Research Journal of Applied Sciences 1 (1-4): 11-15, 2006

Asymptotic Behaviour of an Abstract Delay-Differential Equation

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Abstract: We formulate and analyze the asymptotic behaviour of solutions of a simple retarded functional differential Equation in a suitable Banach space. Conditions that guarantee existence, uniqueness and convergence of solutions for any initial distribution are given.

Key words: Delayed differential equation, asymptotic behaviour, population dynamics, resolvent operator, C_0-semigroup

INTRODUCTION

Three basic approaches are commonly used in solving population dynamics models, namely: classical, variational and abstract. The classical method yields the close form solution if the model is easily tractable. When faced with complex model equations, a major task for a mathematical model builder is at least to ascertain whether or not a solution exists and if so, is it unique? and what is the ultimate behaviour of the system? In general, classical solutions are not to be expected always. In order to circumvent the difficulty in ascertaining whether a solution exists or not, we are left with the abstract and variational formulations. The calculus of variation seeks to combine trial functions into satisfactory approximate solutions Rektorys[1].

Age structure is critical to the evolutionary dynamics of biological populations. Consequently, the Kermack-McKendrick model for age-structured populations and its variants has play an important role in demography and remains the preferred theoretical paradigm in the study of human and animals demography. A brief review of some related works provides the context of this study.

Garroni and Langlais[2] studied an age-structured dependent population diffusion with an external constraint using the variational approach. They considered the existence and uniqueness of solutions under a weak hypothesis and rediscovered all the biologically intuitive properties connecting the densities of the population to other parameters of the problem. Using this same technique with some suitable modifications, Tchuenche[3] studied the existence and uniqueness of a weak solution of a population dynamics problem with an additional structure.

Nevertheless, the abstract formulation is also mathematically challenging and enables us, under some conditions to ascertain the existence of a unique mild solution (by weak or mild, we refer to the solution of an integral equation of the form (6)). The basic theory behind this formulation is the use of semi-group theory, which applies meaningfully to smooth functions in Goldstein[4]. A suitable change of variables transforms most population dynamics problem into linear/non-linear non-homogeneous abstract Cauchy problem of the form,

$$\frac{du(t)}{dt} = Au(t) + F(u(t))u(0) = u_0, \qquad (1)$$

which is well developed in the literature (Pazy). u(t) represents the population density of individuals at time t, with F being a Lipschitz perturbation of the generator of a strongly continuous semigroup A. For simplicity of discussion and without restriction to the generality, we shall not consider Partial Functional Differential Equations (PFDEs) Memory[5], but their ordinary counterpart, because from the normal theory of Functional Differential Equations (FDEs), it is known that the Ordinary Differential Equation (ODE) giving the flow on the centre manifold for retarded FDEs at a singularity can be explicitly given in terms of the original FDE Faria[6]. For a brief review of the elements of the theory of PFDEs, Faria[7].

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Magal[8] discussed an abstract formulation of a population divided into several species and several patches. His model includes a term which may represent intra and inter-specific competition, fisheries and/or migration. He observed that since it is natural to introduce periodic births and mortalities in fisheries problem, this type of evolution problem can best be described at least approximately with some level of certainty by using an abstract formulation, together with the well-known integrated semi-group approach Thieme[9] and Iannelli[10] proved the existence and uniqueness of an age-structured SIS epidemic model with mixed inter/intra-cohort transmission, by reformulating their model equations as an abstract semi-linear equation. A concise analysis of linear and non-linear abstract Cauchy problem can be found in Pazy and Goldstein[11,4], to name a few.

Since only non-negative solutions are biologically significant and no one cares what these equations do when the variables are negative by Rosen and Prüb[12,13] studied a mathematical model of a population consisting of n species with age-specific interactions in a standard convex cone. His analysis is basically tailored towards the equilibrium solutions, with the operator A being linear, but unbounded. He also proved the well-posedness of under mild assumptions on the structural vital (birth and death) rates.

Various authors have used the abstract formalism in order to prove the well-posedness, existence and uniqueness of solutions to various population models Inaba and Iannelli[14,15]. Da Prato and Iannelli[16] derived an abstract setting for a boundary control problem of an age-dependent model and proved the stabilizability result. Here we note that their abstract problem is a variant of the classical Cauchy problem and has the following form:

$$q'(t) = Aq(t) - \eta Dv'(t)\; q(0) = q_0, \tag{2}$$

where v is the control, D is an operator and η is a parameter.

