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Singular Perturbation Problems. A Study by A-Convergence

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Abstract: This research is devoted to singular perturbation problems associated to a thin layer of thickness ε. We study asymptotically as $\varepsilon \to 0$ the elliptic problem $-\text{div}(a_\varepsilon \nabla u_\varepsilon) + b_\varepsilon u_\varepsilon = f$, on$\Omega$ with the condition $u_\varepsilon = 0$ at the boundary $\partial\Omega$. The convergence under consideration is Γ-convergence with the strong topologie of $L^2(\Omega)$.

Key words: Singular perturbation, asymptotic behaviour, variational problem

INTRODUCTION

We are concerned with the study of the following elliptic problem

$$\begin{cases} -\text{div}(a_\varepsilon \nabla u) + b_\varepsilon u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where the coefficients (a_ε) and (b_ε) are taking respectively very small and very high values on a subset $\Sigma_\varepsilon \subseteq \Omega$ (a "thin layer" of thickness ε) and 1 on $\Omega \backslash \Sigma_\varepsilon$. ($\varepsilon$ is a parameter which converges to 0) and f is a given function in $L^2(\Omega)$. The most natural approach for these types of problems relies on *Γ*-convergence of the corresponding functionals. In this study we generalize a well known result due to Sanchez-Palencia (1980).

This problem has been studied in a variational way in Attouch (1984) where the conductivity coefficient a_ε is taking very small values on a small subset of Ω and the coefficient b_ε is equal to 0 and which is known as the conductivity equation. For details for this kind of problems, we refer to Huy and Sanchez-Palencia (1974) and Sanchez-Palencia (1969, 1970, 1974).

A BRIEF INTRODUCTION TO Γ-CONVERGENCE

The convergence under consideration is the Γ-convergence of functions especially designed in order to study convergence of solutions and their corresponding minimization problems: It is a "variational convergence". Such convergence was introduced by De Giorgi and Franzoni (1975) is nowadays a commonly-used tool of the calculus of variations. For further details about Γ-convergence in a general setting we refer to Braides and Defranceschi (1998) and Dal Maso (1993).

In what follows X is a metric space, u an element of X, F a function from X into $[0,+\infty]$. In the applications X will be a space of functions u on some open domain Ω of R^n and F a functional on X.

Definition: 2.1. Let X be a metric space and for $\varepsilon > 0$ let be given $F^\varepsilon : X \to [0,+\infty]$.We say that F^ε Γ-converge to F on X as $\varepsilon \to 0$ and we write: $F = \Gamma - \lim F^\varepsilon$,
if the following two conditions hold
(LB) Lower Bound inequality: For every $u \in X$ and every sequence (u_ε) such that $u_\varepsilon \in u$ in X, there holds

$$\liminf F^\varepsilon(u_\varepsilon) = F(u); \text{ as } \varepsilon \to 0 \quad (2.1)$$

(UB) Upper bound inequality: for every u? X there exists (u_ε) s.t. $u_\varepsilon \to u$ in X and

$$F(u) = \lim F^\varepsilon(u_\varepsilon), \text{ as } \varepsilon \to 0 \quad (2.2)$$

Remark 2.1: Condition (LB) means that whatever sequence we choose to approximate u, the value of $F^\varepsilon(u_\varepsilon)$ is, in the limit, larger than F(u), while condition (UB) implies that this bound is strong, that is, there always exists a recovery sequence (u_ε) which approximates u so that $F^\varepsilon(u_\varepsilon) \to F(u)$. Notice that if (LB) holds, then equality (2.2) can be replaced by

$$\limsup F^\varepsilon(u_\varepsilon) = F(u), \text{ as } \varepsilon \to 0. \quad (2.3)$$

This notion enjoys useful compactness properties and, under suitable equi-coerciveness assumptions, is strong enough to guarantee that minima and minimizers for problems related to $F^\varepsilon \to 0$ converge to the corresponding minima and minimizers for problems related to F. The proof of the "liminf inequality" is usually the most technical part in a Γ-convergence result, while the form of "recovery sequences" gives an insight of the nature of the convergence.

