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Mathematical Modelling of the Stochastic Nonlinear Thermal Processes in Electronic Systems by the Method of Pseudoinverse Matrix

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Abstract: Thermal processes in electronic systems are stochastic and nonlinear. The stochasticity of thermal processes in electronic systems can be caused by both external stochastic environmental factors and internal stochastic factors of the electronic systems. Methods existing in the literature allow modeling thermal processes caused only by external stochastic factors. At the same time, there are no methods that would have the ability to simulate thermal processes, taking into account both nonlinearities and stochasticities of external and internal factors. In this study is developed a method that allows us to simulate stationary stochastic thermal processes in electronic systems taking into account the stochasticity of both external and internal factors and nonlinearity too. The method is based on the concept of a pseudoinverse matrix as well as on the decomposition of a stochastic temperature-dependent matrix of thermal conductivities into a product of two matrices. One of this matrix depends only on temperature and the other matrix depends only on stochastic factors. The application of the proposed method to the design of real electronic systems has shown its adequacy and efficiency sufficient for engineering practice.

Key words: Pseudoinverse matrix, thermal processes, electronic system, nonlinear, stochastic, mathematical model

INTRODUCTION

The development of the multipurpose Electronic Systems (ES) is based on the mathematical and computer modelling of the processes of various physical nature. Moreover, the thermal processes which occur in the operating ES are among the key factors that limit the required operating and reliability parameters of the said ES. Valid and highly-effective mathematical and computer-aided methods for thermal processes modeling are required for the ES to be competitive. Moreover, to meet increasingly strict requirements applicable to modeling validity, it is necessary to develop methods and techniques which are able to ensure a possibility to simulate both nonlinear and stochastic thermal processes occurring in the ES. Mathematical and computer modelling of the thermal processes in the ES exclusively of nonlinear and stochastic factors that define the thermal process results in the development of inappropriate electronic components and ES in terms of their thermal characteristics and consequently in the design errors, the operating parameters that are beyond of the tolerance range, the loss of performance, the reduced reliability and finally, the development of the noncompetitive ES.

The nonlinear character of the thermal processes in the ES is due to the nonlinear relationship between heat exchange and temperature (convective heat flow and emission, thermal conductivity of the materials and environment) (Madera, 2005; Ellison, 2010; Spalding and Taborek, 1983) and due to strong dependency between the power consumption of the electronic components of the ES (Integrated Circuits (IC), Electronic Components (EC)) on temperature (Ellison, 2010; Madera, 2018).

The stochastic behavior of the thermal processes in the ES depends on two groups of factors, i.e., the internal, system-specific factors and the external factors which occur only when the ES is operating and interacting with the environment (Madera and Kandalov, 2016; Chiba, 2012; Wang *et al*., 2015).

The stochastic character of the internal factors occurs due to the statistical technical dispersion of the thermal and electrical parameters of the IC and EC during their manufacturing assembling and installation in the ES. Those factors include the size and the location of the elements in the ES, the thermal resistance of the cases, the value of installation clearances between the elements, contacting inside the ES between the printed board and the installed IC, etc. The external stochastic factors depend, firstly, on the stochastic values of IC and EC power consumption and secondly, on the stochastic characteristics of the environment, namely the stochastic values of the ambient temperature, the air flow rate, the temperature of a cooling liquid at the input of the ES, the humidity and the pressure in the environment.

The stochastic factors, both internal and external, defining the thermal processes in the ES are of interval-stochastic character (Madera and Kandalov, 2016), meaning that a stochastic factor under the consideration is a random variable, obeying a certain probability law within an interval of its values. It is often assumed in the engineering practice of the thermal design of the ES that the interval-stochastic factors are distributed within the interval of their values under the uniform law. The interval-stochastic character defining the thermal processes in the ES results in the interval-stochastic character of the temperature distributions in the ES. The values of the possible limits of the intervals for the changing stochastic factors depend both on the adopted technology for ES manufacturing and installation and on the degree of final testing and screening of the IC and the EC used in the ES. The interval-stochastic character of the factors that define the thermal processes is unavoidable and can be found whatever processes are applied in the course of production assembly or installation of the electronic or design elements in the ES.

The available literature examines the methods that consider either the nonlinearity of the thermal processes or their interval-stochastic character which caused solely by the external stochastic factors (Madera, 2018; Chantasiriwan, 2006; Saleh *et al*., 2007; Srivastava, 2005; Stefanou, 2009; Chiba, 2012). At the same time, only a few papers examine the methods of modeling of the interval-stochastic and nonlinear thermal processes affected both by the external and internal stochastic factors (Madera and Kandalov, 2016; Adomian, 1983; Keller and Antonetti, 1979). This is preliminary due to the fact that the mathematical models for the stochastic thermal processes which are conditioned by the internal factors are much more complicated than those models which consider only external stochastic factors. Although, the methods suggested in studies by Adomian (1983), Madera (2005), Madera and Kandalov (2016) and Rubinstein (2016) which utilize the Stochastic Green's Function, the stochastic operator, the stochastic inverse matrix, the decomposition of the nonlinearities with the Taylor's series, the Monte Carlo method, allows for the examination of the stochastic processes affected by the external and internal stochastic factors, a part of them is very time and memory consuming (e.g., the Monte Carlo method). Though, their application in simple and one-dimensional systems is limited. The other methods have a small validity range due to their specific features and the rest provide low accuracy (Chiba, 2012).

This study offers a method for mathematical modeling of the stationary thermal processes in the ES which involves the nonlinear interval-stochastic character of the thermal processes affected simultaneously by the internal and external stochastic factors. The method is based on the pseudoinverse matrix (Gantmacher, 2010; Horn and Johnson, 2013; Penrose, 1955) and the method for modeling of the interval-stochastic thermal processes with the interval and statistical measures (Madera and Kandalov, 2016). The nonlinearities in the original equations of the mathematical model are solved using Taylor's series expansion with the deduction of the elements of the first order of smallness. Combined modeling of the nonlinearities and the stochastic internal and external factors is based on the decomposition of the matrix of the ES thermal conductivities into the product of two matrices, namely the temperature-dependent matrix and the matrix with only stochastic elements. The resulting method is universal and allows for modeling of the stationary, nonlinear, interval-stochastic thermal processes in the complex ES with accuracy sufficient for the engineering practice of the thermal design of ES.

MATERIALS AND METHODS

Stochastic nonlinear mathematical model: The thermal model of the ES represents a system of N isothermal elements (Madera, 2005; Madera and Kandalov, 2016) which is presented as a graph, containing N+1 nodes (nodes from 1 to N-1 correspond to the ES elements, node N corresponds to the ES case, node N+1 correspond to the environment inside the ES), M paths, N-1 independent sources of the heat flows ($\bullet_i$, i = 1, 2, …, N-1) and two nodes that simulate the initially known temperature of the environment T_e and that of the air flow, coming at the ES input from a premise with T_e ambient temperature.

