New Types of Openness and Closed Graphs in Topological Space

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Abstract: ≺-unequivocally \( \theta \)-continuity function and (≺, \( \theta \))-closed graphs was examined by Chae et al. The goal of this study is to research a few of new portrayal and properties of ≺-unequivocally \( \theta \)-continuity and (≺, \( \theta \))-closed graphs. Besides we characterize new sort of a function called ≺, \( \theta \)-open function which is more grounded than quasi ≺-open and ≺-open and we acquire a few portrayals and properties for it.

Key words: Characterize, properties, portrayals, function, \( \theta \)-continuity, ground

INTRODUCTION

The concept of ≺-open sets was introduced and investigated by Njastad (1965). Latterly, the concept of ≺-unequivocally \( \theta \)-continuity function has studied by Chae et al. (1995). We know from Chae et al. (1995) that the type of ≺-unequivocally \( \theta \)-continuity function is stronger than a unequivocally \( \theta \)-continuity function (Noiri, 1980) and a unequivocally ≺-continues function (Faroo, 1987).

In this study we aim to investigate further properties and characterizations of ≺-unequivocally \( \theta \)-continuity functions as well as \( \theta \)-closed graph (Chae et al., 1995) and new types of function define called ≺, \( \theta \)-open functions which is stronger than quasi ≺-open and hence, unequivocally ≺-open, some characterizations and properties are obtain for it.

Preliminaries: All through this study just \( X \) speaks to a topological space.

Definition 2.1: Let be a subset of a topological space \((X, \tau)\) then is called:

- Regular open if \( A = (\overline{A})^o \) (Njastad, 1965)
- ≺-open if \( A \subset (\overline{A})^o \) (Levine, 1969)
- Semi-open if \( A \subset \overline{A} \) (Levine, 1969)
- \( \theta \)-open if for each \( x \in A \), there exist an open set \( U \) in \( X \) such that \( x \in U \subset \overline{U} \subset A \) (Velicko, 1968)
- \( \theta \)-semi-open if for each \( x \in A \), there exist an semi-open set \( U \) in \( X \) such that \( x \in U \subset \overline{U} \subset A \) (Noiri and Kang, 1984)

The supplements of the sets said above are their individual closed sets.

Definition 2.2: The set \( \overline{A} = \{ p | \exists X: A \subset \overline{X} \subset \varnothing \} \) for each ≺-open set \( H \) containing \( p \).

Definition 2.3: A filter base \( \Psi \) is said to be \( \theta \)-convergent (Velicko, 1968) (resp. \( \theta \)-convergent to a point \( x \) in \( X \) if for each open (resp. ≺-open) set \( G \) containing \( x \), there exist an \( F \in \Psi \) such that \( F \subset X \) (resp. \( F \subset X \)).

Definition 2.4; (Maheshwar, 1982): A subset \( A \) of a topological space \((X, \tau)\) is called a feebly open set in \( X \) if there exist an open set \( U \) such that \( U \cap A = \sigma Cl(U) \) where is the semi-closure operator.

Remark 2.5; (Jankovic, 1985): A subset \( A \) of a topological space \((X, \tau)\) is called ≺-open if and only if it is feebly open. It is notable that for a space \((X, \tau)\), \( X \) can be retopologized by the family \( \tau^a \) of all ≺-open sets of \( X \) (Maheshwar et al., 1982; Thakur, 1980) and furthermore the family \( \tau^a \) of all \( \theta \)-open set of \( X \) (Velicko, 1968) that is \( \tau^a \) (called \( \theta \)-topology) and \( \tau^s \) (called an \( \sigma \)-topology) are topologies on \( X \) and it is clearly that \( \tau^s \subset \tau^a \subset \tau^a \). The family of all ≺-open (resp. \( \theta \)-open and feebly-open) arrangements of \( X \) is indicated by ≺0(X) (resp. \( \theta_0(X) \) and \( \sigma_0(X) \)).

Definition 2.5; (Noiri and Kang, 1984): A function \( f: X \rightarrow Y \) is said to be unequivocally \( \theta \)-continuous if for each \( x \in X \) and each open set \( H \) of \( Y \) containing \( f(x) \), there exist an open set \( G \) of \( X \) containing \( x \) such that \( f(G) \subset H \).

Definition 2.6; (Noiri and Kang, 1984): A function \( f: X \rightarrow Y \) is said to be unequivocally \( \theta \)-continuous if for each open set \( H \) of \( Y \), \( f^{-1}(H) \) is \( \theta \)-open in \( X \) if and only if each closed set \( F \) of \( Y \) \( f^{-1}(F) \) is \( \theta \)-closed in \( X \).

