Journal of Economic Theory 2 (3): 106-111, 2008
ISSN: 1994-8212

The Analytic Properties for a Generalized Abel's Asset Pricing Model

Shaoyong Lai and Ya Qin
College of Economic Mathematics,
Southwestern University of Finance and Economics, 610074, Chengdu, China

Abstract: The derivation of the integral equation for a generalized Abel's asset pricing model, which yields an analytic price-dividend function of one state variable, is established. When the constants appearing in the equation satisfy some inequalities and assumptions, the existence and uniqueness of the solutions for the equation are proved to be true. The analytic property of the solutions is investigated in the complex plane.

Key words: Analyticity, asset pricing model, nonlinear price-dividend function

INTRODUCTION

Asset pricing models have been investigated by many scholars in finance and management sciences. Lucas (1978) developed a consumption asset pricing model in which the risk of an asset can be measured by the covariance of its return with per capita consumption growth. Abel (1990) constructed an asset pricing model under habit formation and explained the equity premium puzzle. Mehra and Prescott (1985) solved the equilibrium asset pricing model with a representative agent of constant relative risk aversion preferences where the growth rate of the endowment followed a simple two-state first order Markov process. Cecchetti *et al*. (1993) examined and Mehra and Prescott (1985), equtiy premium puzzle in an economy field, where the growth rate of the endowment followed a two-state Markov process augmented by independent and identical distribution shocks to the growth rate. Campbell and Cochrane (1999) and Wachter (2006) discussed the equilibrium asset prices for a consumption asset pricing model with habit formation when the endowment followed an independent and identical distribution process.

Researchers have also sought analytic solutions for various asset pricing models. For instance, Burnside (1998) proposed a closed form solution for a standard asset pricing model under the assumption that the growth rate of endowment follows a first order autoregressive process with Gaussian shocks. When the endowment is an independent and identical distribution process (Abel, 1990) derived exact solutions for a consumption asset pricing model to analyze risky asset and one-period interest rate. Calin *et al*. (2005) obtained an analytic price-dividend function of one state variable to an asset pricing model.

In this research, we derive a nonlinear integral equation for a generalized asset pricing model which has a degree n with $n \geq 1$. We will use contraction mapping theorem to prove the existence and uniqueness of the solution for the integral equation under some assumptions. The analytic property of the price-dividend function for the asset pricing model is analyzed. Comparing with the research made in Calin *et al*. (2005), 2 different aspects should be mentioned. Firstly, the integral equation (n = 1) for the asset pricing model presented in Calin *et al*. (2005) is linear, while our model is nonlinear ($n \geq 1$). Secondly, the existence and uniqueness of the price-dividend function for the asset pricing model in Calin *et al*. (2005) were proved by using the Picard series technique. The approach we use in this research is the contraction mapping theorem.

THE GENERALIZED ABEL'S ASSET PRICING MODEL

In this study, we introduce the structure of the Abel's model and its detailed parameterizations and derive the fundamental integral equation for a generalized Abel's asset pricing model.

The Abel's model has the following specific structure, which is described by the utility function:

$$\frac{[c_t / v_t]^{1-\gamma}}{1-\gamma}, \tag{1}$$

where, $v_t = [c^h_{t-1}\ C^{(1-h)}_{t-1}]^\alpha$, c_{t-1} is the consumer's own consumption in period t-1, C_{t-1} is an aggregate per capita consumption in period t-1 and parameters $\gamma>0$ and $\gamma \neq 1$, $\alpha \geq 0$ and $h \geq 0$. Abel parameterized the model by setting α

Corresponding Author: Shaoyong Lai, College of Economic Mathematics, Southwestern University of Finance and Economics, 610074, Chengdu, China

and h to 0 or 1. Note that when $\alpha = 0$, we are in the case of Mehra and Prescott (1985), in which the utility does not depend on habit. When $\alpha = 1$ and $h = 1$, we are in the external habit case in which the habit is external to the individual's choice, or called that we are in the case of relative consumption of catching up with the Joneses. When $\alpha = 1$ and $h = 1$, we are in the internal habit case where an individuals own lagged consumption affects their choice of consumption.