The main variational properties of Γ-convergence are the following

Theorem 2.1: Let F^ε a sequence of functions from X into $[0,+\infty]$ which is Γ-convergent to F

- Stability of minimizing sequences: Let us assume that there exists a minimizing sequence u_ε i.e.,

$F^\varepsilon(u_\varepsilon) = \inf F^\varepsilon(u) + \varepsilon_n$, with $\varepsilon_n \to 0$ which is relatively compact. Then

$\inf F^\varepsilon(u) \to \min F(u)$ as $\varepsilon \to 0$ and every cluster point of the sequence u_ε minimizes F over X.

- Stability under continuous perturbations: For every continuous function G, then

$\Gamma - \lim(F^\varepsilon + F) = F + G$

PROBLEM POSITION

Let Ω be a bounded open of R^N with smooth boudary $\partial\Omega$. For $x' = (x_1, x_2, ..., x_{N-1})$ and $\varepsilon > 0$ (period of a composite, perturbations, or a thickness of a thin layer), we define the following sets

$$\Sigma_\varepsilon = \{x = (x', x_N) \in \Omega / |x_N| < \varepsilon/2\},\ \Omega_\varepsilon = \Omega \backslash \Sigma_\varepsilon$$
$$\Omega_\varepsilon^\pm = \{x \in \Omega_\varepsilon : \pm x_N \geq 0\},$$
$$\Sigma_\varepsilon^\pm = \{x \in \Sigma_\varepsilon : \pm x_N \geq 0\},$$
$$\Omega^\pm = \{x \in \Omega\ : \pm x_N > 0\},\ \Sigma = \Omega \cap \{x_N = 0\}$$
$$\text{and } \partial\Omega^\pm = \partial\Omega \cap \{\pm x_N > 0\}$$

We define the coefficients

$$a_\varepsilon = \begin{cases} 1 & \text{if } x \in \Omega_\varepsilon \\ \lambda_\varepsilon & \text{if } x \in \Sigma_\varepsilon \end{cases}, \text{and}$$

$$b_\varepsilon = \begin{cases} 1 & \text{if } x \in \Omega_\varepsilon \\ 1/\mu_\varepsilon & \text{if } x \in \Sigma_\varepsilon \end{cases}$$

Where λ_ε and μ_ε are positive reals converging to zero as $\varepsilon \to 0$.

We are concerned with the asymptotic behaviour as $\varepsilon \to 0$ of the solution u_ε of the problem (1.1) or in a more general framework with the asymptotic analysis of variational problems of the form

$$\inf\left(F^\varepsilon(u) - \int_\Omega fu dx\ ,\ u \in H_0^1(\Omega) \right)$$

Where

$$\begin{cases} F^\varepsilon(u) = 1/2\int_\Omega (a_\varepsilon|\nabla u|^2 + b_\varepsilon|u|^2)dx \text{ if } u \in H_0^1(\Omega) \\ \qquad +\infty \quad \text{otherwise} \end{cases} \tag{3.1}$$

We show that the limit equation associated with (1.1) as $\varepsilon \to 0$ depends on the parameters k and k' defined by

$$k = \lim \frac{\lambda_\varepsilon}{\varepsilon} \text{ and } k' = \lim \frac{\mu_\varepsilon}{\varepsilon}, \text{ as } \varepsilon \to 0 \tag{3.2}$$

Where k and k' $\in [0,+\infty]$. The lack of coercivity of the solutions in the Sobolev space $H_0^1(\Omega)$ leads to the use of an extension technique (lemma 4.2).

Under some hypothesis we show that the solution of (3.3) or more precise its extension converges to u for the strong topology of $L^2(\Omega)$, where u is a solution of a transmission problem (corollary 4.5). The local character of the Γ-convergence reduces then the problem to a unidimensionnal variationnal one where a test-fonction will be constructed. Drawing inspiration from Sanchez-Palencia (1980) the integral on Σ, has the following form,

$$k\int_\Sigma [u]^2 d\sigma$$

Where $[u] = u^+ - u^-$, is the jump of u across Σ and u^+ (resp. u^-) is the trace of $u|_{\Omega^+}$ (resp. of $u|_{\Omega^-}$) on Σ. So that it is of the following form

$$\int_\Sigma (a(u^+)^2 + b(u^-)^2 + 2cu^+u^-)d\sigma$$

with a = b = k and c = -k. The question is naturally to ask how to know if the functional (Γ-limit) is of the form:

$$F(u) = 1/2\int_{\Omega\backslash\Sigma} (|\nabla u|^2 + |u|^2)dx$$
$$+1/2\int_\Sigma (a(u^+)^2 + b(u^-)^2) + 2cu^+u^-)d\sigma$$

