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A Method of Solving a Class of Nonlinear Integral Equation in the Reproducing Kernel Space

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Abstract: In the study, we will give a method of solving the solution for the nonlinear Volterra-Fredholm integral equation in the reproducing kernel space W(D). The problem on solving the solution of the nonlinear Volterra-Fredholm integral equation is transformed into the problem of solving system of linear equations. The approximate solution converges to exact solution of the nonlinear Volterra-Fredholm integral equation in the sense of $\|.\|_w$ but also in the sense of $\|.\|_C$. In addition, the error of the approximate solution is monotone deceasing and high convergence order $O(\frac{1}{n}+\frac{1}{m})$ in the sense of $\|.\|_W$. Numerical experiments illustrate the method is efficient.

Key words: Nonlinear equation, Volterra-Fredholm integral equation, reproducing kernel space

INTRODUCTION

We shall consider the general nonlinear mixed Volterra-Fredholm integral equation of the form

$$u(t,x)=f(t,x)+\int_0^t\int_\Omega F(t,x,\tau,\xi,u(\tau,\zeta))d\xi d\tau,\quad (t,x)\in[0,T]\times\Omega \tag{1}$$

Where u(t, x) is determined function f(t, x) and F(t, x, τ, ξ, u(τ, ζ)) are analytic functions on D =[0, T] × Ω, where Ω is a closed subset of R^n, n = 1, 2, 3.

Equations of this type arise in the theory of nonlinear parabolic boundary value problems, the mathematical model of the spatiotemporal development of an epidemic and various physical, mechanical and biological problems (Diekman, 1978; Thieme, 1977). The existence and uniqueness of the solution for the Eq. 1 are discussed in (Hacia, 1996; Kauthen, 1989). Significant progress has been made in numerical analysis linear and nonlinear version of the Eq. 1. For the linear case, some methods for numerical treatment are given in (Hacia,1 996; Kauthen, 1982; Guoqiang and Liqing, 1994). For nonlinear case, the literature of integral equations contains few numerical methods (Maleknejad and Hadizadha, 1994) for handling the Eq. 1. In recent years, there has been renewed interest in Eq. 1, such as the time collocation and time discretization methods, the particular trapezoidal Nystrom method (Guoqiang, 1995) the Adomian decomposition method (Maleknejad and Hadizadeh, 1994; Adomian, 1994; Cherruault, 1992) and so on. The present research is motivated by the desire to obtain convenient method of solving the Eq. 1.

In the study, we will give a method of solving the solution for the Eq. 1 in the reproducing kernel space W(D).

SEVERAL THE REPRODUCING KERNEL SPACES

In the study, several the reproducing kernel space are given for solving the solution of the Eq. 1.

- Space

$$W_2^1[a,b]$$

(Ming and Cui, 2004}) is defined by

$$W_2^1[a,b]\triangleq\{u\,\Big|\,\begin{array}{l}u\text{ is one-variable absolutely}\\ \text{continuous function, }u'\in L^2[a,b]\}\end{array}$$

The inner product and the norm in

$$W_2^1[a,b]$$

are defined, respectively by

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$$< u(x), v(x) >_{W_2^1} = \int_a^b [u(x)v(x) + u'(x)v'(x)]dx,$$

$$u(x), v(x) \in W_2^1[a, b], \ \|u\|_{W_2^1} = < u(x), v(x) >^{\frac{1}{2}},$$

$W^1{}_2$ [a, b) is a complete reproducing kernel space (Minggen and Zhongxing, 1986) and its reproducing kernel is given by

$$R_X^{(1)}(y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x+y|-b+a)] \tag{2}$$

Using the definition of the reproducing kernel, it holds that

$$u(x) =< u(y), R_X^{(1)}(y) >_{W_2^1} \text{ for all } u \in W_2^1 [a, b] \text{and fixed } x \in [a, b]$$