Definition 2.7; (Maheshwar, 1983): A function \( f: X \rightarrow Y \) is said to be unequivocally ≺-continuous (resp. faintly continuous (Long and Herrington, 1982), completely ≺- irresolute and unequivocally ≺- irresolute (Faroo, 1987) if for each open (resp. \( \theta \)-open, ≺-open and ≺-open) set \( H \) of \( Y \), \( f^{-1}(H) \) is ≺-open (resp. open, regular open and open) in \( X \).
Definition 2.8; (Noiri, 1973): A function $f: X \to Y$ is said to be semi-open (resp. $\sim$-open (Maheshwari et al., 1983), quasi $\sim$-open (Thivagar, 1991; Abdul Jabbar, 2000), $\theta$-open (Abdul-Jabbar, 2000) weakly $\theta$-open and $\theta^{**}$-open (Ali, 2003) function if the image of each open (resp. open $\sim$-open, open, $\theta$-open and semi-open) set of $G$ of $X$, $f(G)$ is semi-open (resp. $\sim$-open, open, $\theta$-semi-open, and $\theta$-semi-open and open) in $Y$.

Definition 2.9; (Lee et al., 1985): A function $f: X \to Y$ is said to be pre-feebly-open (resp. inequivalently $\sim$-open (Thivagar, 1991), $\sim^{**}$-open (Ali, 2003) function if the image of each $\sim$-open set of $G$ of $X$, $f(G)$ is $\sim$-open in $Y$.

Definition 2.10; (Baker, 1986): Let $A$ be a subset of a topological space $(X, \tau)$ then $A$ is called a $\theta$-neighborhood of a point $x$ in $X$ if there exist an open set $U$ such that $x \in U \subset \bigcup U \subset A$.

Definition 2.11; (Lee et al., 1985): “A function $f: X \to Y$ is said to be $\sim$-open function if for each $x \in X$ and each $\theta$-neighborhood $A$ of $x$, $f(A)$ is $\sim$-neighborhood $f(x)$.”

Definition 2.12; (Singal and Arya, 1969): A space $X$ is said to be practically regular if for each regular closed set of $X$ and each point $x \in X$, there exist disjoint open set $U$ and $V$ such that $x \in U \subseteq V$.

Definition 2.13; (Faro, 1987): A space $X$ is said to be $\sim$-Hausdorff if for any $x, y \in X$, $x \neq y$, there exist $\sim$-open sets $G$ and $H$ such that $x \in G, y \in H, G \cap H \sim \phi$.

Definition 2.14; “A space $X$ is said to be $\sim$-compact (resp. $\sim$-compact (Jankovic et al., 1988) if and only if every cover of $X$ by $\theta$-open (resp. $\sim$-open) sets has a finite subcover”.

Definition 2.15; (Porter and Thomas, 1969): “A subset $A$ of a topological space $(X, \tau)$ is said to be quasi $H$-closed relative to $X$ if $\{E : i \in I_{E}\}$ each cover of $A$ by open sets of $X$, there exist a finite subset $I_{E}$ of $I$ such that $A \subset \bigcup \{E : i \in I_{E}\}$.”

Definition 2.16; (Porter and Thomas, 1969): “A space $X$ is said to be quasi $H$-closed if $X$ is quasi $H$-closed relative to $X$."

Definition 2.17; (Noiri, 1975): A function $f: X \to Y$ is said to be $\theta$-closed (resp. $\theta^{**}$-closed (Long and Herrington, 1977), semi-$\theta$-closed (Dubu et al., 1998), $\theta$-$\theta$-closed (Abdul-Jabbar, 2000), almost inequivalently $\theta$-$\theta$-closed and inequivalently $\theta$-$\theta$-closed graph if and only if for $x \in X$ and each $y \in Y$ such that $y \neq f(x)$, there exist an open (resp. semi-open, open, semi-open and semi-open, semi-open and semi-open) $U$ containing $x$ in $X$ and an open (resp. open, semi-open, open, open and open) set $V$ containing $f(x)$ in $Y$ such that $(U \times V) \cap f^{-1}(U) \sim \phi$ (resp. $(U \times V) \cap f^{-1}(U) = \phi$).

MATERIALS AND METHODS

$\sim$-Inequivalently $\theta$-coherence

Definition 3.1: By Chae et al. (1993) “A function $f: X \to Y$ is said to be $\sim$-inequivalently $\theta$-coherence if for each $x \in X$ and each $\sim$-open set $H$ of $Y$ containing $f(x)$, there exist an open set $U$ of $X$ containing $x$ with the end goal that $f(U) \subset H$.”

Theorem 3.1: For a function $f: X \to Y$, $\tau(Y, \gamma)$ the accompanying proclamations are proportionality:

- $f$ is $\sim$-inequivalently $\theta$-coherence
- $f: (X, \tau(Y, \gamma))$ is unequivalently $\sim$-irresolute

Theorem 3.2: In the event that a function $f: X \to Y$ inequivalently $\theta$-coherence at that point for each $x \in X$ and each $\sim$-open set $H$ of $Y$ containing $f(x)$, there exist a $\theta$-open set $N$ of $X$ containing $x$ with the end goal that $f(N) \subset H$. The evidence of the above theorems are not hard and along these lines, they are precluded.