If $u \in H^1(\Omega\backslash\Sigma)$ and u = 0 on Γ

and $F(u) = +\infty$ if $u \in L^2(\Omega)\backslash H^1(\Omega\backslash\Sigma)$,

Where the constants a, b and c depend on k et k'. u^+ (resp. u^-) are the traces of $u|\Omega^+$, (resp. $u|^-\Omega$) on Σ. The aim of this research is to give an answer to this question. This result is a generalization of those obtained by Attouch (1984) and Sanchez-Palencia (1980) in the case where a = b = k and c = -k.

RESULTS

Our purpose is to establish the result of Γ-convergence of the functional $F^\varepsilon(u)$. From the theorem 2.1, it suffies to find a topology for which the minimizing sequences are relatively compact. In fact, the solutions are not bounded in $H_0^1(\Omega)$ and this is due to the behaviour of $(u_\varepsilon)_{\varepsilon>0}$ inside Σ_ε. To overcome this difficulty we use an extension argument. For the passage to the limit, we reduce the problem to a one-dimensional one (lemma 4.6). From the following proposition we obtain the convergence in $L^2(\Omega)$-strong.

Proposition 4.1: Let λ_ε and μ_ε be two strictly positif real numbers, then, for every minimizing sequence $(u_\varepsilon)_\varepsilon>0$ verifying

a) $u_\varepsilon \to u$ in $L^1(\Omega)$-strong
b) There exist a constant $M>0$, such that

$F^\varepsilon(u_\varepsilon)\le M$, then u_ε converges to u in $L^2(\Omega)$-strong.

The hypothesis b) implies the existence of two constants $M_1 > 0$ and $M_2 > 0$ such that

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + |u_\varepsilon|^2 \le M_1$$
and
$$\int_{\Sigma_\varepsilon} \lambda_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\mu_\varepsilon}|u_\varepsilon|^2 \le M_2$$

The second inequality does not exclude the possibility that the gradient of u_ε maybe infinite and this suggests that the solutions will tend to a limit everywhere exept on Σ. We will show that this limit-function does not lies in $H^1_0\ \Omega_\varepsilon$. We then study the convergence in Ω_ε, after the modification of the solutions in Σ_ε. To be precise, we have the following result

Lemma 4.2: (*extension lemma*): There exists a linear and continuous operator P^+_ε (resp. P^-_ε) from $H^1_\varepsilon(\Omega^+_\varepsilon)$ to $H^1(\Omega^+)$, (resp. from $H^1(\Omega^-_\varepsilon)$) to $H^1(\Omega^-)$ such that:

$$\forall u^\pm_\varepsilon \in H^1(\Omega^\pm_\varepsilon) : P^\pm_\varepsilon u^\pm_\varepsilon = u^\pm_\varepsilon \text{ on } \Omega^\pm_\varepsilon$$

More precisely, μ^+_ε if is any function in $D(\overline{\Omega^+_\varepsilon})$, we define $\tilde{u}^+_\varepsilon$ by modifying $\tilde{u}^+_\varepsilon$ inside Σ^+_ε.
This is expressed by

$$P^+_\varepsilon u^+_\varepsilon = \tilde{u}^+_\varepsilon(x',x_N) = \begin{cases} u^+_\varepsilon(x',x_N) \text{ if } x \in \Omega^+_\varepsilon \\ u_\varepsilon(x',\varepsilon - x_N) \text{ if } 0 \le x_N \le \varepsilon/2 \end{cases}$$

The construction of P_ε for $(-\varepsilon/2\le x_N\le 0)$ is analogous and defined by

$$P^-_\varepsilon u^-_\varepsilon = \tilde{u}^-_\varepsilon(x',x_N) = \begin{cases} u^-_\varepsilon(x',x_N) \text{ if } x \in \Omega^-_\varepsilon \\ u_\varepsilon(x',-\varepsilon - x_N) \text{ if } -\varepsilon/2 \le x_N \le 0 \end{cases}$$