The only source of real income is the dividend taking from the risky security. As in Calin *et al*. (2005), we have $c_t = C_t = D_t$. We take the dividend process for the risky security in the form:

$$D_{t+1} = D_t e^{x_0 + \phi x_t + u_{t+1}} \quad (2)$$

where, $u\sim NID(0, \sigma^2)$ and x_t is the current continuously compounded growth rate of the dividend and follows an AR (1) process $x_{t+1} = x_0 + \phi x_t + u_{t+1}$, in which x_0 is the constant growth rate per period.

It was shown in Abel (1990) that the Euler equation for equity is:

$$\frac{P_t}{D_t} = E_t\left[M_{t+1}\frac{D_{t+1}}{D_t}(1+(\frac{P_{t+1}}{D_{t+1}}))\right] \quad (3)$$

in which the Stochastic Discount Factor (SDF) is given by the form:

$$M_{t+1} = \beta(\frac{D_{t+1}}{D_t})^{-\gamma}(\frac{u_{t+1}}{u_t})^{\gamma-1}(\frac{H_{t+2}}{E_t(H_{t+1})}) \quad (4)$$

where, $\beta\in(0, 1)$ is a subjective discount factor, $u_{t+1} = D^{\alpha}_1$ and:

$$H_{t+2} = 1-\alpha\beta h(\frac{D_{t+2}}{D_{t+1}})^{1-\gamma}(\frac{D_{t+1}}{D_t})^{\alpha(\gamma-1)}, \alpha \ge 0, h \ge 0 \quad (5)$$

Here, we propose the following generalized nonlinear Euler equation for equity:

$$\frac{P_t}{D_t} = E_t[M_{t+1}\frac{D_{t+1}}{D_t}(1+(\frac{P_{t+1}}{D_{t+1}})^n)],\ n \ge 1 \quad (6)$$

It is known that the Euler Eq. (3) together with the intertemporal rate of substitution Eq. (4) is homogeneous of 1° in stock price, current dividends and next period's dividends. However, Eq. (6) with (4) is homogeneous of degree n. We shall discuss the dynamic properties of the solution to the equilibrium generalized price-dividend function, which satisfies Eq. (6). Here, we define the price-dividend function as:

$$p_t(x) = \frac{P_t}{D_t}$$

In order to get the specific form of function $p_t(x)$, as in Calin *et al*. (2005), we drop the subscript for time since, we seek a function of current dividend growth. Substituting Eq. (4) and dividend process Eq. (2) into (6) derives the basic integral equation of the generalized asset pricing model as follows:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\frac{A_0 e^{A_1 x}}{1-A_2 e^{A_1 x}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u-\sigma^2(1-\gamma)]^2} [1-A_2 e^{A_1(x_0+\phi x+u)}][1+p^n(x_0+\phi x+u)]du \quad (7)$$

where,

$$A_0 = \beta e^{x_0(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}$$

$A_1 = (1-\gamma)(\phi-\alpha)$ and $A_2 = \alpha h A_0$. We know that Eq. (7) is a nonlinear model. Now, we give the detailed derivation of Eq. (7).

Recalling that $x_{t+1} = x_0 + \phi x_t + u_{t+1}$, where, $u\sim NID(0,\sigma^2)$ and $D_{t+1} = D_t e^{x_0+\phi x_t+u_{t+1}}$ with $|\phi|<1$, $u_{t+1} = D_t^{\alpha}$ rewriting Eq. (6) in terms of p_t and substituting Eq. (4) and (5) into (6), we have

$$\begin{aligned} p_t &= [\beta(\frac{D_{t+1}}{D_t})^{-\gamma}(\frac{u_{t+1}}{u_t})^{\gamma-1}\frac{H_{t+2}}{E_t[H_{t+1}]}\frac{D_{t+1}}{D_t}(1+p^n_{t+1})] \\ &= \frac{\beta}{E_t[H_{t+1}]}E_t[(\frac{D_{t+1}}{D_t})^{1-\gamma}(\frac{u_{t+1}}{u_t})^{\gamma-1}H_{t+2}(1+p^n_{t+1})] \\ &= \frac{\beta}{E_t[H_{t+1}]}E_t[(\frac{D_{t+1}}{D_t})^{1-\gamma}(\frac{D_t}{D_{t-1}})^{\alpha(\gamma-1)}H_{t+2}(1+p^n_{t+1})] \end{aligned} \quad (8)$$