Theorem 3.3: For a function $f: X \to Y$, $\tau(Y, \gamma)$ the accompanying articulations are proportionality:

- $f$ is $\sim$-inequivalently $\theta$-coherence
- For each point $x \in X$ and each filter base $\Psi$ in $X$ $\theta$-converging to $x$, the filterbase $f(\Psi)$ converges to $f(x)$ in $(Y, \tau(Y))$.
- For each point $x \in X$ and each net $\{x_i\}_{i \in I}$ in $X$ $\theta$-converging to $x$, the net $f(x_i)$ converges to $f(x)$ in $(Y, \tau(Y))$.
- For each point $x \in X$ and each filter base $\Psi$ in $X$ $\theta$-converging to $x$, the filterbase $f(\Psi)$ $\alpha$-converges to $f(x)$ in $(Y, \gamma)$.
- For each point $x \in X$ and each net $\{x_i\}_{i \in I}$ in $X$ $\theta$-converging to $x$, the net $f(x_i)$ converges to $f(x)$.

Proof: (i)$\Rightarrow$(ii)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv)$\Rightarrow$(v) follows, immediately from Definition 3.1 and Theorem 2 of (Chae et al., 1995).

Lemma 3.1; (Andrijevic, 1984): Let $X$ be a topological space and $\alpha \subseteq X$. At that point the accompanying are hold:

- $\alpha Cl(E) = E \cup Cl(\alpha Int(E))$.
- $\alpha Int(E) = E \cap Cl(\alpha Int(E))$. 
Theorem 3.4: For a function \( f: X \rightarrow Y \) the accompanying articulations are comparability:

- \( f \) is \( \sim \)-unequivocally \( \theta \)-coherence
- \( f(\text{Cl}(A)) \subseteq \text{Cl}(f(\text{Int}(\text{Cl}(f(A))))), \) for every subset \( A \) of \( X \)
- \( \text{Cl}(f(E)) \subseteq f(\text{Cl}((\text{Int}(\text{Cl}(f(E))))), \) for every subset \( E \) of \( Y \)
- \( f(\text{Cl}(\text{Int}(\text{Cl}(f(E)))))) \subseteq \text{Int}(f(\text{Cl}(E))), \) for every subset \( E \) of \( Y \)

Proof: This follows from Lemma 3.1 and Theorem 2 of (Chae et al., 1995).

Theorem 3.5: If a function \( f: X \rightarrow Y \) is \( \sim \)-unequivocally \( \theta \)-coherence and if \( E \) is an open subset of \( X \), then \( f(E,E^\circ) \) is \( \sim \)-unequivocally \( \theta \)-coherence in the subspace \( E \).

Proof: Let \( H \) be any \( \sim \)-open subset of \( Y \). Since, \( f \) is \( \sim \)-unequivocally \( \theta \)-coherence. Therefore, by [7, theorem 2], \( f^{-1}(H) \cap \theta^0(X) \), so by Lemma 1.2.9 of (Abdul-Jabbar, 2000) \( (f|E')^{-1}(H) = f^{-1}(H) \cap \theta^0(E) \). This implies that \( f|E:E \rightarrow Y \) is \( \sim \)-unequivocally \( \theta \)-coherence.

Theorem 3.6: For any two functions, \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \), the accompanying are valid:

- \( f \) is \( \sim \)-unequivocally \( \theta \)-coherence and \( g \) is \( \sim \)-continuous, then \( g \circ f \) is unequivocally \( \theta \)-coherence
- \( f \) is faintly continuous and \( g \) is \( \sim \)-unequivocally \( \theta \)-coherence, then \( g \circ f \) is unequivocally \( \sim \)-irresolute

Theorem 3.7: "Let \( f: X \rightarrow Y \) be faintly continuous and \( \theta \)-open function and \( g: Y \rightarrow Z \) be a function. Then \( g \circ f: X \rightarrow Z \) is unequivocally \( \tau \)-continuous if and only if \( g \) is unequivocally \( \theta \)-coherence."

Proof: Let \( g \circ f: X \rightarrow Z \) be \( \sim \)-equivocally \( \theta \)-coherence and \( H \subseteq \Theta^0(Z) \). Then \( (g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H)) \subseteq \Theta^0(X) \). Since, \( f \) is \( \theta \)-open function, \( \text{Int}(\text{Cl}(f(H))) \subseteq \Theta^0(Y) \). Hence, \( g^{-1}(H) \subseteq \Theta^0(Y) \). Thus, \( g \) is \( \sim \)-equivocally \( \theta \)-coherence. It is easy to prove the opposite and is thus omitted.