Under the hypothesis of proposition 4.1, we have

Lemma 4.3: $\tilde{u}^+_\varepsilon$ (resp. $\tilde{u}^-_\varepsilon$) is bounded in $H^1(\Omega^+)$, (resp. in $H^1(\Omega^-)$), the boundedness is independant of ε and we have $\tilde{u}^+_\varepsilon \to u^+$ in $L^2(\Omega^+)$-strong (resp. $\tilde{u}^-_\varepsilon \to u^-$ in $L^2(\Omega^-)$-strong), consequently, $u^+\in H^1(\Omega^+)$ and $u^-\in H^1(\Omega^-)$ i.e., $u \in H^1(\Omega\backslash\Sigma)$. In other words
Γ- lim $F^\varepsilon = +\infty$ on $L^2(\Omega)\backslash H^1(\Omega\backslash\Sigma)$.

We are now in position to estabilish our main result.

Theorem 4.4: We have
$L^2(\Omega)$-Γ - lim $F^\varepsilon(u_\varepsilon) = F(u)$, with

$$F(u) = \begin{cases} 1/2 \int_{\Omega\backslash\Sigma} (|\nabla u|^2 + |u|^2)dx + \\ \int_\Sigma (a(u^+)^2 + b(u^-)^2 + 2cu^+u^-)d\sigma \\ \text{if } u \in H^1(\Omega\backslash\Sigma) \text{ and } u = 0 \text{ on } \Gamma \\ +\infty \text{ if } u \in L^2(\Omega)\backslash H^1(\Omega\backslash\Sigma) \end{cases}$$

It suffices to find the coefficients in the quadratique form, functions of k and k' and find the limit problem equivalent to the minimization of

$$F(u) - \int_\Omega fu\,dx$$

Thus, we have

Corollary 4.5: (*The limit problem*).
u_ε converges in s-L^2 (Ω) to u (solution of the transmission problem), in $H^1(\Omega\backslash\Sigma)$.

$$\begin{cases} -\Delta u + u = f \quad \text{on } \Omega \\ \dfrac{\partial u^+}{\partial x_N} = au^+ + cu^- \quad \text{on } \Sigma \\ \dfrac{\partial u^-}{\partial x_N} = -cu^+ - bu^- \quad \text{on } \Sigma \end{cases}$$

To etabilish the upper bound inequality of definition 2.1, or (2.3) of remark (2.1), it suffices to construct a sequence $(u_\varepsilon)\in H^1_0(\Omega)$ such that $u_\varepsilon \to u$ in s-$L^2(\Omega)$ and lim supF^ε $(u^\varepsilon) \le F(u)$. For the construction of such a sequence, we take $\lambda_\varepsilon \approx k\varepsilon$ and $i_\varepsilon \approx k'\varepsilon$ in the expression

$$J_\varepsilon = \int_{\Sigma_\varepsilon} (\lambda_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\mu_\varepsilon}|u_\varepsilon|^2)dx$$

For a fixed x, let u_ε (x', x_N) = φ(x', x_N /ε) with x_N /ε= t; thus

$$J_\varepsilon = k\int_\Sigma (\int_{-1/2}^{+1/2} (|\varphi'(t)|^2 + \frac{1}{kk'}|\varphi(t)|^2)dt)d\sigma$$

such that
φ (-1/2) = á = u(x', -ε/2) = u^-_ε
and
φ (+1/2) = â = u(x', +ε/2) = u^+_ε

where u^+_ε and are respectively the traces of u on x_N = +ε/2 and on x_N =-ε/2.

We then have a one-dimensional variational problem which depends on two real parameters α and β:

Lemma 4.6: Let

$$f(\alpha,\beta) = \inf_{\varphi\in H^1[-1/2,+1/2]} \left\{\int_{-1/2}^{+1/2} (|\varphi'|^2 + \frac{1}{kk'}|\varphi|^2)dt\right\}$$

