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Generalized Rational Runge-Kutta Method for Integration of Stiff System of Ordinary Differential Equations

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Abstract: This study describes the development, analysis and implementation of generalized implicit rational runge-kutta schemes for integration of stiff system of ordinary differential equations. Its development adopted Taylor and binomial series expansion techniques to generate its parameters. The analysis of its basic properties adopted Dalhquist, A-stability model test equation and the results show that the scheme is, consistent, convergent and A-stable. Numerical results show that the method is accurate and effective.

Key words: Rational, Runge-Kutta, convergent, consistent, effective, error bound, implementation, development, A-stable

INTRODUCTION

A differential equation of the form

$$\dot{y} = f(x, y),\ y(x_0) = y_0,\ a \le x \ge b \tag{1}$$

whose Jacobian $\frac{\partial f}{\partial y}$ possesses eigen values

$$\lambda_j = U_j + iV_j,\ j = 1(1)_n \tag{2}$$

where, $i = \sqrt{-1}$, satisfying the following conditions.

(a) $U_i << 0,\ j = 1\ (1)n$

(b) $\text{Max}\left|U_j(x)\right| >> \min\left|U_j(x)\right|$

or

$$r(x) = \frac{\text{Max}\left|U_j(x)\right|}{\min\left|U_j(x)\right|} >> 1 \tag{3}$$

where, r (x) denotes is the stiffness ratio is called stiff ODEs. For example, the differential equations:

(1) $$y' = \lambda(y\text{-}E(x) + E^1(x),\ y(x_o) = y_o \tag{4}$$

where, E (x) continuously differentiable, λ is a complex constant with Re (λ) <<0, with the exact solution

$$y(x) = E(x) + y_0 e^{\lambda x} \tag{5}$$

consisting of 2 components namely E (x) which is slowly varying in the interval of integration (a, b) and the second component $y_0e^{\lambda x}$ decaying rapidly in the transient phase at the rate of $-1/\lambda$ is stiff.

(2) The system of differential equations of the form.

$$y' = \begin{bmatrix} -0.00005 & 100 \\ -100 & -0.00005 \end{bmatrix} y \tag{6}$$

with

$$y(o) = [1,\ 1]^T,\ 0 \le x \le 10\pi$$

whose solution is obtained as

$$y(x) = e^{-0.0005\ x}\begin{bmatrix} \sin 100x + & \text{Cos } 100x \\ \text{Cos } 100x - & \text{Sin } 100x \end{bmatrix} \tag{7}$$

Whose, transistory phase is the entire interval of integration $0 \le x \le 10\pi$ with 50π as complete oscillation per unit cycle is an ODEs possessing these types of properties are called stiff oscillating ODEs.

Most of the conventional Runge-Kutta schemes cannot effectively solve them because they have small region of absolute stability.

This perhaps motivated Hong Yuanfu (1982) to introduce a rationalized Runge-Kutta scheme of the form

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$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{8}$$

Where:

$$K_i = hf\left(x_n + c_i h,\ y_n + \sum_{i=1}^{S} a_{ij} k_j\right)$$

$$H_i = hg\left(x_n + d_i h,\ z_n + \sum_{i=1}^{S} b_{ij} k_j\right) \tag{9}$$

with

$$g(x_n, z_n) = -Z_n^2 f(x_{n,}\ y_n)$$

subject to the constraints

$$c_1 = \sum_{j=1}^{R} a_{ij}$$

$$d_1 = \sum_{j=1}^{R} b_{ij} \tag{10}$$

Since, the method possesses adequate stability property for solution of stiff ODEs, the papers consider the extension of the scheme to a general step process so that is can serve as a general purpose predictor for multistep schemes.

$$y_{n+m} = \frac{y_{n+m-1} + \sum_{i=1}^{R} W_i K_i}{1 + y_{n+m-1} \sum_{i=1}^{R} V_i H_i} \tag{11}$$

Where:

$$K_1 = hf\left(x_{n+m\text{-}1} + C_i h,\ y_{n+m\text{-}1} + \sum_{j=1}^{R} a_{ij} K_i\right)$$

$$H_i = hg\left(x_{n+m\text{-}1} + d_i h,\ Z_{n+m\text{-}1} + \sum_{j=1}^{R} b_{ij} H_i\right) \tag{12}$$

with

$$g\left(x_{n+km1},\ Z_{n+m\text{-}1}\right) = Z_{n+m-1}^2 f\left(x_{n+m\text{-}1},\ y_{n+m\text{-}1}\right) \tag{13}$$