Space

$$W_2^2[c, d]$$

(Ming and Cui, 2004) is defined by

$$W_2^2[c, d] \triangleq \{u \left| \begin{array}{l} u, u'u'' \text{ are one-variable absolutely} \\ \text{continuous function, } u', u'' \in L^2[c, d] \end{array} \right. \}$$

and endowed it with the inner product and the norm, respectively,

$$< u(x), v(x) >_{W_2^2} = \int_a^b [4u(x)v(x) + 5u'(x)v'(x) + u''(x)v''(x)]dx,$$

$$u(x), v(x) \in W_2^2[c, d], \|u\|_{W_2^2} =< u(x), v(x) >_{W_2^2}^{\frac{1}{2}}$$

Similarly,

$$W_2^2[c,d]$$

is also a complete reproducing kernel $R_x^{(2)}$ (y) space and the reproducing kernel is given in (Minggen and Zhongxing, 1988).

For the Eq. 1, u(t, x) is continuous in first partial derivative on t. We considered the space W (D), where D = [a, b]×[c, d] W (D) is defined by

$$W(D) = \{u(t, x) \left| u(t, x), \frac{\partial u(t, x)}{\partial t} \right.$$

are two-variable complete continuous fuctions,

$$\frac{\partial^{p+q} u(t, x)}{\partial t^p \partial x^q} \in L^2 (D), p = 0, 1, 2, q = 0, 1\}$$

The inner product and norm are defined by, respectively

$$\begin{aligned} &< u_1, u_2 >_W \\ &= \int_D [4u_1(t,x)u_2(t,x) + 5\frac{\partial}{\partial t}u_1(t,x)\frac{\partial}{\partial t}u_2(t,x) \\ &+ \frac{\partial^2}{\partial t^2}u_1(t,x)\frac{\partial^2}{\partial t^2}u_2(t,x) + 4\frac{\partial}{\partial x}u_1(t,x) \\ &\frac{\partial}{\partial x}u_2(t,x) + 5\frac{\partial^2}{\partial x\partial t}u_1(t,x)\frac{\partial^2}{\partial x\partial t}u_2(t,x) \\ &+ \frac{\partial^3}{\partial x\partial t^2}u_1(t,x)\frac{\partial^3}{\partial x\partial t^2}u_2(t,x)]dtdx \end{aligned}$$

and

$$\|u\|_W =< u(x), v(x) >_W^{\frac{1}{2}} \tag{3}$$

Lemma 1:

$$\text{if } u(t, x) = u_1(t)u_2(x), \ u \in W, u_1 \in W_2^2, u_2 \in W_2^1$$
$$\text{then } \|u\|_W = \|u_1\|_{W_2^2} \|u_2\|_{W_2^1} \tag{4}$$

The reproducing kernel is expressed as (Ming and Cui, 2004).

$$R_{(\eta,\xi)}(t,x) = R_\eta^{(2)}(t)R_\xi^{(1)}(x) \tag{5}$$

SOLVING THE EQ. (1) IN THE REPRODUCING KERNEL SPACE

In this study, we will give the method of solving the solution and a representation of the solution of the Eq. 1 in the reproducing kernel space.

Transformation of the nonlinear integral equation: We will discuss on solving the Eq. 1 in the reproducing kernel space W (D), where D = [0, T]×[a, b]. We find there exist first partial derivative of function u(t, x). Apply partial derivative of two sides of (1.1) with respect to t, we have

$$\frac{\partial u(t,x)}{\partial t}=\frac{\partial f(t,x)}{\partial t}+\int_a^b F(t,x,t,\xi,u(t,\xi))d\xi+\int_0^t\int_a^b\frac{\partial F(t,x,\tau,\xi,u(\tau,\xi))}{\partial t}d\xi d\tau \quad (6)$$

with initial values

$$u(0,x)=f(0,x) \quad (7)$$

Obviously, the Eq. 6 is equivalent to (1).