Multiplying the above equation by $E_t[H_{t+1}]$ yields

$$p_t E_t[H_{t+1}] = \beta E_t[(\frac{D_{t+1}}{D_t})^{1-\gamma}(\frac{D_t}{D_{t-1}})^{\alpha(\gamma-1)} H_{t+2}\cdot(1+p^n_{t+1})] \quad (9)$$

Since,

$$\begin{aligned} E_t[(\frac{D_{t+1}}{D_t})^{1-\gamma}] &= E_t[(e^{x_0+\phi x_t+u_{t+1}})^{1-\gamma}] \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(e^{x_0+\phi x_t+u_{t+1}})^{1-\gamma}e^{-\frac{u^2_{t+1}}{2\sigma^2}}du_{t+1} \\ &= e^{x_0(1-\gamma)+\frac{\sigma^2(1-\gamma)^2}{2}}e^{\phi(1-\gamma)x_t} \end{aligned} \quad (10)$$

and $D_t = D_{t-1}e^{x_t}$, we have

$$\begin{aligned} E_t[H_{t+1}] &= E_t[1-\alpha\beta h\cdot(\frac{D_{t+1}}{D_t})^{1-\gamma}(\frac{D_t}{D_{t-1}})^{\alpha(\gamma-1)}] \\ &= 1-\alpha\beta h E_t[(\frac{D_{t+1}}{D_t})^{1-\gamma}](\frac{D_t}{D_{t-1}})^{\alpha(\gamma-1)} \\ &= 1-\alpha\beta h e^{(1-\gamma)+\frac{\sigma^2(1-\gamma)^2}{2}}e^{(\phi-\alpha)(1-\gamma)x_t} \end{aligned} \quad (11)$$

Using

$$\begin{aligned} H_{t+2} &= 1-\alpha\beta h\cdot\left(\frac{D_{t+2}}{D_{t+1}}\right)^{1-\gamma}\left(\frac{D_{t+1}}{D_t}\right)^{\alpha(\gamma-1)} \\ &= 1-\alpha\beta h e^{(1-\gamma)x_{t+2}} e^{\alpha(\gamma-1)x_{t+1}} \\ &= 1-\alpha\beta h e^{(1-\gamma)(x_0+\phi(x_0+\phi x_t+u_{t+1})+u_{t+2})} e^{\alpha(\gamma-1)(x_0+\phi x_t+u_{t+1})} \\ &= 1-\alpha\beta h e^{(1+\phi-\alpha)(1-\gamma)x_0} e^{\phi(\phi-\alpha)(1-\gamma)x_t} e^{(\phi-\alpha)(1-\gamma)u_{t+1}} e^{(1-\gamma)u_{t+2}} \end{aligned} \tag{12}$$

results in

$$\begin{aligned} p_t E_t\,[H_{t+1}] &= \beta E_t\left[\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\left(\frac{D_t}{D_{t-1}}\right)^{\alpha(\gamma-1)} H_{t+2}\cdot(1+p_{t+1}^n)\right] \\ &= \frac{\beta}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(1-\gamma)(x_0+\phi x_t+u_{t+1})} e^{\alpha(\gamma-1)x_t} \\ &\times [1-\alpha\beta h e^{(1+\phi-\alpha)(1-\gamma)x_0} e^{\phi(\phi-\alpha)(1-\gamma)x_t} e^{(\phi-\alpha)(1-\gamma)u_{t+1}} e^{(1-\gamma)u_{t+2}}] \\ &\times [1+p_t^n(x_0+\phi x_t+u_{t+1})] e^{-\frac{u_{t+1}^2}{2\sigma^2}} e^{-\frac{u_{t+2}^2}{2\sigma^2}} du_{t+2}\, du_{t+1} \\ &= \frac{\beta e^{(1-\gamma)x_0+\frac{\sigma^2(1-\gamma)^2}{2}} e^{(\phi-\alpha)(1-\gamma)x_t}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}[1-\alpha\beta h e^{x_0(1-\gamma)+\frac{\sigma^2(1-\gamma)^2}{2}} e^{(\phi-\alpha)(1-\gamma)(x_0+\phi x_t+u_{t+1})}] \\ &\times [1+p_t^n(x_0+\phi x_t+u_{t+1})] e^{-\frac{1}{2\sigma^2}[u_{t+1}-\sigma^2(1-\gamma)]^2} du_{t+1} \end{aligned} \tag{13}$$