In the spirit of Ademiluyi and Babatola (2000) the scheme is classified into:

- Explicit if $a_{ij} = 0$, $b_{ij} = 0$, for $j \geq i$.
- Semi-implicit if $a_{ij} = 0$, $b_{ij} = 0$ for $j > i$.
- Implicit if $a_{ij} \neq 0$, $b_{ij} \# 0$ for at least one $j > i$.

Derivation of the method: In this study, the parameters V_i, W_i, C_i, d_i, a_{ij}, b_{ij} are to be determined from the system of non-linear equations generated by adopting the following steps:

- Obtain the Taylor series expansion of Ki's and Hi's about point (x_{n+m-1}, y_{n+m-1}) for i =1 (1)R.
- Insert the series expansion into Eq. 11.
- Combine terms in equal powers of h and compare the final expansion with the Taylor series expansion of y_{n+m-1} about (x_n, y_n) in the power series of h.

The number of parameters normally exceeds the numbers of equations, but in the spirit of King (1966), Gill (1951) and Blum (1952), these parameters are chosen as to ensure that (the resultant computation method has:

- Adequate order of accuracy.
- Minimum local truncation error bound.
- Large interval of absolute stability.
- Minimum computer storage facilities requirement.

One-step one-stage schemes: By setting m = 1 and R = 1, in Eq. 11 the general one-step one-stage scheme is of the form

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} \tag{14}$$

Where:

$$K_1 = hf\left(x_n + c_1 h,\ \ y_n + a_{11} K_1\right)$$

$$H_1 = hg\left(x_n + d_1 h,\ \ z_n + b_{11} H_1\right) \tag{15}$$

$$g\left(x_n\ \ z_n\right) = -Z_n^2 f(x_n, y_n) \tag{16}$$

and

$$Z_n = 1/y_n \tag{17}$$

with the constraints

$$c_1 = a_{11}$$
$$d_1 = b_{11} \tag{18}$$

The binomial expansion theorem of order one on the right hand side of Eq. 11 yields

$$y_{n+1} = y_n + W_1 k_1 - y_n^2 V_1 H_1 + \text{(higher order terms)} \tag{19}$$

While, the Taylor series expansion of y_{n+1} about y_n gives

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y'' + \frac{h^3}{6}y_n^{(3)} + \frac{h^4}{24}y_n^{(4)} + 0h^5 \quad (20)$$

Adopting differential notations

$$\begin{aligned} y'_n &= f_n \\ y''_n &= f_x + f_n f_y = Df_n \\ y_n^{(3)} &= f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy} + f_y\left(f_x + f_n f_y\right) \\ &= D^2 f_n + f_y Df_n \\ y_n^{(4)} &= f_{xxx} + 3f_n f_{xxy} + 3f_n^2 f_{xyy} + f_n^3 f_{yyy} + \\ & f_y\left(f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}\right) + \left(f_x + f_n f_y\right) \\ & \left(3f_{xy}\right) + \left(3f_n f_y + fy^2\right) \\ &= D^3 f_n + f_y D^2 f_n + 3Df_n Df_y \\ & + f_y{}^2 Df_n \end{aligned} \quad (21)$$

substitute Eq. 21 into Eq. 20, we have

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}Df_n + \frac{h^3}{3!}\left(D^2 f_n + f_y Df_n\right) + \frac{h^4}{4!}\left(D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n\right) + 0h^5 \quad (22)$$

Similarly the Taylor series expansion of K_1 about $(x_n y_n)$ is

$$K_1 = h\begin{pmatrix} f_n + \left(c_1 hf_x + a_{11}k_1 f_y\right) + ½ \\ \left(c_1^2 h^2 f_{xx} + 2c_1 ha_{11}k_1 f_{xy}\right) + a_{11}k_1^2 f_{yy} \end{pmatrix} + 0(h^2) \quad (23)$$