The mathematical model, describing the static nonlinear and stochastic thermal processes which is presented as the matrix of the stochastic temperatures $T_i(\bullet)$ where, i = 1, 2, …, N+1 means isothermal elements of the thermal model, for each • • • (where, • is the simple events from the sample space •), appears as follows (Madera, 2005):

$$AG(T,\omega)A^T.T(\omega)=\Phi(T,\omega)+AG(T,\omega).T_a(\omega),\omega\in\Omega \qquad (1)$$

where, $T(\bullet) = (T_1(\bullet), T_2(\bullet), ..., T_{N+1}(\bullet))^T$-N+1 is the vector of the stochastic unknown temperatures in the nodes of the thermal model; A(N+1)×M is the incident matrix of the thermal mode graph; $G(T, \bullet) = diag(g_1(T, \bullet).g_2(T, \bullet), ..., g_M(T, \bullet))$ is the diagonal nonlinear stochastic M×M matrix for the known stochastic temperature-dependent functions of the path's thermal conductivity $g_k(T, \bullet)$, k = 1, 2, ..., M; $\bullet(T, \bullet) = (\bullet_1(T_1, \bullet), \bullet_2(T_2, \bullet), ..., \bullet_{N-1}(T_{N-1}, \bullet),$

$0, 0)^T$-N+1 is the vector of the known stochastic temperature-dependent functions of the heat sources (IC and EC power consumptions) $\bullet_i(T_i, \bullet)$, i = 1, 2, ..., N-1; $T_\bullet(\bullet) = (0, 0, ..., T_e(\bullet), T_e(\bullet))^T$ is the M-vector of the known stochastic temperatures $T_e(\bullet)$ of the external environment and the flow fed into the ES with the known mathematical expectation $\bar{T}_e$ and dispersion D_{T_e} ;• is the space of the simple events • on the probabilistic space {• , U, P}, U is the •-algebra of the subset • ; P is the probability per U (Feller, 1968); $(*)^T$ is the transposition operation.

In general, the power of the heat sources $\bullet_i(T_i, \bullet)$, i = 1, 2, ..., N-1 depends on the temperature T_i in the ith node of the thermal model graph which is explained by the effect of the thermal feedback in the IC and the EC (Madera, 2018). Madera (2018), the effect of the thermal feedback in the ith active element (the IC and the EC) can be simulated by connecting two new model elements to i node of the graph of the thermal model, namely the thermal conductivity $g_i(T_i, \bullet)$, i = 1, 2, ..., M which temperature dependence matches to that of the power of the ith active element $\bullet_i(T_i, \bullet)$ and the heat source with $\bullet_i(\bullet)$, capacity independent of the temperature. The introduction to the thermal model graph of the new model elements, simulating the thermal feedback does not change the structure of the mathematical model (1) and the form of the diagonal matrix of the thermal conductivities G(T, •).

Equation 1 can be significantly simplified, if the temperatures in the nodes of the thermal model $T_i(\bullet)$ will be reckoned from the temperature of the environment $T_e(\bullet)$. By denoting the excess of the temperature in the nodes of the thermal model graph over the temperature on the environment as $\bullet_i(\bullet) = T_i(\bullet)-T_e(\bullet)$, i = 1, 2, …, N+1 (Madera, 2010; Ellison, 2011) and making use the expression $A^T\bullet(\bullet) = A^TT(\bullet)-T_\bullet(\bullet)$, instead of Eq. 1 we will obtain the equation expressed in terms of temperature rise vector $\bullet(\bullet) = T(\bullet)-T_a(\bullet) = (\bullet_1(\bullet), \bullet_2(\bullet), ..., \bullet_{N+1}(\bullet))^T$:

$$AG(\theta,\omega)A^T.\theta(\omega) = \Phi(\omega), \omega \in \Omega \qquad (2)$$

Let's consider the structure of the thermal conductivities $g_k(\bullet, \bullet)$, k = 1, 2, ..., M in the matrix G(• , •) of Eq. 2. The thermal conductivities $g_k(\bullet, \bullet)$ can be caused by various types of heat transfer, namely the thermal convection occurred between the ES elements and the internal or external ES environment, thermal radiation in the ES elements and in the elements and the environment as well as by the thermal conduction between the contacting elements. In all cases, the heat flow J_{ij} between i and j elements with $\bullet_i$ and $\bullet_j$ temperatures which transfer the heat between themselves and the environment is described by Newton's law (Madera, 2005; Ellison, 2010; Spalding and Taborek, 1983) $J_{ij}^{cond, conv, rad} = g_{ij}^{cond, conv, rad}(\theta_i - \theta_j)$ where, $g_{ij}^{cond, conv, rad}$ is the thermal conductivity between two elements or between an element and the environment occurred as a result of thermal conduction, convection and radiation. In turn, the thermal conductivity $g_{ij}^{cond, conv, rad}$ is expressed as $g_{ij}^{cond, conv, rad} = \alpha_{ij}^{cond, conv, rad} S_{ij}$ where, $\alpha_{ij}^{cond, conv, rad}$ is the heat-exchange coefficient between two elements related to thermal conduction, convection and radiation and S_{ij} is the effective surface area of two elements i and j which exchange the heat between each other or the contact surface area of two elements i and j in case of conduction.

Generally, speaking, the heat-exchange coefficients related to convection, $\bullet^{conv}_{ij}$, radiation $\bullet^{rad}_{ij}$ and conduction $\bullet^{cond}_{ij}$ depend on the temperature of i and j elements are stochastic in nature and can be expressed as follows: for natural convection $\bullet^{conv}_{ij}(\bullet_i, \bullet_j, \bullet) = A^{conv}_{ij}(\bullet).(\bullet_i-\bullet_j)^n$, for thermal radiation $\bullet^{rad}_{ij}(\bullet_i, \bullet_j, \bullet) = A^{rad}_{ij}(\bullet).(\bullet^4_i-\bullet^4_j)^n/(\bullet_i-\bullet_j)$, for thermal conduction $\bullet^{cond}_{ij}(\bullet) = \bullet_{ij}(\bullet)/\bullet_{ij}(\bullet)$ where, $A^{conv}_{ij}(\bullet)$, A^{rad}_{ij} are the stochastic variables, n is the index of power in the heat exchange law, $\bullet_{ij}(\bullet)$ and $\bullet_{ij}(\bullet)$ are the stochastic variables, describing the conductivity and the material thickness in the contact layer between elements (Madera, 2005; Madera and Kandalov, 2016; Ellison, 2010; Spalding and Taborek, 1983).

It's also worth considering that the thermal processes in the IC depend on the IC temperature and are of stochastic character. Actually, the thermal resistance of the IC case, i.e., junction-to-case thermal resistance $R_{JC}(\bullet)$ and the case temperature $\bullet_C(\bullet)$ are stochastic in nature while the IC power consumption $\bullet(\bullet, \bullet)$ is stochastic and temperature-dependent. Therefore, the temperature of the IC chip $\bullet_J(\bullet)$ will also be stochastic and will be defined by the equation $\bullet(\bullet_J, \bullet) = g_{JC}(\bullet)(\bullet_J(\bullet)-\bullet_C(\bullet))$ where $g_{JC}(\bullet) = R^{-1}_{JC}(\bullet)$ is the stochastic thermal conductivity of the IC case.