Theorem 3.8: If \( g: Y \rightarrow Z \) be a one to one \( \sim \)-open function on Yonto and \( g \) is \( \sim \)-equivocally \( \theta \)-continuous. Then \( f \) is unequivocally \( \theta \)-coherence.

Proof: Suppose \( g \) is \( \sim \)-open function. Let \( H \) be an open subset of \( Y \), since, \( g \) is one to one and onto, then the set \( g(H) \) is an \( \sim \)-open subset of \( Z \), since, \( g \circ f \) is \( \sim \)-equivocally \( \theta \)-coherence, it follows that \( (g \circ f)^{-1}(g(H)) = f^{-1}((g \circ f)^{-1}(g(H))) = f^{-1}(H) \) is \( \sim \)-open in \( X \). Thus, \( f \) is unequivocally \( \theta \)-continuous.

Theorem 3.9: If \( X \) is almost regular and \( f: X \rightarrow Y \) is completely \( \sim \)-irresolute function \( f \) is \( \sim \)-equivocally \( \theta \)-coherence.

Proof: Let \( H \) be an \( \sim \)-open subset of \( Y \), since, \( f \) is completely \( \sim \)-irresolute function, then \( f^{-1}(H) \) is regular open in \( X \) and from the fact that a space \( X \) is almost regular if and only if for each \( x \in X \) and each regular open set \( f^{-1}(H) \) containing \( x \), there exist a regular open set \( U \) such that \( x \in \text{cl}(U) \cap f^{-1}(H) \). Therefore, \( \theta \)-open in \( X \) and by [7, theorem 2], \( f \) is \( \sim \)-equivocally \( \theta \)-continuous.

Lemma 3.2: (Chae et al., 1986): Let \( \{X_i, \lambda \in \Delta \} \) be a family of spaces and \( U_{\lambda i} \), subset of \( X_i \), for each \( i = 1, 2, \ldots, n \). Then \( U = \bigcap_{i=1}^{n} U_{\lambda i} \times \text{Int}(X_i) \) is \( \sim \)-open in \( U_{\lambda i} \times X_i \) if and only if \( U_{\lambda i} \times \Theta^0(X_i) \) for each \( i = 1, 2, \ldots, n \).

Theorem 3.10: Let \( g_i: X_i \rightarrow Y_i \) be a function for each \( \lambda \in \Delta \) and \( g: \bigcap_{i=1}^{n} X_i \rightarrow Y_i \), a function defined by \( g(x_i) = \{g_i(x_i)\} \) for each \( x_i \in X_i \). If \( f \) is \( \sim \)-equivocally \( \theta \)-coherence, then \( g \circ f \) is \( \sim \)-equivocally \( \theta \)-coherence for each \( \lambda \in \Delta \).

Proof: "Let \( \beta \in \Delta \) and \( V_\beta \subseteq \Theta^0(Y_\beta) \). Then, by Lemma 3.2, \( V_\beta \subseteq \Pi_{\lambda \in \Delta} Y_i \) is \( \sim \)-open in \( \Pi_{\lambda \in \Delta} X_i \) and \( g^{-1}(V_\beta) = \bigcap_{i=1}^{n} X_i \subseteq \Pi_{\lambda \in \Delta} Y_i \) is \( \sim \)-open in \( \Pi_{\lambda \in \Delta} X_i \). From Lemma 3.2, \( g^{-1}(V_\beta) \subseteq \Theta^0(X_i) \). Therefore, \( g \circ f \) is \( \sim \)-equivocally \( \theta \)-coherence.

Remark 3.1: It was known in [6, example 2.2] that \( V \subseteq \Theta^0(X \times Y) \) may not, generally, be a union of sets of the form \( A \times B \) in the product space \( X \times Y \) where \( A \subseteq \Theta^0(X) \) and \( B \subseteq \Theta^0(Y) \). Therefore, the converse of Theorem 3.10 may not be true, generally.

Theorem 3.11: Let \( g: X \rightarrow Y \) be a function for each \( \lambda \in \Delta \) and \( Y_i \) is a regular space.

Proof: Let \( x \) be any point in \( X \) and \( H \) be any \( \sim \)-open set in \( X \) containing \( f(x) - x \), then by Lemma 3.2, \( H \times Y \) is \( \sim \)-open in \( Y_i \times Y \) which contain \( (x_i, x) \). Since, \( g \) is \( \sim \)-equivocally \( \theta \)-continuous, there exist an open set \( U \) containing \( x \) such that \( g(\text{Cl}(U)) \subseteq H \times Y \). Then \( f(\text{Cl}(U)) \subseteq f(\text{Cl}(U)) \subseteq H \times Y \). Therefore, \( f(\text{Cl}(U)) \subseteq H \times Y \). Hence, \( f \) is \( \sim \)-equivocally \( \theta \)-coherence. Similar statement for \( f \), is \( \sim \)-equivocally \( \theta \)-coherence.