Collecting coefficients of equal powers of h, equation can be rewritten in the form

$$K_1 = hA_1 + h^2B_1 + h^3D_1 + 0h^4 \quad (24)$$

Where:

$$\begin{aligned} A_1 &= f_n,.\ B_1 = c_1\ (f_x + f_n f_y) = C_1\ Df_n \\ D_1 &= c_1 B_1 f_y + ½\, C_1^2\ (f_{xx} + 2f_n f_{xy} + f_n^2\ f_{yy}) \\ &= C_1\ Df_n f_y + ½\, C_1^2\ D^2 f_n \end{aligned} \quad (25)$$

In a similar manner, expansion of H_1 about (x_n, z_n) yields

$$H_1 = hN_1 + h^2M_1 + h^3R_1 + 0h^4 \quad (26)$$

Where:

$$\begin{aligned} N_1 &= g\ (x_n, z_n) = g_n \\ M_1 &= d_1(g_x + g_n g_z) = d_1 Dg_n \\ R_1 &= d_1^2\left(g_z Dg_n + ½\, D^2 g_n\right) \end{aligned} \quad (27)$$

$$\begin{aligned} & g_n = \frac{-f_n}{y_n^2},\ g_x = \frac{-f_x}{y_n^2},\ g_{xx} = \frac{-f_{xx}}{y_n^2} \\ & g_z = \frac{-2f_n}{y_n} + f_y,\ g_{xz} = \frac{-2f_x}{y_n} + f_{xy} \\ & g_{xxz} = \frac{-2f_{xx}}{y_n} + f_{xxy},\ g_{zz} = -2f_n - y_n^2 f_{yy} \\ & g_{xzz} = -2f_x - 2y_n^2 f_{xyy,} \\ & g_{zzz} = 4y_n^2 f_y + 6y_n^2 f_{yy} + y_n^4 f_{yyy} \end{aligned} \quad (28)$$

Substitute Eq. 28 into Eq. 27, we obtained

$$N_1 = \frac{-f_n}{y_n^2},\ M_1 = \frac{-d_1}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n}\right), \quad (29)$$

$$R_1 = \frac{d_1^2}{y_n^2}\left[\left(\frac{-2f_n}{y_n} + f_y\right)\left(Df_n + \frac{f_n^2}{y_n}\right)\right] + ½\left(D^2 f_n - \frac{2f_n}{y_n}\left(\frac{f_n{}^2}{y_n} + f_x\right)\right]$$

Using Eq. 25 and 26 in 19, to get

$$\begin{aligned} y_{n+1} &= y_n + W_1\left(hA_1 + h^2B_1 + h^3D_1 + 0h^4\right) \\ & -y_n^2\left(V_1\left(hN_1 + h^2M_1 + h^3R_1 + 0h^4\right)\right] \\ &= y_n\left(W_1A_1 - y_n^2 V_1 N_1\right)h + \left(W_1B_1 - y_n^2 V_1 M_1\right) \\ & h^2 + \left(W_1D_1 - y_n^2 V_1 R_1\right)h^3 + 0h^4 \end{aligned} \quad (30)$$

Comparing the coefficients of the powers of h and h^2 in Eq. 22 and 30, we obtained

$$\begin{aligned} W_1A_1 - y_n^2 V_1 N_1 &= f_n \\ W_1B_1 - y_n^2\ V_1 M_1 &= Df_n \end{aligned} \quad (31)$$

Adopting the values of A_1, N_1, B_1 and M_1 as earlier defined, we have system of non-linear simultaneous Equation

$$W_1 + V_1 = 1,\ W_1C_1 + V_1d_1 = ½ \quad (32)$$

with the constraints

$$a_{11} = c_1 \quad b_{11} = d_1 \tag{33}$$

With local truncation error

$$T_{n+1} = \frac{1}{6}\left(D^2 f_n + f_y D f_n\right)\left(-\tfrac{1}{2}C_1^2 W_1 - \tfrac{1}{2}V_1 d_1^2\left(\frac{2f_n}{y_n} - \frac{2f_n f_y}{y_n}\right)\right) \tag{34}$$

Here, we have more umknowns than the number of equations, hence some of them have to be declare free as $W_1 = ½$, $V_1 = ½$ to obtain