Hence, it follows that the thermal conductivities $g_k(\bullet, \bullet)$ in the diagonal matrix of the thermal conductivities G(• , •) can be normally presented as the product of two conductivities $g_k(\bullet, \bullet) = g_{k,s}(\bullet).g_{k,\bullet}(\bullet(\bullet))$, k = 1, 2, ..., M, one of which namely the conductivity $g_{k,s}(\bullet, \bullet)$ is solely stochastic while another one $g_{k,\bullet}(\bullet(\bullet))$ depends solely on the stochastic temperature • (•). In this case, the diagonal matrix of the conductivities G(• , •) can be also presented as the product of two diagonal matrices $G_s(\bullet)$ and $G_\bullet(\bullet(\bullet))$ as follows:

$$G(\theta, \omega) = G_s(\omega).G_\theta(\theta(\omega)), \omega \in \Omega \qquad (3)$$

where, $G_s(\bullet) = diag(g_{1,s}(\bullet).\ g_{2,s}(\bullet), ..., g_{M,s}(\bullet))$ is the stochastic diagonal M×M matrix with the stochastic temperature-dependent elements; G.(• (•) = diag($g_{1,}$. ((• •), $g_{2,}$.(•), ..., $g_{M,\ s}$(•)) is the diagonal M×M matrix depending only on temperature. The matrix G(•, •) presented as the product (3) of two matrices G_s(•) and G.(• (•)) allows deriving the following nonlinear stochastic matrix Eq. 2 (• ••):

$$AG_s(\omega)G_\theta(\theta(\omega))A^T\theta(\omega)=\Phi(\omega),\omega\in\Omega \quad (4)$$

The solution of (Eq. 4) which is both nonlinear and stochastic, comes across the insuperable difficulties. Meanwhile, the matrix AG_s(•)G.(• (•))A^T in the set of (Eq. 4) has a specific structure. Namely (N+1)×M matrix of A incidences of the thermal model graph consists only of elements 0, 1, -1 while the matrix of the thermal conductivities G(•, •) in (Eq. 3) is the diagonal matrix and is the product of the diagonal matrices. As it is shown below, the structure of (Eq. 4) allows for the application of the pseudoinverse matrix concept (Gantmacher, 2010; Horn and Johnson, 2013; Penrose, 1955) to search for the stochastic solution of equation . The decomposition (3) of the stochastic matrix G(•, •) into the product of two diagonal matrices G_s(•) and G.(• (•)) allows for the separation of the stochastic and temperature-dependent variables.

Method for defining the statistical measures of the stochastic temperatures based on the pseudoinverse matrix concept: As is known, any stochastic process is completely described by its probability distribution laws of all possible orders at any specific time. However, it is impossible to define them for, so, complicated equations. At the same time, the engineering practice of the thermal design of the ES does not require the knowledge of probability distribution laws (Pugachev, 1984; Feller, 1968) as it is sufficient to be able to find the statistical measures of the stochastic temperatures, $•_i$(•), i = 1, 2, ..., N+1 with a degree of accuracy required for practical needs, namely:

Mathematical expectations $\bar{\theta}_i = E\{\theta_i(\omega)\}$ where, E{·} is the expectation operator; dispersions $D_{\theta i} = E\left\{\left(\overset{0}{\theta}_i(\omega)\right)^2\right\}$ where, $\overset{0}{\theta}_i(\omega)=\theta_i(\omega)\text{-}\bar{\theta}_i(t)$ is the centered stochastic temperature difference with zero expectation $E\left\{\overset{0}{\theta}(\omega)\right\}=0$ and mean-square deviation $\sigma_{\theta i}=\sqrt{D_{\theta i}}$; covariances $K_{\theta i,\theta j}=E\left\{\overset{0}{\theta}_i(\omega)\overset{0}{\theta}_i(\omega)\right\}$ between the stochastic temperatures $•_i$ = $•_i$(•)• $•_j$ = $•_j$(•) of i and j elements (i, j = 1, 2, ..., N+1) of the thermal model.

The statistical measures $\bar{\theta}_i$, $D_{•i}$, $•_{•i}$, $K_{•i,•j}$, i, j = 1, 2, ..., N+1, completely define the intervals [$•_{Bot,\ i}$, $•_{Up,\ i}$] for measuring the real temperature values of the ES elements, occurring during the operation. The lower $•_{Bot\ i}$ and upper limits $•_{Up,i}$ of the interval [$•_{Bot,i}$, $•_{Up,i}$] are defined as $•_{Bot,i}$ = $\bar{\theta}_i$-• ·$•_{•i}$ and $•_{Up,i}$ = $\bar{\theta}_i$-• ·$•_{•i}$ where • is the coefficient which is defined in accordance with the Chebyshev's $tyP\left\{\overset{0}{\theta}_i(\omega)\le•\cdot\sigma_{\theta i}(t)\right\}\ge 1\text{-}1/•^2$ inequality (Madera and Kandalov, 2016; Madera, 2017).

To identify the sough statistical measures of the stochastic temperatures of the ES and their variation intervals, it is necessary to have the equations, describing the sough statistical measures.

Determination of a stochastic solution of the matrix stochastic equation using the concept of a pseudoinverse matrix: The method which is developed in this study is based on the pseudoinverse matrix concept, decomposition (3) of the conductivity matrix and the method of deriving the equations for the statistical measures developed in studies (Madera and Kandalov, 2016). The pseudoinverse matrix, also known as the Moore-Penrose generalized inverse matrix (Gantmacher, 2010; Horn and Johnson, 2013; Penrose, 1955) has the following essence.

If the matrix equation Ax = y contains the square and non-singular matrix $A = \{a_{ij}\}^n_1$, then the said matrix has the inverse matrix A^{-1} and the unique solution $x = A^{-1}y$. If the matrix A is square but singular or $A = \{a_{ij}\}_{n\times m}$ is the rectangular n×m matrix (where n and m is the number of rows and columns), then the matrix A has no inverse matrix, however, there can be created the only one matrix A^+ which is called the pseudoinverse matrix that makes it possible to find the best approximate solution $x^0 = A^+y$, $x^0 = (x^0_1, x^0_2, ..., x^0_m)^T$ of the equation Ax = y in terms of the minimum squared residual norm (Euclidian norm), i.e.:

$$\min_x\|y\text{-}Ax\|^2 = \min_x\sum_{i=1}^{n}\left|y_i\text{-}\sum_{j=1}^{m}a_{ij}x_j\right|^2$$

Attained when $x = x^0$ while the vector of the best approximate solution x^0 has the least length, i.e., $\|x\|^2 = x^T{\cdot}x$ = min where, $x^T{\cdot}x$ is the scalar product of the vector x. It's worth noting that if the matrix A is square and non-singular, then its inverse A^{-1} coincides with the pseudoinverse matrix A^+. The random rectangular n×m matrix A of r = min {n, m} rank can be always presented as the product of two rectangular matrices A = BC , namely n×r matrix B and r×m matrix C, also known as the skeleton decomposition of the matrix A (Gantmacher, 2010; Horn and Johnson, 2013). As a result, the pseudoinverse matrix A^+ will be described as (Gantmacher, 2010) $A^+ = C^+B^+$ where, $C^+ = (CC^T)^{-1}$ and $B^+ = (B^TB)^{-1}B^T$. Moreover, despite the fact that the skeleton decomposition A = BC does not explicitly define the terms B and C, the expression $A^+ = C^+B^+$ defines the only one pseudoinverse matrix with any skeleton decompositions.