Setting

$$A_0 = \beta e^{x_0(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2},\quad A_1 = (1-\gamma)(\phi-\alpha)$$

and $A_2 = \alpha h A_0$, we know that Eq. (13) is equivalent to:

$$\begin{aligned} (1-A_2 e^{A_1 x_t})p_t(x_t) &= \frac{A_0 e^{A_1 x_t}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u_{t+1}-\sigma^2(1-\gamma)]^2} \\ &[1-A_2 e^{A_1(x_0+\phi x_t+u_{t+1})}][1+p_t^n(x_0+\phi x_t+u_{t+1})]du_{t+1} \end{aligned} \tag{14}$$

dividing Eq. (14) by $1-A_2 e^{A_1 x_t}$ and dropping the subscript for time, we have Eq. (7).

EXISTENCE AND UNIQUENESS OF PRICE-DIVIDEND FUNCTION FOR EQ. (7) WITH $A_2 = 0$

The problem in the proof of existence and uniqueness of nonlinear price-dividend function for Abel's asset pricing model recognized by Abel is the possibility of a negative marginal utility of consumption as reflected in the term $1-A_2 e^{A_1 x}$. For the external habit case, this doesn't present an issue since, h = 0, so, we have $A_2 = 0$. In this study, by making use of a fixed point theorem, we will prove the existence and uniqueness of solutions for Eq. (7) with $A_2 = 0$ in the space of continuous functions. In fact, for n = 1 and $A_2 = 0$, the existence and uniqueness of price-dividend function for the Eq. (7) was obtained in Calin *et al*. (2005) by using the Picard series technique.

Definition: Let G be a real-valued space that consists of all continuous functions f such that $|f(x)| < Me^{k|x|}$ with norm defined by:

$$\|f(x)\| = \sup_{x\in R}|f(x)e^{-k|x|}| \le M$$

where, M and K are positive constants.

We will use the contraction mapping theorem to prove that Eq. (7) with $A_2 = 0$ exists a unique solution in space G. When $A_2 = 0$, Eq. (7) is equivalent to

$$\begin{aligned} p(x) &= \frac{A_0 e^{A_1 x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u-\sigma^2(1-\gamma)]^2} \\ &[1+p^n(x_0+\phi x+u)]du \end{aligned} \tag{15}$$

Letting, $t = x_0+\phi x+u$, $\psi(x) = x_0+\phi x+\sigma^2(1-\gamma)$ yields

$$\begin{aligned} p(x) &= \frac{A_0 e^{A_1 x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u-\sigma^2(1-\gamma)]^2} \\ &[1+p^n(x_0+\phi x+u)]du \\ &= A_0 e^{A_1 x} + \frac{A_0 e^{A_1 x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[t-\psi(x)]^2} p^n(t)dt \end{aligned} \tag{16}$$

Theorem: If

$$A_0 = \beta e^{x_0(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}$$

is suitable small, k and M satisfy

$$A_0 < \frac{M}{2},\quad k > |A_1| + kn|\phi| \tag{17}$$

$$A_0 n M^{n-1} e^{\frac{\sigma^2 k^2 n^2}{2}}[e^{kn(x_0+\sigma^2(1-\gamma))} + e^{-kn(x_0+\sigma^2(1-\gamma))}] < \frac{1}{2} \tag{18}$$

then there exists a unique solution in space G to satisfy integral Eq. (16).