Lemma 3.3: Let \( X_1, X_2, \ldots, X_n \) be \( n \) topological spaces and \( \mu = \Pi_{i=1}^{n} X_i \). Let \( E \subseteq \Theta^0(X_i) \) for \( i = 1, 2, \ldots, n \), then \( \Pi_{i=1}^{n} E \subseteq \Theta^0(\mu) \).
Proof: Let \((x_1, x_2, ..., x_n)\) be an element of \(\prod_i E_i\), then \(x_i \in E_i\) for \(i = 1, 2, ..., n\). Set \(U_i = \prod_j E_j\) for \(i = 1, 2, ..., n\). Therefore, \((x_1, x_2, ..., x_n) \in U_1 \times U_2 \times ... \times U_n\). Then \(f((x_1, x_2, ..., x_n)) = f(x_1) \times f(x_2) \times ... \times f(x_n) = \prod_i E_i \times \prod_i E_i \times ... \times \prod_i E_i = \prod_i E_i \times \prod_i E_i = (\prod_i E_i, \prod_i E_i)\) is \(\theta\)-open set in \(\prod_i E_i\).

**Theorem 3.12:** Let \(X_1, X_2, ..., X_n\) and \(Z\) be topological spaces and \(\prod_i E_i \to Z\). If given any point \(p\) of \(X_1, X_2, ..., X_n\), \(X_p\) be \(n\) topological spaces and \(\prod_i E_i\), and given any \(\alpha\)-open set \(U\) containing \(f(p)\), there exist \(\theta\)-open set \(E_i\) in \(X_i\), for \(i = 1, 2, ..., n\), such that \(p \in E_i\) and \(f(E_i) \subseteq U\). Then \(f\) is \(\alpha\)-equivocally \(\theta\)-coherence.

Proof: Let \(p \in X\) and \(U\) be any \(\alpha\)-open set in \(Z\) containing \(f(p)\), there exist \(\theta\)-open set \(E_i\) in \(X_i\), for \(i = 1, 2, ..., n\), such that \(p \in X\) and \(f(E_i) \subseteq U\). Since, \(E_i \in \theta(X)\) for \(i = 1, 2, ..., n\). Therefore, by Lemma 3.3, \(E_i \in \theta(\prod_i E_i)\), for \(i = 1, 2, ..., n\). Thus, \(f\) is \(\alpha\)-equivocally \(\theta\)-coherence.

**RESULTS AND DISCUSSION**

\(\alpha\theta\)-Open Function: In this area, new kind of function called \(\alpha\theta\)-open function study and we discover some portrayal and properties for it.

**Definition 4.1:** A function \(f: X \to Y\) is called \(\alpha\theta\)-open if and only if for each \(\alpha\)-open set \(G\) in \(Y\), \(f^{-1}(G) \in \theta(X)\). Let the open set \(G\) that takes over quickly that each \(\alpha\theta\)-open function is quasi \(\alpha\)-open and thus, unequivocally \(\alpha\)-open, the opposite is not valid as observed from the accompanying illustration.

**Example 4.1:** Let \(X = \{a, b, c, d\}\) and \(Y = \{x, y, \{a\}, \{e\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}\). The identity function \(i: X \to Y\) is unequivocally \(\alpha\)-open and not is \(\alpha\theta\)-open function, since, \(\{a\} \in \alpha \theta(X, \tau)\) but \(i(\{a\}) = \{a\} \notin \theta(Y, \tau)\). We discover a few portrayals and properties of \(\alpha\theta\)-open function.

**Theorem 4.1:** For any bijection function \(f: X \to Y\), the accompanying are proportionate:

- The inverse function is \(\alpha\)-equivocally \(\theta\)-coherence
- \(f: X \to Y\) is \(\alpha\theta\)-open function

The following lemmas are used in sequel.

**Lemma 4.1:** (Abdul-Jabbar, 2000): The accompanying is valid, for each subset \(E\) of \(X\):

\[
X / \text{Cl}_E(E) = \text{Int}_E(X / E)
\]

**Lemma 4.2:** The accompanying is valid for every subset \(E\) of \(X\):

\[
X / \text{Cl}_E(E) = \alpha \text{Int}_E(X / E)
\]

**Theorem 4.2:** For a function \(f: X \to Y\) the accompanying are equal:

- \(f\) is \(\alpha\theta\)-open function
- \(f(\alpha \text{Int}(E)) = \alpha \text{Int}(f(E))\), for each subset \(E\) of \(X\)
- \(f(\alpha \text{Int}(f^{-1}(W))) = \alpha \text{Int}(f^{-1}(W))\), for each subset \(W\) of \(Y\)