(I) $V_1 = W_1 = ½$ $c_1 = a_{11} = ¾$, $d_1 = b_{11} = ¼$

Substituting these values in Eq. 14 we obtain a family of one-step, one stage schemes of the form

$$y_{n+1} = \frac{y_n + \frac{1}{2}K_1}{1 + \frac{y_n}{2}H_1} \tag{35}$$

Where:

$$K_1 = hf\,(x_n + ¾\,h,\ y_n + ¾\,K_1) \quad H_1 = hg\,(x_n + ¼\,h,\ z_n + ¼\,H_1) \tag{36}$$

Also with

(ii) $V_1 = ¾$, $W_1 = ¼$, $d_1 = c_1 = ½$, $a_{11} = b_{11} = ½$

Eq. (14) becomes

$$y_{n+1} = \frac{y_n + \frac{1}{4}K_1}{1 + \frac{3}{4}y_n H_1} \tag{37}$$

Where:

$$K_1 = hf\,(x_n + ½h,\ y_n + ½\,K_1) \quad H_1 = hg\,(x_n + ½\,h,\ z_n + ½\,H_1) \tag{38}$$

With:

$W_1 = 1/3$, $V_1 = 2/3$, $a_{11} = C_1 = 1/3$, $b_{11} = d_1 = 7/12$

$$y_{n+1} = \frac{y_n + \frac{1}{3}K_1}{1 + \frac{2}{3}y_n H_1} \tag{39}$$

Where:

$$K_1 = hf\,(xn + \tfrac{1}{3}h,\ yn + \tfrac{1}{3}K_1) \quad H_1 = hg\,(xn + \tfrac{7}{12}h,\ xn + \tfrac{7}{12}H_1) \tag{40}$$

Two step one-stage schemes: By setting M = 2 and R = 1, we obtained 2 step, one-stage schemes of the general form.

$$y_{n+2} = \frac{y_{n+1} + W_1 K_1}{1 + y_{n+1}\ V_1 H_1} \tag{41}$$

Where:

$$K_1 = hf\,(x_{n+1} + c_1 h,\ y_{n+1} + a_{11}\,K_1) \quad H_1 = hg\,(x_{n+1} + d_1 h,\ z_{n+1} + b_{11}\,H_1) \tag{42}$$

With the constraints

$$c_1 = a_{11},\ d_1 = b_{11} \tag{43}$$

Adopting Binomial expansion technique and ignoring higher order terms, than one we obtained

$$y_{n+2} = y_{n+1} + W_1 K_1 - y_{n+1}^2\ V_1 H_1 + \text{higher order terms} \tag{44}$$

Expanding K_1 using Taylor series expansion

$$K_1 = hAP_1 + h^2 B + h^3 D_1 + 0h^4 \tag{45}$$

Where:

$$A_1\ \text{-}f_{n+1},\ B_1 = c_1 Df_{n+1}, \quad D_1 = C_1 f_y Df_{n+1} + \tfrac{1}{2}C_1^2 D^2 f_{n+1} \tag{46}$$

In a similar manner, the Taylor series expansion of H_1 about (x_{n+1}, z_{n+1}) yields

$$H_1 = hN_1 + h^2 M_1 + h^3 R_1 + 0h^4 \tag{47}$$

Where,

$$N_1 = \frac{-f_{n+1}}{y_{n+1}^2},\ M_1 = \frac{-d_1}{y_{n+1}^2}\left(Df_{n+1} + 2\frac{f_n^2}{y_n}\right)$$

$$R_1 = \frac{-d_1^2}{y_{n+1}^2}\left[\left(\frac{-2f_{n+1}}{y_{n+1}}f_y\right)\left(Df_{n+1} + \frac{f_{n+1}^2}{y_{n+1}}\right) + \tfrac{1}{2}\left(D^2 f_{n+1}\frac{-2f_{n+1}}{y_{n+1}}\left(\frac{f_{n+1}^2}{y_{n+1}} + f_x\right)\right)\right] \tag{48}$$

Using (46) and (48) in Eq. 49, we have

$$y_{n+2} = y_{n+1} + W_1\ (hA_1 + h^2 B_1 + h^3 D_1 + 0h^4) - y_{n+1}^2\ (hN_1 + h^2 M_1 + h^3 R_1 + 0h^4) \tag{49}$$