The definition of operator L: In Eq. 6, the operator

$$L: W(D)\to W_2^1[a,b]$$

is defined by

$$(Lu)(x)\triangleq\frac{\partial u(t,x)}{\partial t}\Big|_{t=0} \quad (8)$$

If let t = 0 in Eq. 6, then we obtain operator equation

$$(Lu)(x)=G(x) \quad (9)$$

Where

$$G(x)=\frac{\partial f(t,x)}{\partial t}\Big|_{t=0}+\int_a^b F(0,x,0,\xi,u(0,\xi))d\xi \quad (10)$$

It is easy to prove the operator L is bounded linear operator.

Obviously, the solution of Eq. 6 satisfies the form (8). But we must point out the Eq. 9 is not equivalent to Eq. 6. Here, we will solve the solution of (6) in virtue of the definition of the operator .

Solving the solution of Eq. 6: In order to obtain the representation of the exact solution of Eq. 6, let

$$\varphi_i(y)=R_{y_i}^{(1)}(y)$$

where

$$\{y_i\}_{i=1}^{\infty}$$

is dense in the interval. From the definition of the reproducing kernel, we have

$$<v(x),\varphi_i(x)>=v(x_i).$$

Let L^* denotes the conjugated operator of L from W (D) to

$$W_2^1[a,b]$$

Lemma: The bounded operator L* from

$$W_2^1[a,b]$$

to W (D) is expressed by

$$(L^*\varphi_i)(t,x)=\frac{\partial R_t^{(2)}(\eta)}{\partial\eta}\Big|_{\eta=0}R_x^{(1)}(y_i)\quad i=1,2,\cdots \quad (11)$$

Proof: From the definition of \$L\$ and the properties of the reproducing kernel

$$R_{(t,x)}(\eta,\xi)=R_t^{\{(2)\}}(\eta)R_x^{\{(1)\}}(\xi)$$

for any mixed t and x we have

$$\begin{aligned}(L^*\varphi_i)(t,x)&=<(L_y^*R_{y_i}^{\{(1)\}}(y))(\eta,\xi),R_t^{\{(2)\}}(\eta)R_x^{\{(1)\}}(\xi)>_W\\&=<(R_{y_i}^{\{(1)\}}(y)),L_{(\eta,\xi)}R_t^{\{(2)\}}(\eta)R_x^{\{(1)\}}(\xi)(y)>_{W_2^1}\\&=(LR_t^{\{(2)\}}(\eta)R_x^{\{(1)\}}(\xi))(y_i)\\&=\frac{\partial R_t^{\{(2)\}}(\eta)}{\partial\eta}\Big|_{\eta=0}R_x^{\{(1)\}}(y_i),\qquad i=1,2,\cdots\end{aligned}$$

Where the subscript x by operator L_x indicates that the operator L applies to functions of x.

We write

$$\psi_i(t,x)=(L^*\varphi_i)(t,x)$$

and

$$\bar{\psi}_i(t,x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(t,x)$$

Where β_{ik} are coefficients of Gram-Schmidt orthonomalization. Then the span for

$$\{\bar{\psi}_i(t,x)\}_{i=1}^{\infty}$$

is a subspace of W (D)

$$\text{span}(\{\overline{\psi}_i(t,x)\}_{i=1}^{\infty}) = \{u(t,x) \mid u(t,x) = \sum_{i=1}^{n} c_i \overline{\psi}_i(t,x), c_i \in R, n \in N\} \quad (12)$$

Let *S* be the closure of this subspace

$$S = \overline{\text{span}(\{\overline{\psi}_i(t,x)\}_{i=1}^{\infty})}$$

And S⊥ denote orthcomplement space of S in W (D). Using the following method, we can obtain an orthonormal basis

$$\{\overline{\rho}_i(t,x)\}_{i=1}^{\infty} \text{ of } S^{\perp}$$

Take a set of points

$$B = \{p_j(\eta_j, \xi_j)\}_{i=1}^{\infty}$$

as a dense set of region D = [0, 1]×[a, b] and put

$$\rho_j(t,x) = R_{\eta_j}^{((2))}(t) R_{\xi_j}^{((1))}(x), \quad j = 1, 2, \cdots \quad (13)$$