Proof: (a) \(\Rightarrow\) (b) Suppose \(f\) is \(\alpha\theta\)-open function and \(E \subseteq X\). Since, \(\alpha \text{Int}(E) = f(\alpha \text{Int}(E)) = f(\alpha \text{Int}(f^{-1}(W)))\), and hence, \(f(\alpha \text{Int}(E)) = \alpha \text{Int}(f(E))\). Let \(W \subseteq Y\). Then \(f^{-1}(W) \subseteq X\), therefore, we apply (b), we obtain \(f(\alpha \text{Int}(f^{-1}(W))) = \text{Int}(f^{-1}(W))\). Then \(\alpha \text{Int}(f^{-1}(W)) = f(\text{Int}(f^{-1}(W)))\). (c) \(\Rightarrow\) (d): let \(W \subseteq Y\), then apply (c) to \(W \subseteq Y\), we get \(\alpha \text{Int}(f^{-1}(W)) = f(\text{Int}(Y \cap f^{-1}(W)))\). Then \(\alpha \text{Int}(f^{-1}(W)) = \alpha \text{Int}(f^{-1}(W))\). Therefore, \(f^{-1}(W) \subseteq X\). Which completes the proof.

**Remark 4.1:** Let \(f: X \to Y\) be a bijection function. Then, \(f\) is \(\alpha\theta\)-open function if and only if \(f(F) \in \theta(Y, \tau)\), for each \(\alpha\)-closed set \(F\) in \(X\).

**Theorem 4.3:** If \(Y\) is regular space, then each \(\alpha\theta\)-open function is \(\alpha\theta\)-open.

Proof: Given \(G\) a chance to be any \(\alpha\)-open subset of \(X\), then it is semi-open. Since, \(f\) is \(\alpha\theta\)-open function. Therefore, \(f(G)\) is open in \(X\). But \(Y\) is regular space, then by [1, Lemma 1.2.8] \(f(G)\) is \(\theta\)-open in \(Y\). Which completes the proof.

**Theorem 4.4:** In the event that \(f: X \to Y\) is \(\theta\)-open function and \(E \subseteq X\) is an open set in \(X\), at that point the \(f(E) \subseteq Y\) is \(\alpha\theta\)-open function.

Proof: Let \(H\) be any \(\alpha\)-open set in the open subspace \(E\). At that point, by [15, Theorem 3.7], \(H\) is \(\alpha\)-open in \(X\). Since, \(f\) is \(\alpha\theta\)-open function. In this way, \(f(H)\) is \(\theta\)-open in \(Y\). Hence, \(f(E) \subseteq \alpha\theta\)-open function.
Theorem 4.5: Given \( f: X \to Y \) be a function and \( \{E_\alpha: \alpha \in \mathcal{V}\} \) be an open cover of \( X \). If the restriction \( f|_{E_\alpha} \), \( \alpha \in \mathcal{V} \), is \( \sim_0 \)-open function for each \( \alpha \in \mathcal{V} \), then \( f \) is \( \sim_0 \)-open function.

Proof: Give \( H \) a chance to be any \( \sim_0 \)-open set in \( X \). In this manner, by [15, Theorem 3.4], \( H \cap E_\alpha \) is \( \sim_0 \)-open in the subspace \( E_\alpha \) for each \( \alpha \in \mathcal{V} \). Since, \( f|_{E_\alpha} \) is \( \sim_0 \)-open function \( (f|_{E_\alpha})(H \cap E_\alpha) \) is \( \theta \)-open in \( Y \) and hence, \( f(H) = \bigcup \{f|_{E_\alpha}(H \cap E_\alpha): \alpha \in \mathcal{V}\} \). This demonstrate \( f \) is \( \sim_0 \)-open function.

Remark 4.1: Unmistakably \( \theta \)-compact and quasi \( H \)-closed equivalent from theorem 2.11 of (Ahmed and Yunis, 2002).

Theorem 4.6: In the event that \( f: X \to Y \) is \( \sim_0 \)-open function and \( f(F) \) is \( \sim_0 \)-compact relative to \( Y \), then \( F \) is \( \sim_0 \)-compact subspace relative to \( X \).

Proof: Let \( \{E_\alpha: \alpha \in \mathcal{V}\} \) be an open cover of \( F \), then \( \{f(E_\alpha): \alpha \in \mathcal{V}\} \) is cover for \( f(F) \). Since, \( f \) is \( \sim_0 \)-open function.