Simplifying, we have

$$y_{n+2} = y_{n+1} + (W_1 A_1 - y_{n+1}^2 V_1 N_1)h + (W_1 B_1 - y_{n+1}^2 V_1 M_1)h^2 + (W_1 D_1 - y_{n+1}^2 V_1 R_1)h^3 + 0h^4 \tag{50}$$

Adopting Taylor series expansion of y_{n+2} about (x_{n+1}, y_{n+1}) to get

$$y_{n+2} = y_{n+1} + hf_{n+1} + \frac{h^2}{2}Df_{n+1} + \frac{h^3}{3!}\left(D^2 f_{n+1} + f_y Df_n\right) + 0h^4 \tag{51}$$

Where:

$$f_{n+1} = f(x_{n+1},\ y_{n+1})$$
$$Df_{n+1} = f_x + f_{n+1} f_y \quad (52)$$
$$D^2 f_{n+1} = f_{xx} + 2f_{n+1} f_{xy} + f_{n+1}^2 f_{yy}$$

Comparing the coefficient of the powers of h in Eq. 51 and 52, we obtained

$$W_1 + V_1 = 1$$
$$W_1 C_1 + V_1 d_1 = ½ \quad (53)$$

Subject to the constraints equations

$$a_{11} = C_1$$
$$b_{11} = d_1 \quad (54)$$

Here, we have six unknowns with four equation, choosing the values of V_1 and W_1, we obtained.

(i) $V_1 = W_1 = ½$, $C_1 = d_1 = ½$, $a_{11} = b_{11} = ½$ Eq. 42 yields

Two step one-stage family of formula

$$y_{n+2} = \frac{y_{n+1} + \frac{1}{2} K_1}{1 + \frac{y_{n+1}}{2} H_1} \quad (55)$$

Where:

$$K_1 = hf\left(x_n + \tfrac{1}{2}h,\ \ y_n + \tfrac{1}{2}K_1\right)$$
$$H_1 = hg\left(x_n + \tfrac{1}{2}h,\ \ z_n + \tfrac{1}{2}H_1\right) \quad (56)$$

while for

(ii) $V_1 = ¼$, $W_1 = ¾$, $C_1 = ½$, $d_1 = ½$ $a_{11} = b_{11} = ½$

Eq. 42 becomes

$$y_{n+2} = \frac{y_{n+1} + \frac{3}{4} K_1}{1 + \frac{y_{n+1}}{4} H_1} \quad (57)$$

Where:

$$K_1 = hf\left(x_n + \tfrac{1}{2}h,\ \ y_{n+1} + \tfrac{1}{2}K_1\right)$$
$$H_1 = hg\left(x_{n+1} + \tfrac{1}{2}h,\ \ z_{n+1} + \tfrac{1}{2}H_1\right) \quad (58)$$

Next, we access the basic properties of this family of methods.

THE BASIC PROPERTIES OF THE METHOD

The basic properties required of a good computational method for stiff ODEs includes consistency, convergence and a-stability.

Consistency: A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve (Ademiluyi, 2001).

To prove that Eq. 11 is consistent. Recall that

$$y_{n+k} = \frac{y_{n+k-1} + \sum_{i=1}^{R} W_i K_i}{1 + y_{n+k-1} \sum_{i=1}^{R} V_i H_i} \quad (59)$$

subtract y_{n+k-1} on both sides of Eq. 60

$$y_{n+k} - y_{n+k-1} = \frac{y_{n+k-1} + \sum_{i=1}^{R} W_i K_i}{1 + y_{n+k-1} + \sum_{i=1}^{R} V_i H_i} - y_{n+k-1}$$
$$= \frac{y_{n+k-1} + \sum_{i=1}^{R} W_i K_i - y_{n+k-1}\left(1 + y_{n+k-1} \sum_{i=1}^{R} V_i H_i\right)}{1 + y_{n+k-1} + \sum_{i=1}^{R} V_i H_i} \quad (60)$$

$$\frac{\sum_{i=1}^{R} W_i Ki - y_{n+k-1}^2 \sum V_i H_i}{1 + y_{n+k-1} \sum_{i=1}^{R} V_i H_i} \quad (61)$$