A variety of problems in differential equations (abstract/functional differential equations, age-dependent population models with and without delay) can be written as semilinear Cauchy problems with a Lipschitz perturbation of a closed linear operator which is not densely defined, but satisfies the resolvent estimates of the Hille-Yosida Thieme[17]. An illustration of this approach can be found in Thieme[17], where he analyses an age-structured population dynamics which models the growth of a plant population reproducing both by layers and by seeds. Our approach is somehow different from what is commonly found in the literature in the sense that we apply the inverse Laplace transform to the resolvent operator in order to prove the main Theorem. We wish to show that the origin together with its neighborhood is an absorbing state, while all other states outside the origin's neighbourhood are transient. The result obtained herein is a particular case of the one obtained previously by one of us Liadi[18]. An extension of both results can be found in Tchuencheb.

MODEL FRAMEWORK

Recognizing the role that resolvent and semigroup theories play in population dynamics and epidemiology, we consider the problem of analyzing the following delayed abstract functional differential Eq.

$$\dot{x} = Ax + F(x_t), x(0) = x_0;\; t \geq 0 \tag{3}$$

where A is a mapping from $D(A) \subset E$, x_t is the section of t of the function x, such that $x_t(s) = x(t+s)$, $s \in [-\gamma, 0]$, $\gamma > 0$ and E is the underlying Banach space, while F: $C([-\gamma,0]; E) \rightarrow E$ is a bounded linear operator.

Here we ask: which conditions guarantee that for any initial solution x_0 of (3), the solution x(t) converges?

Motivation: For any initial distribution starting near the origin, the solution of (3) for $t \geq 0$ converges asymptotically to zero.

Suppose that $\bar{x}$ is a unique steady state solution of (3). Then, the substitution $u = x - \bar{x}$ yields

$$\left.\begin{aligned} \dot{u}(t) &= Au(t) + F(u_t) + h(u) \\ u(0) &= u_0 \end{aligned}\right\} \tag{4}$$

where, A is a bounded linear operator and h(u) denotes the nonlinear term, which may represent the effect of the environment on the species (food shortage, overcrowding,…). In stochastic terms, h(u) may be referred to as `noise'.

Let L be the Laplace transform operator and denote

$$L\{u(t)\} \text{ by } u(\xi) L\{F(u_t)\} = f(\xi) \text{ and } L\{L(u)\} = L(u(\xi)).$$

Then, the following estimate holds Arino[19]

$$f(\xi) = 0\left(e^{|\xi|\gamma}|u(\xi)|\right), \quad \xi \rightarrow -\infty \tag{5}$$

By the variation of parameters formula, Eq. 4 becomes

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-\tau)}(F(u_\tau) + h(u(\tau)))d\tau \tag{6}$$

an integral equation which in the sense of Da Prato and Sinestrari[20] provides a generalized notion of solution.

Taking the Laplace transform of (6) with respect to t, with ξ as the transform variable Waston and Sawashima[21,22], we obtain:

$$u(\xi) = (\xi I - A)^{-1}\{u_0 + f(\xi) + h(u(\xi))\} \tag{7}$$

Theorem 1: If

- $|h(u(t))| \le \delta\, |u(t)|$, for $\delta > 0$ and
- The bounded inverse operator (ξ-IA) has a resolvent $H(\xi)$ whenever $\alpha < \mathrm{Re}(\xi) < 0$, then, the solution $x(t) \epsilon$ E converges asymptotically to zero.

Proof: From (7), we have:

$$|u(\xi)| \le \left|(\xi I - A)^{-1}\right|\left\{|u_0| + |f(\xi)| + h(u(\xi))\right\}$$

By the estimate in (5), we have:

$$|u(\xi)| \le \left|(\xi I - A)^{-1}\right|\left\{|u_0| + e^{|\xi|\gamma}|u(\xi)|\cdot\varepsilon(\xi) + \delta|u(\xi)|\right\}$$

Hence,

$$\left(I - \left(|\xi I - A|^{-1}\right)\left(e^{|\xi|\gamma}\varepsilon(\xi) + \delta\right)\right).|u(\xi)| \le |\xi I - A|^{-1}|u_0| \tag{8}$$