Conversely, let \( G \) be any \( \sim_0 \)-open subset of \( X \) and put \( S = Y \setminus f \). Then \( X \setminus S \) is \( \sim_0 \)-closed set containing \( f(G) \). By hypothesis, there exist a \( \theta \)-closed set \( M \) in \( Y \) containing \( S \) such that \( f(M) \subseteq X \setminus S \). Thus, we have \( f(M) \subseteq Y \setminus M \). On the other hand, we have \( f(G) = Y \setminus S \subseteq Y \setminus M \) and hence, \( f(G) \subseteq Y \setminus M \). Consequently, \( f(G) \) is \( \theta \)-open in \( Y \) in \( \sim_0 \)-open function.

Function with \( (\sim_0, \theta) \)-closed graph: In this area, we examine new properties of \( (\sim_0, \theta) \)-closed graph (Chae et al., 1995). Definition 5.1 (Chae et al., 1995). Let, \( f(G) = \{(x, f(x)): x \in \mathcal{X}\} \) be the graph of \( f: X \to Y \), then is said to be \( (\sim_0, \theta) \)-closed with respect to \( X \setminus Y \), if for each point \( (x, f(x)) \in \mathcal{G}(f) \), there exist an open set \( U \) and an \( (\sim_0, \theta) \)-open set \( H \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap H = \emptyset \). The accompanying diagram is a growth of the graph 4.1.1 of (Abdul-Jabbar, 2000). None of the suggestions is reversible (Fig. 1).

Example 5.1: Let \( X = \{a, b, c\} \) and, \( \tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\} \), then the function \( f(X, \tau) \subseteq (Y, \tau) \) defined as \( f(x) = a \), for each \( x \in \mathcal{X} \) has \( \theta \)-closed graph which has not \( (\sim_0, \theta) \)-closed graph.

Theorem 5.1: If \( f: X \to Z \) is a function with \( (\sim_0, \theta) \)-closed graph and \( f(X) \subseteq \equiv \)-equivalently \( \theta \)-coherence functions, then the set \( \{(x, y): f(x) = g(y)\} \) is \( \theta \)-closed in \( X \setminus Y \).

Proof: Let \( E = \{(x, y): f(x) = g(y)\} \). If \( (x, y) \in X \setminus Y \setminus E \), then \( f(x) \neq g(y) \). Hence, \( (x, g(y)) \in \{X \setminus Y \setminus E \setminus g(y) \} \). Since, \( f \) has \( (\sim_0, \theta) \)-closed graph. Therefore, there exist open set \( U \) and \( (\sim_0, \theta) \)-equivalently \( \theta \)-coherence of \( g \) implies that there is an open set \( V \) of \( X \) such that \( g(V) \subseteq H \). Therefore, we have \( f(U) \setminus g(V) = \emptyset \). This established that \((f(U) \setminus g(V)) \cap E = \emptyset \) which implies that \( (x, y) \notin \text{Cl}_E E \). hence, \( E \subseteq Y \setminus X \).

Corollary 5.1: If is an Hausdorff space and \( f, g: X \to Y \) are \( (\sim_0, \theta) \)-equivalently \( \theta \)-coherence functions, then the set \( \{(x, y): f(x) = g(y)\} \) is \( \theta \)-closed in \( X \setminus Y \).
Theorem 5.2: If \( f : X \to Y \) is any function with \( \theta \)-closed point inverses such that the image of each closure of open set is \( \sim \)-closed, then has \( \sim \)-closed graph.

Proof: Let \( (x, y) \in X \times Y(f) \). Then \( x \in f^{-1}(y) \) and since, \( f^{-1}(y) \) is \( \theta \)-closed, there exist an open set \( U \) containing \( x \) such that \( \overline{U} \cap f^{-1}(y) = \varnothing \). It follows that \( f(\overline{U}) \) is \( \sim \)-closed therefor. Because, there is an \( \sim \)-open set \( H \) in \( Y \) containing \( y \) such that \( f(\overline{U}) \cap H = \varnothing \). Thus, \( f(\overline{U}) \) has \( \sim \)-closed.

Theorem 5.3: Let \( f : X \to Y \) be given function with \( \sim \)-closed graph, then for each \( x \in X \), \( \{ f(x) \} = \cap \{ \text{Cl}(f(\overline{U})) : U \text{ is an open set of } X \} \).

Proof: Let the graph of the function be \( \sim \)-closed. Then it is claimed that for each \( x \in X \), \( \{ f(x) \} = \cap \{ \text{Cl}(f(\overline{U})) : U \text{ is an open set of } X \} \).

For if not, so, let \( y \neq f(x) \) such that \( y \notin \cap \{ \text{Cl}(f(\overline{U})) : U \text{ is an open set of } X \} \). Which implies that \( y \notin \text{Cl}(f(\overline{U})) \) for each open set of \( x \); it means that, for each \( \sim \)-open set \( V \) of \( y \) in \( Y \), \( V \cap f(\overline{U}) \neq \varnothing \). Thus, we obtain that \( (x, y) \notin f(\overline{U}) \) and there exist \( U \) and \( V \) such that \( V \cap f(\overline{U}) \neq \varnothing \) implies that \( (x, y) \) is contradiction. Thus, \( y = f(x) \).