Let us use the concept of a pseudoinverse matrix in order to obtain the equation, defining the statistical masures of the stochastic temperature vector •(•) using the method that has been developed by Madera and Kandalov (2016). Let's multiply both parts of (Eq. 4) by the transposed A^T incident matrix. We will obtain the following:

$$A^T A G_s(\omega) G_\theta(\theta(\omega)) A^T \theta(\omega) = A^T \Phi(\omega),\ \omega \in \Omega \quad (5)$$

The matrix $B = A^T A$ is square and singular and has no inverse matrix. However, its pseudoinverse matrix can be created and it is possible to obtain the best approximate solution of (Eq. 5) as follows:

$$G_s(\omega) G_\theta(\theta(\omega)) A^T \theta(\omega) = B^+ A^T \Phi(\omega),\ \omega \in \Omega \quad (6)$$

taken in terms of the minimum squared residual norm for each • •• . To obtain an explicit expression in order to define the pseudoinverse matrix B^+, let's consider the square singular matrix $B = A^T A$ which is the product of two rectangular matrices, namely M×(N+1) matrix A^T and (N+1)×M matrix A. As the number of nodes does not exceed the number of paths in the thermal model graph of the real ES, then the condition N+1• M will be always met for the A incident matrix, meaning that the rank r of B matrix meets the relationship r• M and $A^T A$ product is the skeleton decomposition of B matrix. Then, B matrix will have the only one pseudoinverse B^+ matrix defined as follows:

$$B^+ = A^T \left(AA^T\right)^{-1} \left(AA^T\right)^{-1} A \quad (7)$$

By multiplying the left part of Eq. 6 by the inverse matrix G^{-1}_s(•), we will obtain:

$$G_\theta(\theta(\omega)) A^T \theta(\omega) = G_s^{-1}(\omega) B^+ A^T \Phi(\omega) \quad (8)$$

where, G^{-1}_s(•) = diag($g^{-1}_{1,s}$(•), $g^{-1}_{2,s}$(•), ..., $g^{-1}_{M,s}$(•)) is the inverse diagonal matrix stochastic M×M matrix. It's worth noting that as the matrix G^{-1}_s(•) is diagonal, its elements can be found in an explicit form as $g^{-1}_{k,s}$(•) where the elements $g_{k,s}$(•), k = 1, 2, ..., M, of the stochastic matrix G_s(•) are the known source data. Moreover, the elements $g_{k,s}$(•) and $g^{-1}_{k,s}$(•), k = 1, 2, ..., M, of the matrices G_s(•) and G^{-1}_s(•) as well as the elements $•_i$(•), i = 1, 2, ..., N+1, of the stochastic power vector •(•) are statistically independent of each other.

In stochastic matrix (Eq. 8) all nonlinearity is centralized in the left part and the stochastic property is centralized in the right part which allows considering the nonlinearity and the stochastic property in Eq. 9 separately and independently of each other.

To obtain the equations with respect to the statistical measures, let's linearize the nonlinear matrix G.(•(•)) in Eq. 8 by using Taylor's series expansion method with the deduction of the elements of the first order of smallness developed in study (Madera and Kandalov, 2016; Madera, 2017). As a result, we will obtain the matrix stochastic equation with respect to the centered temperature N+1-vector $\overset{0}{\theta}(\omega) = \theta(\omega) - \bar{\theta}$ ($\bar{\theta}$ is the vector of the mathematical expectations of the temperatures):

$$G_\theta(\bar{\theta}) A^T \bar{\theta} + V(\bar{\theta}) \overset{0}{\theta}(\omega) = G_s^{-1}(\omega) B^+ A^T \Phi(\omega), \omega \in \Omega \quad (9)$$

where, $\overset{0}{\theta}(\omega) = \left(\overset{0}{\theta}_1(\omega), \overset{0}{\theta}_2(\omega), ..., \overset{0}{\theta}_{N+1}(\omega)\right)^T$ N+1 is the vector of the stochastic centered temperatures in the nodes of the thermal model; $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_{N+1})^T$ N+1 is the vector of the mathematical expectations of the temperatures in the nodes of the thermal model; $G_\theta(\bar{\theta}) = \text{diag}\left(g_{1,\theta}(\bar{\theta}), g_{2,\theta}(\bar{\theta}), ..., g_{M,\theta}(\bar{\theta})\right)$ is the diagonal M×M matrix with the elements that depend on the mathematical expectations of the temperatures $\bar{\theta}$; $V(\bar{\theta}) = (\partial J_i / \partial \theta_j)_{\bar{\theta}} - M \times (N+1)$ is the Jacobian matrix, i = 1, 2, ..., M, j = 1, 2, ..., N+1 which elements are equal to the partial derivatives of the heat flows J_i with respect to the temperatures $•_j$, being taken at the values equal to the mathematical expectations $\bar{\theta}$; J_i are the elements of the M vector of J = G.(•(•))A^T•(•), equal to the heat flows in the paths i = 1, 2, ..., M of the thermal model graph of the ES.

The assessment of the error occurred in the course of linearization of the heat flows shows the following (Madera and Kandalov, 2016). For the heat flow occurred as a result of natural convection $J^{con•}_{ab}$ between two elements a and b or the element a and the liquid b with the temperatures T_a and T_b, we will have the following relationship J^{conv}_{ab} = $•^{con•}_{ad} S_{ab}(T_a-T_b)$• • T^{n+1},0<n• 1 where, • T = T_a-T_b. It is possible to show that the relative deviation $\varepsilon = \left|\Delta \overset{0}{T} / \Delta \bar{T}\right|$ of the centered temperature difference $\Delta \overset{0}{T}$ from its mathematical expectation $\Delta \bar{T}$ satisfies the inequality $\varepsilon \le \sqrt{2\delta_0 / n(n+1)}$ where, $•_0$ is the relative error occurred as a result of replacement of the convective flow J^{conv}_{ab} with its approximate value during Taylor's series expansion with the elements of the first order of smallness remained in place. For example, if the relative error of linearization of the convective heat flow is $•_0$• 5%, then the tolerable relative deviation will make • 80% for natural convection, obeying 1/8 power law, • 57% in case of 1/4 power law, •• 80% in case of 1/3 power law. In absolute figures, this means that, for example, under 1/4 power law and $\Delta \bar{T}$ = 40°C the tolerable change in the centered difference of the temperatures will make • T• 23%°C in case of convection.