Proof: We define the operator T as follows

$$(Tp)(x) = A_0 e^{A_1 x} + \frac{A_0 e^{A_1 x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[t-\psi(x)]^2} p^n(t)dt \tag{19}$$

For arbitrary p (x) in space G, we derive

$$
\begin{aligned}
|Tp(x)| &\leq \left| A_0 e^{A_1 x}\left(1+\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|}\frac{p^n(t)}{e^{kn|t|}}\right)dt\right| \\
&\leq \left| A_0 e^{A_1 x}\left(1+\frac{M^n}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|}\right)dt\right| \\
&\leq \Big| A_0 e^{A_1 x}\Big(1+\frac{M^n}{\sqrt{2\pi}\sigma}[e^{\psi(x)kn+\frac{\sigma^2k^2n^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[t-(\psi(x)+\sigma^2 kn)]^2} \\
&\quad dt + e^{-\psi(x)kn+\frac{\sigma^2k^2n^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[t-(\psi(x)-\sigma^2 kn)]^2} dt]\Big)\Big| \\
&\leq \Big| A_0 e^{A_1 x}(1+M^n e^{\frac{\sigma^2k^2n^2}{2}}[e^{\psi(x)kn}+e^{-\psi(x)kn}])\Big|
\end{aligned} \tag{20}
$$

Applying assumptions Eq. (17) (18) and $\psi(x) = x_0 + \phi x + \sigma_2 (1-\gamma)$ to Eq. (20), we obtain

$$
\left| Tp(x)e^{-k|x|} \right| \leq \left| A_0\left(1+M^n e^{\frac{\sigma^2k^2n^2}{2}}\begin{bmatrix} e^{kn(x_0+\sigma^2(1-\gamma))}+ \\ e^{-kn(x_0+\sigma^2(1-\gamma))}\end{bmatrix}\right)\right|
$$

$$
\leq \frac{M}{2}+\frac{M}{2} < M
$$

from which we have

$$
\| Tp \| \leq M \tag{21}
$$

For arbitrary $p_1(x)$ and $p_2(x)$ in space G, we have

$$
\begin{aligned}
\left| Tp_1(x) - Tp_2(x) \right| &= \left| \frac{A_0 e^{A_1 x}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|} \right. \\
&\quad \left. \frac{p_1(t)-p_2(t)}{e^{k|t|}}\sum_{j=0}^{n-1}\frac{p_1(t)^{n-1-j}p_2(t)^j}{e^{(n-1)k|t|}}dt \right| \\
&\leq \| p_1 - p_2 \| \frac{|A_0 e^{A_1 x}| nM^{n-1}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2} e^{kn|t|}dt \\
&\leq \| p_1 - p_2 \| \left| A_0 e^{A_1 x} nM^{n-1} e^{\frac{\sigma^2k^2n^2}{2}}[e^{\psi(x)kn}+e^{-\psi(x)kn}] \right|
\end{aligned}
$$

which derives

$$
\begin{aligned}
\frac{|Tp_1(x)-Tp_2(x)|}{e^{k|x|}} &\leq \| p_1 - p_2 \| \\
&\quad \frac{\left| A_0 e^{A_1 x} nM^{n-1} e^{\frac{\sigma^2k^2n^2}{2}}[e^{\psi(x)kn}+e^{-\psi(x)kn}] \right|}{e^{k|x|}} \\
&\leq \| p_1 - p_2 \| \left| A_0 nM^{n-1} e^{\frac{\sigma^2k^2n^2}{2}}\left[e^{kn(x_0+\sigma^2(1-\gamma))}+e^{-kn(x_0+\sigma^2(1-\gamma))}\right] \right| \\
&\leq \frac{1}{2} \| p_1 - p_2 \|
\end{aligned} \tag{22}
$$

It follows from Eq. (22) that,

$$
\| Tp_1 - Tp_2 \| \leq \frac{1}{2} \| p_1 - p_2 \| \tag{23}
$$

By the contraction mapping theorem, from Eq. (21) and (23), we complete the proof.