But

$$K_i = hf\left(x_{n+k-1} + C_i h,\ \ y_n + \sum_{j=1}^{R} a_{ij} k_j\right)$$
$$H_i = hg\left(x_{n+k-1} + d_i h,\ \ Z_n + \sum_{j=1}^{R} b_{ij} H_j\right) \quad (62)$$

$$y_{n+k} - y_{n+k-1} = \frac{\sum_{i=1}^{R} h W_i f\left(x_{n+k-1} + c_i h,\ \ y_{n+k} + \sum_{j=1}^{R} a_{ij} K_j\right) - y_{n+k-1}^2 \sum_{j=1}^{R} V_i hg\left(x_{n+k-1} + d_1 h,\ \ Z_{m+k-1} + \sum_{j=1}^{R} b_{ij} H_j\right)}{1 + y_{n+k=1} \sum Vihg\left(x_{n+k-1} + d_1 h,\ \ Z_{m+k-1} \sum_{j=1}^{R} b_{ij} H_j\right)} \quad (63)$$

Dividing all through by h and taking limit as h tends to zero on both sides to have

$$\frac{\lim_{h\to o} y_{n+k} - y_{n+k-1}}{h} = \sum_{i=1}^{R} W_i f(x_{n+k-1}, y_{n+k-1}) - y_{n+k-1}^2 \sum V_1 g(x_{n+k-1}, y_{n+k-1}) \quad (64)$$

but

$$g(x_{n+k-1}, Z_{n+k-1}) = \frac{1}{y_{n+k-1}^2} f\left(x_{n+k-1}, y_{n+k-1}\right) \quad (65)$$

then

$$Lt \frac{y_{n+k} - y_{n=k-1}}{h} = \sum_{i=1}^{R} (W_i + V_i) f\left(x_{n+k-1}, y_{n+k-1}\right),$$

$$\text{but} \left(\sum_{L=1}^{R} W_i + V_i\right) = 1$$

$$y'(x_{n+k-1}) = f\left(x_{n+k-1}, y_{n+k-1}\right) \quad (66)$$

Hence, the method is consistent.

Convergence: Since the proposed scheme is one -step and it has been proved to be consistent then it is convergent by Lambert (1973).

Stability properties: To examine the stability property of this schemes we apply scheme Eq. 11 to Dalhquist (1963) stability scalar test initial value problem.

$$y' = \lambda y, \; y(x_o) = y_o \quad (67)$$

to obtained a difference equation

$$y_{n+k} = \mu(z) y_{n+k-1} \quad (68)$$

with the stability function

$$\mu(z) = \frac{1 + ZW^T (I\text{-}ZA)^{-1} e}{1 + ZV^T (I\text{-}ZB)^{-1} e} \quad (69)$$

Where:

$$W^T = (W_1, W_2 \ldots\ldots\ldots\ldots W_r)$$
$$V^T = (V_1, V_2 \ldots\ldots\ldots\ldots V_r)$$

To illustrate this, we consider the one-step, one-stage scheme

$$y_{n+1} = \frac{y_n + W_1 k_1}{1 + y_n V_1 H_1} \quad (70)$$

Where:

$$K_1 = hf(x_n + c[_1 h, y_n + a_{11} K_1)$$
$$H_1 = hg(x_n + d_1 h, z_{n+} b_{11} H_1) \quad (71)$$

Applying Eq. 70 to the stability test Eq. 67 we obtain the recurrent relation

$$y_{n+1} = \mu(z) y_n \quad (72)$$

Where:

$$\bar{\mu}(z) = \frac{1 + W_1 Z(1 - a_{11} Z)}{1 - V_i Z(1 + b_{11} Z)} \quad (73)$$

The difference Eq. 72 will produce a convergent and stable approximation to equation if

$$|\mu(z)| = \left|\frac{1 + \frac{1}{4} Z}{1 - \frac{3}{4} Z}\right| \prec 1 \quad (74)$$

Simplifying the in equality (74), we obtain (-∞ = z = 0). Hence the scheme is A-stable because the interval of absolute stability is (-∞, 0).

NUMERICAL COMPUTATIONS AND RESULTS

In order to access the performance of the schemes, the following sample problems were solved.