The assessment of the linearization error for thermal radiation allows for the following conclusions to be made. Thermal radiation flow J^{rad}_{ab} between two elements a and b with the temperatures T_a and T_b is described as follows J^{rad}_{ab} = $\bullet^{rad}_{ab}S_{ab}(T_a\text{-}T_b)\bullet T^4_a$ $\text{-}T^4_b$. If $\bullet_0$ is the relative linearization error, then it is possible to demonstrate that the relative deviation $\varepsilon = \left|\overset{0}{\Delta T}/\Delta\bar{T}\right|$ of the centered temperature $\overset{0}{T} = T\text{-}\bar{T}$ with respect to its mathematical expectation $\bar{T}$ will meet the condition $\leq\sqrt{\delta_0/6}$. For example, in case of the linearization error for the thermal radiation flow $\bullet\bullet$ 50%, the tolerable relative deviation • will be 9%. In other words, if the absolute temperature of the element is $\bar{T}$ = 400 K (maximum value for the ES), then, the tolerable range of a random deviation of the element's temperature shall not exceed 36°C in absolute units.

The analysis of the magnitude of the errors occurred as a result of linearization of the heat flows with Taylor's series while retaining the elements of the first order of smallness allows for the conclusion that the applied method makes it possible to model the thermal processes in the course of ES thermal design with accuracy sufficient for the engineering practice.

Equations for the statistical measures of the stochastic temperatures: Stochastic matrix (Eq. 9) is a linear equation with respect to the vector of the centered temperatures $\overset{0}{\theta}(\omega)$ but is nonlinear with respect to the vector of the mathematical expectations of the temperatures $\bar{\theta}$. That being said, the right part of (Eq. 9) contains only stochastic terms while the left side contains only temperature-dependent nonlinear terms. This allows obtaining the matrix equations for the statistical measures of the stochastic temperature vector $\overset{0}{\theta}(\omega)$, namely for the vector of mathematical expectations $\bar{\theta}$ and the covariance matrix K••.

To derive the equation with respect to the vector of the mathematical expectations $\bar{\theta} = (\bar{\theta}_1(\omega), \bar{\theta}_2(\omega), ..., \bar{\theta}_{N+1}(\omega))^T$ of the stochastic temperatures, let's apply the expectation operator to (Eq. 9) and use statistic independence of the elements in the stochastic matrix $G^{-1}_s(\bullet)$ and the stochastic vector • (•) of each other. We will get:

$$G_\theta(\bar{\theta})A^T\bar{\theta} = \bar{G}^{-1}_s(\omega)B^+A^T\bar{\Phi} \tag{10}$$

where, $\bar{G}^{-1}_s = \text{diag}(\bar{g}^{-1}_{1,s}, \bar{g}^{-1}_{2,s}, ..., \bar{g}^{-1}_{M,s})$ is the inverse diagonal M×M matrix, depending on the mathematical expectations of the thermal conductivities defining explicitly.

Matrix (Eq. 10) is nonlinear with respect to an unknown vector of the mathematical expectations of the temperatures $\bar{\theta}$ and it can be solved using the known numerical techniques (Rheinboldt, 1998). It's worth noting that the inverse matrix $G^{-1}_\theta(\bar{\theta})$ is diagonal one and is defined explicitly, i.e., $G^{-1}_\theta(\bar{\theta}) = \text{diag}(g^{-1}_{1,s}(\bar{\theta}), g^{-1}_{2,s}(\bar{\theta}), ..., g^{-1}_{M,s}(\bar{\theta}))$. To define the covariance matrix $K_{\theta\theta} = E\left\{\overset{0}{\theta}(\omega)\overset{0}{\theta}{}^T(\omega)\right\}$, let's deduct (Eq. 10) from (Eq. 9), we will obtain the following:

$$V(\bar{\theta})\overset{0}{\theta}(\omega) = \overset{0}{W}(\omega)$$

where, $\overset{0}{W}(\omega) = W(\omega)\text{-}\bar{W} = G^{-1}_s(\omega)B^+A^T\Phi(\omega)\text{-}\bar{G}^{-1}_sB^+A^T\bar{\Phi}$ is the stochastic centered vector with zero mathematical expectation. By multiplying the right part of the equation $V(\bar{\theta})\overset{0}{\theta}(\omega) = \overset{0}{W}(\omega)$ by the transposed equation $\overset{0}{\theta}{}^T(\omega)V^T(\bar{\theta}) = \overset{0}{W}{}^T(\omega)$ and applying the product to the definition of the covariance matrix $K_{\theta\theta} = E\left\{\overset{0}{\theta}(\omega)\overset{0}{\theta}{}^T(\omega)\right\}$, we will obtain the matrix expression to define the covariance matrix of the stochastic temperature vector • (•):

$$V(\bar{\theta})K_{\theta\theta}V^T(\bar{\theta}) = K_{ww}$$

where, $K_{WW} = E\left\{\overset{0}{W}(\omega)\overset{0}{W}{}^T(\omega)\right\}$ is the covariance (N+1)×(N+1) matrix of the stochastic centered N+1 vector $\overset{0}{W}(\omega)$. The resulting expression can be presented in a more convenient form, i.e. as the matrices $AV(\bar{\theta})$ and $V^T(\bar{\theta})A^T$ are square and nondegenerate, they have inverse matrixes. Therefore, we are able to obtain the explicit expression to define the covariance matrix K••:

$$K_{\theta\theta} = \left(AV(\bar{\theta})\right)^{-1}K_{ww}\left(V^T(\bar{\theta})A^T\right)^{-1} \tag{11}$$

In this way, (Eq. 10) defines the vector of mathematical expectations $\bar{\theta}$ and (Eq. 11) defines the covariance matrix K••. Solving (Eq. 10) and further calculations using (Eq. 11) allows defining mathematical expectations $\bar{\theta}_i$, dispersions $D_{\bullet i}$ and mean-square deviations $\sigma_{\theta i} = \sqrt{D_{\theta i}}$ of the stochastic temperatures $\bullet_i(\bullet)$, i = 1, 2, ..., N+1 which are then used to calculate the intervals $\left[\theta_{Bot,i}, \theta_{Up,i}\right] = \left[\bar{\theta}\text{-}\bullet\cdot\sigma_{\theta i}, \bar{\theta}\text{+}\bullet\cdot\sigma_{\theta i}\right]$ for the changing real values of the temperatures of the ES elements.

