 1$ (when the sheet shrinks in the x-direction only) and $m = 2$ (when the sheet shrinks axisymmetrically) when $s = 0.5$, $\lambda = -1.05$ and $\phi = 0.1$. The first (upper branch) solution and second (lower branch) solution are illustrated with solid and dashed lines, respectively. It is seen that the boundary layer thickness of Cu nanoparticles is larger compared to Ag nanoparticles. Further, it can be observed in Fig. 4 that the heat transfer rate at the surface for Ag nanoparticles is higher than Cu. The boundary layer thickness for the second (lower branch) solution is seen to be larger than the first (upper branch) solution in both figures. Both velocity and temperature profiles displayed in Fig. 3 and 4 satisfy the far field boundary conditions asymptotically and thus, support the validity of the dual solutions obtained in this study.

Finally, a stability analysis was performed by solving an unknown eigenvalue $\gamma$ on Eq. 22-23 along with the boundary conditions Eq. 24. The smallest eigenvalues $\gamma$ for some values of $\gamma$ and $s$ are shown in Table 5. From the table, it can be seen that the upper branch solutions have positive eigenvalues $\gamma$ while the lower branch have negative eigenvalues $\gamma$, thus, we conclude that the first (upper branch) solution is stable while the second (lower branch) solution is unstable.

**CONCLUSION**

The problem of boundary layer flow and heat transfer near the stagnation point past a permeable stretching/shrinking sheet in a nanofluid is investigated. The nonlinear ordinary differential equations are solved numerically for 2 types of nanoparticles which are copper (Cu) and silver (Ag) in the base fluid of water with Prandtl number kept constant to Pr = 68750. Results of the skin friction coefficients, local Nusselt numbers as well as the velocity and temperature profiles are presented and discussed for different values.
of the governing parameters. Dual solutions are found for a certain range of shrinking and suction parameters and therefore, a stability analysis has been performed to determine which solution is stable and physically realizable. It can be concluded that the first (upper branch) solution is stable while the second (lower branch) solution is unstable.

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