Theorem 5.4: Let \( f : X \to Y \) be a function with \( \sim \)-closed graph. If is quasi \( H \)-closed in \( X \), then \( f(E) \) is \( \sim \)-closed in \( Y \).

Proof: Let \( E \) be a quasi \( H \)-closed in \( X \). Suppose that \( f(E) \) is not \( \sim \)-closed in \( Y \). Let \( y \notin f(x) \) for each \( x \in E \). Since, \( y \notin f(E) \). Therefore, there exist \( \overline{U_0} \) \( \sim \)-open set \( B_0 \) containing \( x \) and \( y \), respectively such that \( f(\overline{U_0}) \cap B_0 = \varnothing \), for each \( x \in E \). The family \( Q = \{ U_0 : x \in E \} \) is an open cover of \( E \). Since, \( E \) is quasi \( H \)-closed, there exist a finite subfamily \( \{ U_0,...,U_{m0} \} \) of \( Q \) such that \( V \in \bigcup_{i=1}^{m0} U_{i} \). Put \( H = \cap_{i=1}^{m0} U_{i} \). Then:

\[ f(E) \cap H \subset \bigcap_{i=1}^{m0} f(U_{i}) \cap E \subset \bigcap_{i=1}^{m0} (f(U_{i}) \cap E) = \varnothing \]

Since, \( H \) is an \( \sim \)-open set containing \( y \), \( \exists \sim \)-Cl(\( f(E) \)). Therefore, \( \sim \)-Cl(\( f(E) \)).

Corollary 5.2: The image of any quasi \( H \)-closed space in any space is \( \sim \)-closed under functions with \( \sim \)-closed graphs.

Theorem 5.4: Let \( f : X \to Y \) be a given function. Then is \( f \) is \( \sim \)-closed graph if and only if for each filter base \( \Psi \) in \( X \) \( \sim \)-converges to some \( p \) in \( X \), \( f(\Psi) \sim \)-converges to some \( q \) in \( Y \), \( f(p) = q \).

Proof: Suppose that \( \exists \), then \( \sim \)-closed graph and \( \Psi \) be a filter based in \( X \) such that \( \Psi(0) \sim \)-converges to \( p \) and \( f(\Psi(0)) \sim \)-converges to \( q \). If \( f(p) = q \), then \( (q, p) \notin f(\sim) \). Thus, there exist open set \( U \subset X \) and \( \sim \)-open set \( V \subset Y \) containing \( p \) and \( q \), respectively, such that \( \overline{U} \cap f(\sim) = \varnothing \). Since, \( \sim \)-converges to \( p \) and \( f(V) \sim \)-converges to \( q \), there exist an \( E \subset \Psi \) such that \( E \subset U \) and \( f(E) \subset V \). Consequently, \( \overline{U} \cap f(\sim) = \varnothing \) which is a contradiction.

Conversely, assume \( f(\sim) \) that is not \( \sim \)-closed graph. Then, there exist a point \( (p, q) \notin f(\sim) \) such for each open set \( U \subset X \) and \( \sim \)-open set \( V \subset Y \) containing \( p \) and \( q \), respectively, such that \( \overline{U} \cap f(\sim) = \varnothing \). Define:

\[ \Psi_1 = \{ \overline{U} : U \subset X \text{ is an open set containing } p \text{ and } \alpha \in V \} \]
\[ \Psi_2 = \{ V : \beta \in \text{ is an } \sim \text{ open set containing } q \} \]
\[ \Psi_3 = \{ E(\alpha, \beta) : E(\alpha, \beta) = \{ \overline{U} \times V \} \subset G(f), (\alpha, \beta) \in V \times V \} \]
\[ \Psi = \{ \Psi(\alpha, \beta) : (\alpha, \beta) \in V \times V \} \text{ where } \Psi(\alpha, \beta) = \{ x \in U \subset X, (x, f(x)) \in (\Psi)^{*}(\alpha, \beta) \} \}

Then, \( \Psi \) is a filter base in \( X \) with property that \( \Psi \sim \)-converges to \( p \) and \( f(\Psi) \sim \)-converges to \( q \) and \( f(p) \neq q \).

Corollary 5.3: A function \( f : X \to Y \) be has \( \sim \)-closed graph if and only if for each set \( X \) in \( X \) such that \( X \sim \)-open \( X \) and \( f(x) \sim \)-open \( Y \), \( f(x) = q \).

CONCLUSION

This study briefly described the \( \theta \)-open function, quasi \( \theta \)-open and \( \theta \)-open their properties in this research.

REFERENCES