Equation 10 and 11 have been obtained to define the statistical measures $\bar{\theta}$ and K•• of the stochastic vector of the excess in the temperatures against the ambient temperature, i.e., for the vector $\bullet(\bullet) = (\bullet_1(\bullet), \bullet_2(\bullet), ..., \bullet_{N+1}(\bullet))^T$. However, we are seeking for the absolute stochastic temperatures $T(\bullet) = (T_1(\bullet), T_2(\bullet), ..., T_{N+1}(\bullet))^T$ and their statistical measures $\bar{T} = E\{T(\omega)\}$ and $K_{TT} = E\left\{\overset{0}{T}(\omega)\overset{0}{T}{}^T(\omega)\right\}$. The temperatures $\bullet_i(\bullet)$ and $T_i(\bullet)$ in the nodes of the thermal model i = 1, 2, ..., N+1 are interrelated via. the expression $\bullet_i(\bullet) = T_i(\bullet)\text{-}T_e(\bullet)$ which can be used to find the expressions to calculate the statistical measures $\bar{T}_i$ and D_{ti}, defining with the above mentioned statistical measures $\bar{\theta}_i$ and $D_{\bullet i}$ namely, $\bar{T}_i(\omega) = \bar{\theta}_i + \bar{T}_e$, $D_{Ti} = D_{\theta i} + D_{Te}$.

RESULTS AND DISCUSSION

Application of method and discussion: Let's consider a specific example of the method using the ES (Fig. 1a) which is an Electronic Module (EM) with three Integrated Circuits (IC) of different types mounted on a Multilayer Printed Board (MPB). The energy consumed by the ICs heats the chip and the body of the ICs. At the same time, the ICs mounted on the same MPB interact with each other creating the conductive heat transfer and with the environment creating the convection heat transfer. A part of the heat flow is released by the heat-releasing surface of the IC body into the environment with convection. Another part of the thermal flow is transferred from the IC body to the MPB with conduction both through the air slot between the IC and the MPB and the IC outputs welded to the MPB. The thermal flows that come from the

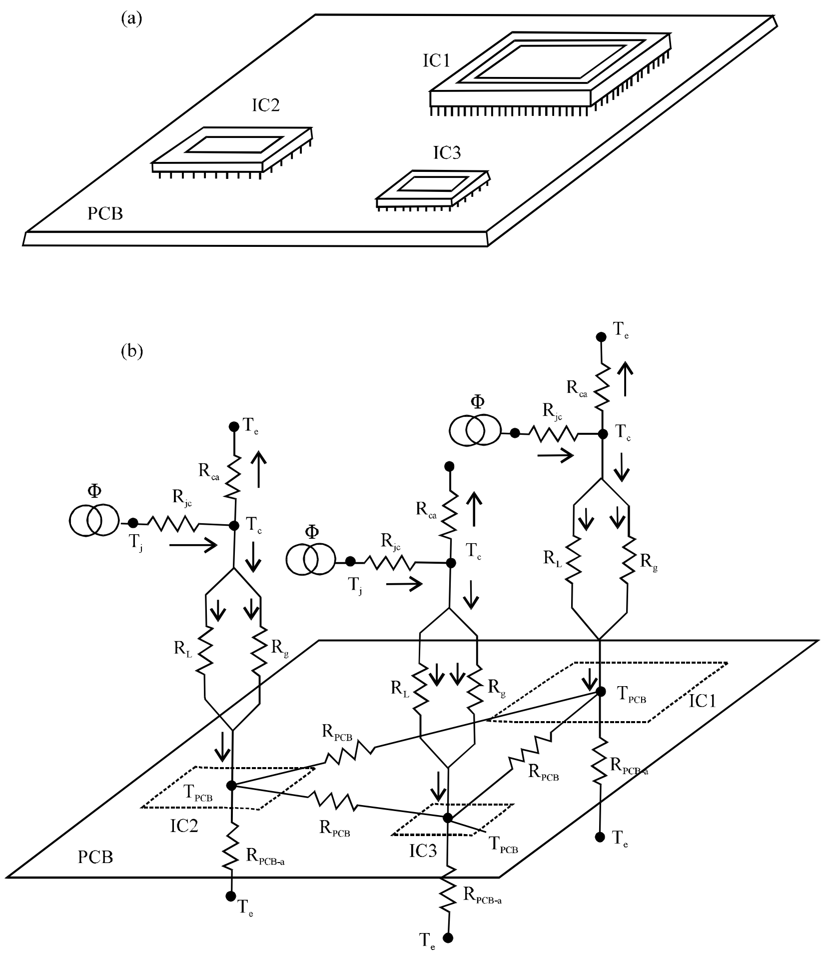

Fig. 1: Electronic system (a) which includes three integrated circuits IC1-IC3, soldered to a multilayer PCB circuit board and the graph of ES thermal Model (b) Designations: • is power consumption IC; R_{jc} is thermal resistance of the hull MC; R_g is the thermal resistance of the gap between the IC case and the PCB; R_L is thermal resistance of the leads of case of IC; R_{ca} is thermal resistance from the surface of the IC case to the environment; R_{PCB} is thermal resistance of conductive heat exchange on the PCB between two neighboring IC; $R_{PCB\text{-}a}$ is thermal resistance from the external surface of the PCB in the area of IC placement into the environment; T_j, T_c, T_{PCB}, T_e are the temperatures of the IC chip, the IC case, the PCB in the area of the IC location, the environment, respectively

MPB are distributed across the MPB structure, creating the conductive heat transfer between all ICs and MPB and convective heat transfer between the EM surface and the environment.

The EM is a standard interchangeable component that should be replaced completely, if one of the ICs welded to the MPB fails. Despite the fact that all ICs in the EM are identical their external and internal factors which identify the thermal behavior of the EM are of interval stochastic nature which is determined by the statistical technological dispersion of the thermal and electric parameters of the IC during their manufacturing assembling and mounting in the EM.

The heat exchange processes in the ES under the consideration are modeled using the graph of the thermal model with 9 nodes and 18 paths (Fig. 1b). In the EM under the consideration, there are the following interval stochastic factors that determine the heat processes in the EM:

- Consumption capacity of the ith IC, (i = 1, 2, 3), $\bullet_i(\bullet)\bullet[\bullet_{Bot,\ i}, \bullet_{up,\ i}]$ where, $\bullet_{Bot,\ i}$ and $\bullet_{up,\ i}$ are the lower and upper limits of the range of dispersion of the consumption capacity $\bullet_i(\bullet)$
- Thermal resistance of the body of the ith IC (i = 1, 2, 3), $R_{jc,\ i}(\bullet)\bullet[R_{Bot,\ jc,\ i}, R_{Up,\ jc,\ i}]$ where, $R_{Bot,\ jc,\ i}$ and $R_{Bot,\ jc,\ i}$ where are the lower and upper limits of the range of dispersion of the IC body thermal resistance $R_{jc,\ i}(\bullet)$
- Thermal resistance of the air slot $R_{g,\ I}(\bullet)\bullet[R_{Bot,\ g,\ i}, R_{up,\ g,\ i}]$ between the body of the ith IC (i = 1, 2, 3) and the MPB where, $R_{Bot,\ g,\ i}$ and $R_{up,\ g,\ i}$ are the lower and upper limits of the range of dispersion of the thermal resistance of the air slot between the body of the IC and the MPB $R_{g,\ i}(\bullet)$
- Temperature of the environment $T_e(\bullet)\bullet[T_{Bot,e}, T_{up,\ ei}]$ where, $T_{bot,\ e}$ and $T_{up,\ ei}$ are the lower and upper limits of the range of dispersion of the temperature $T(\bullet)$