Problem 1: Consider the stiff system of ODEs

$$Y' = AY \quad (75)$$

Where:

$$A = \begin{bmatrix} 1.0 & -4.99 & 0 \\ 0 & -5.0 & 0 \\ 0 & 2.0 & -12 \end{bmatrix} \quad (76)$$

With initial condition y (o) = (2, 1, 2), 0 = x = 1 and theoretical solution

$$\begin{pmatrix} y_1(x) & = e^{-x} + & e^{-5x} \\ y_2(x) & = e^{-5x} & \\ y_3(x) & = e^{-5x} + e^{-12x} & \end{pmatrix} \quad (77)$$

Using step size h = 0.01 the method is implemented and the results are as shown in Table 1.

Problem 2: The second sample problem considered is the stiff system of initial values problems of ODEs shown in Table 2.

$$y = \begin{pmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 \\ 0 & 0 & -9.0 & 0 \\ 0 & 0 & 0 & -10.0 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, y(0) = [1\,1\,1\,1] \quad (78)$$

Table 1: Numerical result of k-step implicit rational runge-kutta schemes for solving stiff systems of ordinary differential equations

		Y1	Y2	Y3
X_n	Control step size (h)	E1	E2	E3
		0.1980099667D+01	0.9706425830D+00	0.8869204674D+00
0.3000000000D-01	0.3000000000D-01	0.8291942688D-09	0.3281419103D-07	0.8161313500D-05
		0.1885147337D+01	0.8379203859D+00	0.4917945068D+00
0.1774236000D+00	0.1771470000D-01	0.9577894033D-01	0.3422855333D-08	0.5357828618D-06
		0.1791235536D+01	0.7191953586D+00	0.2663621637D+00
0.3307246652D+00	0.1046033532D-01	0.11050933794D-10	0.35587255336D-09	0.3474808041D-07
		0.1694213422D+01	6088845946D+00	0.1365392880D+00
0.4977858155D+00	0.6176733963D-02	0.1269873096D-11	0.3655098446D-10	0.2146555961D-08
		0.1556933815D+01	0.4729421983D+00	0.4953161076D-01
0.7512863895D+00	0.3647299638D-01	0.1425978891D-08	0.3505060447D-07	0.1010194837D-05
		0.1435390902D+01	0.3709037123D+00	0.1867601194D-01
0.9951298893D+00	0.2153693963D-01	0.1594313570D-09	0.3316564301D-08	0.4481540687D-07

Table 2: Numerical result of k-step implicit rational runge-kutta schemes for solving stiff systems of ordinary differential equations

		Y1	Y2	Y3	Y4
X	Control step size	E1	E2	E3	E4
		0.9950124792D+00	0.9900498337D+00	0.9139311928+00	0.9048374306D+00
0.3000000000D-01	0.3000000000D-01	0.2597677629D-10	0.4145971344D-09	0.2617874150D-05	0.3971726602D-05
		0.9708623323D+00	0.9425736684D+00	0.5872698932D+00	0.5535451450D+00
0.1774236000D+00	0.1771470000D-01	0.3078315380D-11	0.4788947017D-10	0.2005591107D-06	0.2890213078D-06
		0.9402798026D+00	0.8841261072D+00	0.3300866691D+00	0.2918382654D+00
0.3694667141D+00	0.1046033532D-01	0.3621547506D-12	0.5454525720D-11	0.1355160001D-07	0.1829417523D-07
		0.9144602205D+00	0.8362374949D+00	0.1999708940D+00	0.1672231757D+00
0.5365278644D+00	0.6176733963D-02	0.4285460875D+13	0.6268319197D-12	0.9915873955D-09	0.1265158728D-08
		0.8693495443D+00	0.7557686301D+00	0.8044517344D-01	0.6079796167D-01
0.8400599835D+00	0.3647299638D-01	0.4961209221D-10	0.6922001861D-09	0.5087490103D-06	0.5899525189D-06

CONCLUSION

Generalized Rational runge-kutta methods for the integration of stiff system of ODEs has been proposed. Theoretically, it has been showed that the scheme is consistent, convergent and A-stable. Numerical results and theoretical showed that the schemes are accurate and effective. Also from the above results the error is very minimal this implies that the scheme is very accurate.

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