The solution $\varphi_{á,â}$ of the above problem exists in H^1 [-1/2, +1/2], where

i) $$\varphi_{\alpha,\beta}(t) = \frac{\beta\exp\delta\text{-}\alpha\exp(\text{-}\delta)}{\exp(2\delta)\text{-}\exp(\text{-}2\delta)}\exp(2t\delta) + \frac{\alpha\exp\delta\text{-}\beta\exp(\text{-}\delta)}{\exp(2\delta)\text{-}\exp(\text{-}2\delta)}\exp(-2t\delta)$$
with $\delta = 1/(2\sqrt{kk'})$

ii) f(α, β) = $A\alpha^2 + B\beta^2 + 2C\alpha\beta$, where,

$$A = B = \frac{1}{\sqrt{kk'}}\frac{\cosh 2\delta}{\sinh 2\delta}$$

and $$C = \frac{-2}{\sqrt{kk'}}\frac{1}{\sinh 2\delta}$$

To prove the convergence we use the following trace lemma

Lemma 4.7: Let $u^\pm_\varepsilon = u_\varepsilon(x',\pm\varepsilon/2)$ which converges to $u^\pm(x',0)$ $u^\pm(x',0)$ for the weak topology of $H^1(\Omega^\pm)$ then converges to $u_\varepsilon(x',\pm\varepsilon/2)$ for the strong topology of $L^2(\Sigma)$

Proof of theorem (4.4): We proceed into two steps, in the first step, we suppose u "*smooth*", then in a second step, we achieve by using a density argument. Let u be in $H^1(\Omega\backslash\Sigma)$ such that $u|_\Gamma = 0$. We put

$$\begin{cases} u & \text{if } x_N > \varepsilon/2 \\ \varphi_{\alpha,\beta}(x_N/\varepsilon) & \text{otherwise} \end{cases} \quad (4.1)$$

by application of *lemma 4.3* in the particular case where u^ε = u and we use then the result of *lemma 4.2*, we prove the convergence of the fonctionnelle J_ε

$$\text{to } \int_\Sigma [a(u^+)^2 + b(u^-)^2 + 2cu^+u^-]d\sigma$$

$$\text{when } \varepsilon \to 0 \text{ where } a = b = k(2\delta)\frac{\cosh 2\delta}{\sinh 2\delta}$$

$$\text{and } c = -k(2\delta)\frac{1}{\sinh 2\delta}$$

Remark 4.1: The function defined in (4.1) is, in fact, nothing but, the test-function in the weak formulation of problem (II_ε) it can also be written in the form

$$u_\varepsilon = A_\varepsilon \cosh(\frac{x_N}{\varepsilon\sqrt{kk'}}) + B_\varepsilon \sinh(\frac{x_N}{\varepsilon\sqrt{kk'}})$$

Step two: We estabilish now the inequality

$$F = s\text{-}L^2(\Omega)\text{-}\lim\inf_\varepsilon F^\varepsilon$$

We prove that for every converging sequence v_ε of $H^1_0(\Omega)$, $v_\varepsilon \to u$ in $L^2(\Omega)$-strong, we have $F(u) \le \liminf F^\varepsilon(v^\varepsilon)$. The idea is to compare via a subdifferential inegality $F^\varepsilon(v^\varepsilon)$ with $F^\varepsilon(u^\varepsilon)$, which from the step a), converges to F(u). We obtain $F^\varepsilon(v^\varepsilon) \ge F^\varepsilon(u^\varepsilon) + I_\varepsilon$, where

$$I_\varepsilon = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(v_\varepsilon - u_\varepsilon) + u_\varepsilon(v_\varepsilon - u_\varepsilon) + \int_{\Sigma_\varepsilon} k\varepsilon\nabla u_\varepsilon \nabla(v_\varepsilon - u_\varepsilon) + \frac{1}{k'\varepsilon}u_\varepsilon(v_\varepsilon - u_\varepsilon)$$

We establish at the end of the proof that $I_\varepsilon \to 0$, by using the *lemmas 3.2* and *3.3*. The theorem is then completely proved.

CONCLUSION

We have proved , in fact, that the functional F(u) is a variational limit of the sequence $F^\varepsilon(u_\varepsilon)$, i.e., F(u) is the Γ-limit of $F^\varepsilon(u_\varepsilon)$, as ε → 0. In this case, we have proved that u is not a function of $H^1_0(\Omega)$. The domain of the Γ-limit F is $H^1(\Omega^+ \cup \Omega^-)$. We have also find the particular case of Sanchez-Palencia, when k' = +∞ , then a = b = k and c = -k. Note that for the organization of this paper and for the well understand, most of the lemmas are easy to prove. So that we have only focus our main idea to the essential part of this paper. It is interesting to notice that it is possible to extend this study by applying the main idea to many and various problems that already exist in many situations, such as homogenization.

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