Determining factors are:

- Thermal resistance of the heat-releasing surface of the body of the ith IC (i = 1, 2, 3) $R_{ca,\ i}$
- Thermal resistance of the body outputs $R_{L,\ i}$ of the ith IC (i = 1, 2, 3) welded to the MPB
- Thermal resistance of the conduction occurred across the MPB body between all ICs $R_{PCB,\ i\text{-}j}$, (i, j = 1, 2, 3)
- Thermal resistance of the convective heat exchange between the MPB surface where the ith IC is mounted (i = 1, 2, 3) and the environment $R_{PCB\text{-}a,\ i}$

The heat exchange processes in the EM under the consideration (Fig. 1a) are modelled using the thermal model graph (Fig. 1b) and are described by the interval stochastic (Eq. 2) where $\bullet(\bullet) = (\bullet_1(\bullet), \bullet_2(\bullet), ..., \bullet_9(\bullet))^T$ is the desired vector of the thermal gradient in the graph's nodes of the thermal model; A is 9×8; matrix of incidences of the thermal model graph; $G\bullet(\bullet) = \text{diag}(g_1\bullet(\bullet), g_2\bullet(\bullet), ..., g_{18}\bullet(\bullet))$ is 18×18 diagonal interval stochastic matrix of the thermal conduction of the paths $g_k\bullet(\bullet) = R^{-1}_k\bullet(\bullet)$, k = 1, 2, ..., 18, representing known temperature-dependent and interval stochastic functions; $\bullet(T, \bullet) = (\bullet_1(\bullet), \bullet_2(\bullet), \bullet_3(\bullet), ..., 0)^T$ is the vector (with the length of 9) of the known stochastic consumption capacities $\bullet_1(\bullet), \bullet_2(\bullet), \bullet_3(\bullet)$, IC1, IC2, IC3.

The thermal conductions $g_{ca,\ I}(\bullet) = R^{-1}_{ca,\ I}(\bullet)$ and $g_{PCB\text{-}a,i}(\bullet) = R^{-1}_{PCB\text{-}a,\ i}(\bullet)$ in the paths of the thermal model graph (Fig. 1b) are determined by the natural convection occurred between the surfaces of the body of the ith IC $(g_{(ca,\ i)})$ and the MPB $(g_{PCB\text{-}a,\ i})$ and the environment, therefore, they are temperature-dependent and are defined by the following expressions (Madera, 2005; Ellison, 2010).

Where $A_{ca,\ i}$, $A_{PCB\text{-}a,\ i}$ are the coefficients determined by the environment with which the convective heat exchange occurs; $S_{ca,\ I}$, $S_{PCB\text{-}a,\ I}$ are the surfaces of the body of the i-the IC (i = 1, 2, 3) and MPB where the i-th IC is mounted where the heat exchange with the environment occurs; n is the power coefficient equal to 1/3, 1/4 or 1/8 depending on the law that describes the convective thermal exchange process, form and spatial orientation of the object (Madera, 2005; Ellison, 2011).

The thermal conductions of the body of the i-th IC $g_{jc,\ i}(\bullet) = R^{-1}_{jc,\ i}(\bullet)$ and the air slot between the body of the i-th IC and the MPB $g_{g,\ i}(\bullet) = R^{-1}_{g,\ I}(\bullet)$, i = 1, 2, 3 are of interval stochastic nature and do not depend on temperature while the thermal conductions $g_{ca,\ I}(\bullet)$ and $g_{PCB\text{-}a,\ i}(\bullet)$ on the contrary are temperature-dependent and determined. Therefore, the diagonal matrix of the thermal conductions $G(\bullet, \bullet)$ in Eq. 2 can be presented as a product of two diagonal matrices $G_s(\bullet)$ and $G_\bullet(\bullet)$, one of $G_s(\bullet)$ which is of interval stochastic nature and does not depend on temperature and another, $G_\bullet(\bullet)$ is determined and temperature-dependent one. In other words, $G(\bullet, \bullet) = G_s(\bullet)\cdot G_\bullet(\bullet), \bullet\bullet\bullet$.

The statistical measures (mean vector $\bar{\theta}$ and complete correlation matrix $K_{\bullet\bullet}$) of the stochastic vector $\bullet(\bullet)$ are determined by matrix (Eq. 10) and (11). The statistical measures of the stochastic temperatures of IC1, IC2, IC3 chips (mathematical expectations $\bar{T}_{i(PIM)}$, dispersions $D_{i(PIM)}$ and mean-square deviations $\bullet_{i(PIM)}$, i = 1, 2, 3) calculated with the Pseudoinverse Matrix (PIM) method described herein at different temperatures of the external environment are shown in Table 1. The statistical measures of the stochastic temperatures of IC1, IC2,

Table 1: The statistical measures $(\bar{T}_i, \sigma_i)$ and intervals of the real temperature values for the IC1-IC3 chips calculated with the PIM and M-C methods at temperatures of the external environment 25, 55 and 85°C

	$\bar{T}_i$ (°C)		• $_i$ (°C)			
IC number	PIM method	M-C method	PIM method	M-C method	Relative error (•, %)	Intervals of the real temperature (°C)
Temperature of the external environment T_e = 25°C						
IC1	80.148	78.537	3.146	3.212	2.3	[70.7-89.6]
IC2	49.269	47.793	2.026	1.806	4.5	[43.2-55.3]
IC3	48.347	46.432	2.403	2.129	5.9	[41.1-55.6]
Temperature of the external environment T_e = 55°C						
IC1	110.148	108.5189	3.146	3.199	1.6	[100.7-119.6]
IC2	79.269	77.78	2.026	1.809	2.8	[73.2-85.3]
IC3	78.347	76.502	2.403	2.149	3.4	[71.1-85.6]
Temperature of the external environment T_e = 85°C						
IC1	140.148	138.590	3.146	3.204	1.2	[130.7-149.6]
IC2	109.269	107.8	2.026	1.805	2.0	[103.2-115.3]
IC3	108.347	106.482	2.403	2.132	2.5	[101.1-115.6]