DIFFERENTIABILITY OF THE GENERALIZED PRICE-DIVIDEND FUNCTION

Although, we proved the existence and uniqueness of the generalized price-dividend function under some conditions, there is a little information about differentiability of the price-dividend function. Many works on differentiability deal with the policy and value functions from generic programming problems. For instance, Calin *et al*. (2005) developed conditions for the price-dividend function to have an infinite order derivative. In the case of the generalized Abel's asset pricing model, we can also obtain the information about the differentiability of the generalized price-dividend function by focusing on the specific integral equation.

Theorem: Provided that conditions Eq. (17) and (18) hold, we get that the solution to the integral Eq. (15) is infinitely differentiable for all dividend growth $x \in (-\infty, +\infty)$.

Proof: It only needs to show that the equality given below holds for all positive integer m.

$$
\frac{d^m}{dx^m}\int_{-\infty}^{\infty} p^n(t) e^{-\frac{1}{2\sigma^2}[t-\psi(x)]^2} dt = \int_{-\infty}^{\infty} p^n(t)\frac{\partial^m}{\partial x^m}[e^{-\frac{1}{2\sigma^2}[t-\psi(x)]^2}]dt \tag{24}
$$

The positive constant M and k which satisfies $|p(t)| \leq Me^{k|t|}$ can be found through the definition of G. The m-th partial derivative of

$$
e^{-\frac{1}{2\sigma^2}[t-\psi(x)]^2}
$$

with respect to x is expressed as

$$
e^{-\frac{1}{2\sigma^2}[t-\psi(x)]^2}\sum_{j=1}^{m} f_j(x)t^j \tag{25}
$$

in which the $f_j(x)$ are polynomials. By induction, it is easy to find that $t^j \leq j!e^{|e|}$ for every integer $j \geq 0$ and for all $t \in (-\infty, +\infty)$. Using

$$
\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(t-\psi(x))^2+(nk+1)t} dt = e^{\frac{(nk+1)^2\sigma^2}{2}+(nk+1)\psi(x)} \tag{26}
$$

we know the right hand side of Eq. (24) is uniformly convergent. Therefore, we complete the proof of Theorem.

ANALYTICITY OF THE GENERALIZED PRICE-DIVIDEND FUNCTION

The power series representation of f (x) can be complexified by substituting z = x + iy for x. If the function f (x) is analytic, we get

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \text{ for } |z-z_0|<\delta,\ z_0=x_0+i_0 \quad (27)$$

where, $f^{(k)}(z)$ denotes the kth order complex derivative of f(z). We know that the usual calculus formulas for real differentiation also hold true for complex differentiation. Now we separate the function f (z) into its real and imaginary parts by writing f (z) = f_1 (x, y) + if_2 (x, y). Hence, we get

$$\begin{cases}\frac{\partial f}{\partial x}=\frac{\partial f_1}{\partial x}+i\frac{\partial f_2}{\partial x},\\ \frac{\partial f}{\partial y}=\frac{\partial f_1}{\partial y}+i\frac{\partial f_2}{\partial y}\end{cases} \quad (28)$$

From the above 2 equations, we can conclude that if f (x) is an analytic function, it must satisfy the pair of partial differential equations

$$\frac{\partial f_1}{\partial x}=\frac{\partial f_2}{\partial y} \text{ and } \frac{\partial f_1}{\partial y}=-\frac{\partial f_2}{\partial x} \quad (29)$$

which, are called the Cauchy-Riemann equations (Ahlfors, 1979). This leads to the following characterization: f is said to be analytic in Ω if it satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}}=\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=0 \quad (30)$$

After complexifying the generalized price-dividend function, we identify the largest domain Ω in the complex plane where, the generalized price-dividend function is analytic. In the end, we must investigate the radius of convergence of the Taylor series of the generalized price-dividend function about a point x from the boundary of Ω.

Theorem: If p (x) belongs to G and satisfies (15) or (16), then the generalized price-dividend function p(x) is analytic in the open interval (-∞, +∞) and its Taylor series expansion about a point x_*∈ (-∞, +∞) has an infinite radius of convergence.