Let $(\xi I\text{-}A)^{-1}$ exist whenever $\mathrm{Re}(\xi) > \sigma$ for some $\sigma < 0$. Also, assume that the unique resolvent of $(\xi I - A)^{-1}(e^{|\xi|}.\varepsilon(\xi)+\delta)$ exists when. Thus, from Eq. 8 we have:

$$|u(\xi)| \le |\xi I - A|^{-1}|u_0| + H^*\left(|\xi I - A|^{-1}|u_0|\right)$$

with H^* being the unique resolvent

$$\left(\xi I - A\right)^{-1}\left(e^{|\xi|\gamma}\cdot\varepsilon(\xi) + \gamma\right) = H, \text{ say}$$

Then:

$$H^* = H + H^*H.$$

That is,

$$H^* = H(I-H)^{-1}$$

$$= H + o(H^2), \quad \text{as} \quad \|H\| \to 0, \tag{6}$$

but:

$$H^* = (\xi I - H)^{-1}(e^{|\xi|\gamma}\cdot\varepsilon(\xi) + \delta)$$
$$\{\frac{I}{\xi} + o(\frac{1}{\xi^2})\}(e^{|\xi|\gamma}\cdot\varepsilon(\xi) + \delta), \text{as} |\xi| \mapsto \infty \tag{7}$$

From Eq. 6 and 7,

$$H^* = \frac{e^{|\xi|\gamma}}{\xi}\varepsilon(\xi) + \frac{I}{\xi}\cdot\delta + o\left(\frac{1}{\xi^2}\right), \tag{8}$$

provided $|\xi|$ is large enough, with $|\xi| >> \delta$.
Let

$$\tilde{H}^* = U(t+\gamma) \tag{9}$$

where '~' represents the operation of Laplace inversion and

$$U(t+\gamma) = \begin{cases} 0 & , \ t < -\gamma \\ 1 & , \ t \ge -\gamma. \end{cases} \tag{10}$$

Let, $J(\xi) = (\xi I\text{-}A)^{-1}$ then for $\alpha < \sigma$,

$$J(\xi) = \xi^{-1}I + o(\xi^{-2}) \quad \text{as} \quad |\xi| \to \infty, \quad \alpha \le \mathrm{Re}(\xi) < 0.$$

If,

$$J(t) = o(e^{\alpha t}) \text{as}\, t \to \infty$$

Then,

$$u(t) \le e^{\alpha t}|u_0| + e^{\alpha t}|u_0|\int_{-\infty}^{t} e^{-\alpha\tau}U(\tau+\gamma)d\tau = e^{\alpha t}|u_0| +$$
$$e^{\alpha t}|u_0|\int_{-\gamma}^{0} e^{-\alpha\tau}d\tau = \left(1 + \frac{1}{\alpha}e^{\alpha\gamma} - \frac{1}{\alpha}\right)e^{\alpha t}|u_0|, \quad \forall\ t \ge 0. \tag{11}$$

Hence,

$$\lim_{t\to\infty}|u(t)| = 0. \tag{11}$$

Thus, the origin is an absorbing state and this terminates the proof.

In the study of structured population models, both linear and non-linear, the theory of semigroups has turned out to be a very powerful tool Sawashima[22]. Semigroups associated with solutions of structured population dynamics models have an important property: they are positive Gyllenberg[23]. This is biologically relevant since negative solutions are not important Watson[21]. Thus, the theory of semigroups plays a crucial role in the analysis of asymptotic properties of evolution equations. It also applies naturally to homogeneous dynamical systems Iannelli[10].

Let $T(t), t \geq 0$ be a linear C_0-semigroup with generator A on a Banach space E. $T(t)u_0$ may represent the size distribution of the population at time t, given the initial size distribution $u_0 \in E$ Gyllenberg[23]. A concise illustration can be found in Tchuenchea[24].

In the course of time, scientists were led to consider systems related to the Navier-Stokes equation for a homogeneous incompressible fluid in connection with concrete problems in natural sciences. Such systems can formally be reduced to an equivalent equation given by

$$\frac{du(t)}{dt} = Au(t) + F(u(t)) + gu(0) = u_0, \tag{12}$$

where g may represent `noise effects'. Indeed, the evolution of some mathematical models for the propagation of bacteria may have the same dynamics as Eq. 12.

Below, we propose a technique for investigating a simple abstract retarded functional differential equation with control $v \in L^2(0,\Omega)$, Ω is known as the time duration. The time lag t is assumed constant here for simplicity reasons. Thus, consider the following Eq.

$$\frac{du(t)}{dt} = Au(t) + F(t,u(t-\tau),v(t))u(0) = u_0, \tag{13}$$

This model equation covers many interesting problems, among which, heat, wave and beam equations are typical examples. The mild form of Eq. 13 is given by the following delayed control evolution Eq.