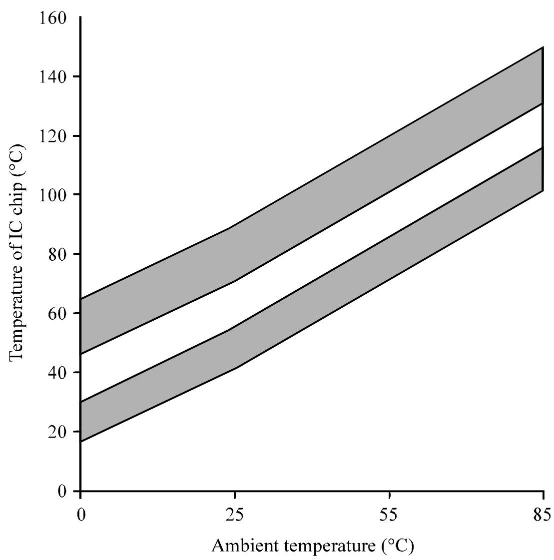

Fig. 2: The intervals which will include the real temperature values for the IC1-C3 chips versus ambient temperatures

IC3 chips $(\bar{T}_{i(M\text{-}C)}, D_{i(M\text{-}C)}, \sigma_{i(M\text{-}C)}, i = 1, 2, 3)$ calculated with the Monte-Carlo method (M-C) with 10 thousand samples are provided in Table 1 for comparison purposes. The relative error (• %) of the results delivered with the pseudoinverse matrix method and the Monte-Carlo method has been evaluated with $\delta,\% = |\bar{T}_{i(M-C)} - \bar{T}_{i(PIM)}|/\bar{T}_{i(M-C)} + 3|\sigma_{i(M-C)} - \sigma_{i(PIM)}|/\bar{T}_{i(M-C)}$ expression. The intervals which will include the real temperature values for the IC1, IC2, IC3 chips in the ES under the consideration, calculated using the pseudoinverse matrix method at different temperatures of the external environment are shown in the last column in Table 1 and Fig. 2.

The comparison of the statistical measures of the IC chip temperatures calculated with the pseudoinverse matrix and Monte-Carlo methods shows that the maximum measure of inaccuracy does not exceed 5.9% which is an admissible measure of inaccuracy for modeling the thermal processes in the ES. Moreover, according to the research studies, the measure of inaccuracy of the statistical measures delivered with the PIM method depends significantly on the condition number of the thermal conduction matrix $x(G) = \|G\| \cdot \|G^{-1}\|$ and decreases with the reduction of x(G). The application of the PIM method also allows for the significant reduction (by several times) in the computing time as compared to the Monte-Carlo method.

CONCLUSION

The method described in this study has proved to be effective in computer-aided and mathematical modeling of the thermal processes in the complex ES. The measure of inaccuracy of the method does not exceed 6% and is sufficient for engineering purposes. The research shows that the measure of inaccuracy of the pseudoinverse matrix method is decreasing gradually with the reduction of the condition number of the thermal conduction matrix G and it takes much less time to compute the results with the said method than to obtain them with the Monte-Carlo method.

The pseudoinverse matrix method is universal and very promising for solving complicated non-linear stochastic matrix equations like $AG(X, \bullet)A^T \cdot (X, \bullet) = Y(\bullet)$, • •• in which G is a diagonal matrix as these equations are used to describe not only the thermal processes in the ES but physical processes as well. This is due to the fact that the matrix equation AGX = Y with G diagonal matrix describes the processes in all graphs like that shown in Fig. 1b, provided that there is a similarity between the processes under the consideration and the electric processes in the electric circuits.

The pseudoinverse matrix method allows deriving an explicit expression for the inverse matrix H^{-1} of the stochastic non-linear matrix $H = AG(X, \bullet)A^T$ which elements depend on the desired X vector, thus, creating its core advantage. In turn, the possibility to represent the stochastic non-linear diagonal matrix G (X, •) as a product of two diagonal matrices $G_s(\bullet)\cdot G_X(X)$ one of which $G_s(\bullet)$ is stochastic and another, $G_X(X)$ depends on the desired X vector, allows to detach the stochastic and desired variables, thus, delivering an explicit solution for the stochastic and nonlinear matrix simultaneous equations $AG(X, \bullet)A^T \cdot X(\bullet) = Y(\bullet)$, • •• .

If the matrix equation $AG(\bullet)A^T \cdot X(\bullet) = Y(\bullet)$, • •• is a linear stochastic equation and the matrix of the system does not depend on the desired X variables, a stochastic solution can be represented in explicit form as follows AG $X(\bullet) = (AA^T)^{-1}AG^{-1}(\bullet)B^+A^TY(\bullet)$ with the pseudoinverse matrix B^+ calculated with expression (7). At that the stochastic measures $\left(\overline{X} = E\{X(\omega)\}, K_{XX} = E\left\{\overset{0}{X}{}^{0T(\omega)}(\omega)X\right\}\right)$ of the stochastic vector X(•) can be easily obtained on the basis of the solution for X(•) above.

If equation $AG(X)A^T \cdot X = Y$ is not a stochastic one but depends on the desired vector X in non-linear form as well the pseudoinverse matrix method also allows to obtain a simple solution for the non-linear equation in the explicit form. This is due to the fact that the matrix $G^{-1}(X)$which is a diagonal matrix of $G(X) = \mathrm{diag}(g_1(X), g_2(X), ..., g_M(X))$ is also diagonal and can be defined in the explicit form, namely as $G^{-1}(X) = \mathrm{diag}\left(g_1^{-1}(X), g_2^{-1}(\overline{\theta}), ..., g_M^{-1}(\overline{\theta})\right)$. This makes it simple to obtain the numerical solutions for the non-linear matrix equations. For example, in case of solution for AG (X) $A^T \cdot X = Y$ obtained with the iteration method, the iteration process with regard to the desired vector X at k and k-1 iteration steps will take the following explicit form $X^{(k)} = (AA^T)^{-1}AG^{-1}(X^{(k-1)})B^+A^TY$. The condition for convergence of the iteration process can be also expressed explicitly as follows $\left\|(AA^T)^{-1}A\frac{\partial G^{-1}(X)}{\partial(X)}B^+A^T\overline{\Phi}\right\| < 1$ where, $\|\cdot\|$ is a matrix norm (Horn and Johnson, 2013). At that, the derivative matrix $\bullet G^{-1}(X)/\bullet X$ will also be diagonal and its explicit expression $\bullet G^{-1}(X)/\bullet X = \mathrm{diag}(\bullet g^{-1}_1(X)/\bullet X, \bullet g^{-1}_2(X)/\bullet X, ..., \bullet g^{-1}_M(X)/\bullet X)$ will allow to simplify the investigation of the convergence of the simultaneous equations.

The developed method has no limitations both in terms of probability distribution functions for the stochastic factors and parameters, determining the thermal processes in the ES and the stochastic characteristics of the environment. The method applied to the stochastic nonlinear thermal process occurred in the ES with three ICs welded to the printed board and thermally interacting with each other and the external environment has proved to be effective and is characterized by the accuracy sufficient for practical application. The research has been performed within the project (0065-2019-0001) implemented under the government order for fundamental scientific research (GP 14).

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