Proof: In order to prove Theorem, we only need to show that the function

$$Q(z)=\int_{-\infty}^{\infty}[e^{-\frac{1}{2\sigma^2}[t-\psi(z)]^2}]\,p^n(t)dt \quad (31)$$

is analytic in the entire complex plane C. It requires us to show that Q (z) has continuous first-order partial derivatives which satisfies the Cauchy-Riemann equation

$$\frac{\partial Q}{\partial x}+i\frac{\partial Q}{\partial y}=0 \quad (32)$$

Let $z_0 = x_0 + iy_0$ be a given point in C, since ψ (z) is bounded on any compact subset of C, there exists L>0 depending on z_0 such that

$$|e^{-\lambda[t-\psi(z)]^2}|\le L e^{-(\lambda/2)t^2},\ \lambda>0, t\in(-\infty,+\infty), |z-z_0|\le 1 \quad (33)$$

In fact, we have

$$\frac{\partial}{\partial x}[e^{-\lambda[t-\psi(z)]^2}]=2\lambda[t-\psi(z)]e^{-\lambda[t-\psi(z)]^2}\frac{\partial\psi(z)}{\partial x} \quad (34)$$

and

$$\frac{\partial}{\partial y}[e^{-\lambda[t-\psi(z)]^2}=2\lambda[t-\psi(z)]e^{-\lambda[t-\psi(z)]^2}\frac{\partial\psi(z)}{\partial y} \quad (35)$$

Using Eq. (33-35), the hypothesis on p (t) and the Lebesgue dominated convergent theorem, we obtain that the partial derivatives of Q (z) with respect to x and y are well-defined near every point z_0∈C. Therefore, we can pass the differentiation inside the integral sign. Since, $e^{-\lambda(t-\psi(z))^2}$ is analytic, so it satisfy the Cauchy-Riemann equation

$$(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})e^{-\lambda(t-\psi(z))^2}=0 \quad (36)$$

which results in

$$\begin{aligned}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})Q(z)&=\int_{-\infty}^{\infty}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})(e^{-\lambda(t-\psi(z))^2})p^n(t)dt\\&=0\cdot p^n(t)dt=0\end{aligned} \quad (37)$$

We complete the proof.

CONCLUSION

As many scholars to study the asset pricing models, we have investigated the following 3 aspects for the generalized asset pricing model. Firstly, the components of the asset pricing models are incorporated into an integral equation that maps the unknown future price-dividend function into the current price-dividend function. Secondly, the equilibrium generalized price-dividend function is the solution to an integral equation which exists a unique solution. Finally, the pricing kernel and dividend process are analytic on a given set. However, the conditions expressed by Eq. (17) and (18) need to be simplified. Comparing the analytic solution with numerical approximation of the price-dividend function also needs to be done in a suitable classical space.

REFERENCES

Abel, A.B., 1990. Asset prices under habit formation and catching up with the Jones. Am. Econ. Rev., 80 (1): 38-42.

Ahlfors, L.V., 1979. Complex analysis. McGraw-Hill, New York.

Burnside, C., 1998. Solving asset pricing models with gaussian shocks. J. Econ. Dyn. Contr., 22 (3): 329-340.

Calin, O.L., Y. Chen, T.F. Cosimano and A.A. Himonas, 2005. Solving asset pricing models when the price-dividend function is analytic. Econometrica, 73 (7): 961-982.

Campbell, J.Y. and J. Cochrane, 1999. By force of habit, a consumption-based explanation of aggregate stock market behavior. J. Polit. Econ., 107 (2): 205-251.

Cecchetti, S.G., P. Lam and N.C. Mark, 1993. The equity premium and the risk-free rate. J. Monet. Econ., 31 (1): 21-45.

Lucas, R.E., 1978. Asset prices in a pure exchange economy. Econometrica, 46 (10): 1429-1445.

Mehra, R. and E.C. Prescott, 1985. The equity premium: A Puzzle. J. Monet. Econ., 15 (2): 145-161.

Wachter, J.A., 2006. A consumption-based model of the term structure of interest rate. J. Finan. Econ., 79 (3): 365-399.