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-\tau)}F(t,u(t-\tau),v(t))d\tau, \tag{14}$$

where e^{At} is a C_0-semigroup on the underline Banach space E, in which the state space u(t) is defined, F: $[0,T] \times E \times V \to E$ is a given map with V being a metric space in which the control v takes on values. Also, Eq. 14 can be written as:

$$u(t) = T(t)u_0 + \int_0^t T(t-\tau)F(t,u(t-\tau),v(t))d\tau \tag{15}$$

If $u \in D(A)$, then, $t \to T(t)$ is differentiable. Initial-value problems are not always globally well-posed, even with weak assumptions. Nevertheless, (13) is well-posed if the resolvent set $\rho(A) \neq \emptyset$, or equivalently,

$\| T(t)u_0 \| \leq Me^{wt} \| u_0 \|, t \in [0,\Omega], w \leq 0, M > 0$. Here, $\|.\|$ is the usual L^1-norm and for brevity, we write L^1 for short. If $F(t,u(t-\tau), v(t))$ is Fréchet or Gâteaux differentiable, Lipschitz or uniformly Hölder continuous in its variables and locally Bochner integrable, then, by defining the left-hand side of (15) as an operator (Su)(t), say, we can show after some little algebra (the method of proof can be found in Goldstein and Tchuenchea[4,24] that the solution u(t) of (13) is unique. The condition of well-posedness given by $\|T(t)\| \leq Me^{wt}\|u_0\|$, relates naturally to Eq. 11, where

$$M \doteq 1 + \frac{1}{\alpha}e^{\alpha\gamma} - \frac{1}{\alpha}, \text{and } w = \alpha.$$

Theorem 2: If $\|T(t)\| = O(e^{\lambda t})$ and $F: L^1([0,A] \times \Omega) \to L^1([0,A] \times \Omega)$ is Lipschitzian, then Eq. 13 has a unique bounded weak solution, provided there exist constants $M > 0$ and $w < 0$.

Proof: In order to prove this Theorem, we first assume that $\| F(t,u,v,)\| \leq c\|u\|$, where c is a constant depending on the parameters t, u and the control v. Applying the usual L^1-norm to (15), we have:

$$\| u(t) \|_{L^1} \leq \| T(t)u_0 \|_{L^1} + \int_0^t \| T(t-\tau)F(t,u(t-\tau),v(t)) \| d\tau$$

$$\leq Me^{wt} \| u_0 \| + Me^{wt} \int_0^t e^{-w\tau} \| F(t,u(t-\tau),v(t)) \| d\tau \tag{16}$$

By a change of variable ($t-\tau = s$) and some little algebraic manipulations, Eq. 16 becomes

$$\| u(t) \|_{L^1} \leq Me^{wt} \| u_0 \| - cM \int_0^t e^{ws} \| u(s) \| ds \tag{17}$$

Instead of applying the well-known classical Gronwall's Lemma, which is not evident in this case, we find the limit of Eq. 17 as $t \to +\infty$.

First, we assume (17) may take the form $\| u(t) \|_{L^1} \Box cMe^{wt} \| u_0 \| \Box cM[u(t)e^{wt} \Box u_0]$, so that,

$$\lim_{t \to \infty} \| u(t) \| \Box c \Box u_0. \tag{18}$$

Hence, if any solution starts in the neighbourhood of the initial distribution u_0, then, at the long run, such a solution may eventually gets back to u_0, if $c \Box 1$ and may depart from it if c□1. Thus, c is a threshold parameter (0□c□1). If, then equations (11') and (18) coincide.

CONCLUSION

We began this study by applying the theory of resolvent to show that solutions of abstract delayed differential equations converge asymptotically to zero if starting in the neighbourhood of the origin. Also, using the theory of semigroups of operators, we have shown that solutions starting in the neighbourhood of the initial distribution will converge to it if the constact c, which depends on the parameters of the equation equals unity. These solutions are locally stable[24]. For values of c □ 1, solutions depart from the initial distribution. The origin, as well as u_0 are points of attraction. In a future study, we shall attempt to establish conditions under which solutions starting outside the neighbourhood of the origin move towards u_0 and vice-versa, as well as condition for which such solutions are unstable or move away from their initial path.

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