Where

$$R_{\eta_j}^{(2)}(t) R_{\xi_j}^{(1)}(x)$$

is the reproducing kernel of W (D). we begin to proceed generalized Schimidt orthonormalization for the function system

$$\{\overline{\rho}_i(t,x)\}_{i=1}^{\infty}$$

about orthonormal system

$$\{\overline{\psi}_i(t,x)\}_{i=1}^{\infty}$$

that is

$$\overline{\rho}_{i+1} = \frac{\rho_{i+1} - \sum_{k=1}^{\infty}(\rho_{i+1}, \overline{\psi}_k)\overline{\psi}_k - \sum_{k=1}^{i}(\rho_{i+1}, \overline{\rho}_k)\overline{\rho}_k}{\| \rho_{i+1} - \sum_{k=1}^{\infty}(\rho_{i+1}, \overline{\psi}_k)\overline{\psi}_k - \sum_{k=1}^{i}(\rho_{i+1}, \overline{\rho}_k)\overline{\rho}_k \|_W}, \quad i = 1, 2, \cdots$$

Where

$$\sum_{i=1}^{j-1} 0 \text{ as soon as } j = 1 \text{ and put}$$

$$\overline{\rho}_j(t,x) = \sum_{k=1}^{\infty} \beta_{jk} \psi_k(t,x) + \sum_{m=1}^{j} \beta_{jk}^{*} \rho_m(t,x), \; j = 1, 2, \ldots, \quad (14)$$

Since

$$\{\overline{\psi}_i(t,x)\}_{i=1}^{\infty}$$

and

$$\{\overline{\rho}_j(t,x)\}_{j=1}^{\infty}$$

are orthonormal basis of S and S⊥, respectively.

$$\{\overline{\psi}_i(t,x)\}_{i=1}^{\infty} \cup \{\overline{\rho}_j(t,x)\}_{j=1}^{\infty}$$

is an orthonormal basis of W (D).

Lemma: Take a set of points

$$B = \{p_j(\eta_j, \xi_j)\}_{j=1}^{\infty}$$

as a dense set of region D = [0, 1]×[a,b]. Then

$$\{\overline{\rho}_j(0,x)\}_{j=1}^{\infty}$$

given by (13) is acomplete system in

$$W_2^1[a,b]$$

Theorem 1: Take $\{s_k\}_{k=1}^{\infty}$ as a dense set of interval [a, b]. Assume u(t, x) be the solution of Eq. 6, then the solution is expressed as

$$u(t,x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} G(s_k) \overline{\psi}_i(t,x) + \sum_{j=1}^{\infty} \alpha_j \overline{\rho}_j(t,x) \quad (15)$$

Where G(x) are given functions in (8) and α_j satisfy

$$u(0,x) \triangleq f(0,x) = \sum_{i=1}^{\infty} \sum_{K=1}^{i} \beta_{ik} G(s_k) \overline{\psi}_i(0,x) + \sum_{j=1}^{\infty} \alpha_j \overline{\rho}_j(0,x) \quad (16)$$

We will obtain the approximate solution by truncating the series

$$u_{n,m}(t,x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} G(s_k) \overline{\psi}_i(t,x) + \sum_{j=1}^{m} \alpha_j \overline{\rho}_j(t,x) \quad (17)$$

and α_j satisfy

$$u_{n,m}(0,x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} G(s_k) \overline{\psi}_i(0,x) + \sum_{j=1}^{m} \alpha_j \overline{\rho}_j(0,x) \quad (18)$$

Proof: Assume u(t, x) be the solution of Eq. 6 and $u(t, x) \in W(D)$. Let u(t, x) be expanded in Fourier series

$$u(t,x) = \sum_{i=1}^{\infty} < u(t,x), \bar{\psi}_i(t,x) > \bar{\psi}_i$$

$$(t,x) + \sum_{j=1}^{\infty} < u(t,x), \bar{\rho}_j(t,x) > \bar{\rho}_j(t,x)$$

by the orthonormal basis of.

$$\{\bar{\psi}_i(t,x)\}_{i=1}^{\infty} \cup \{\bar{\rho}_i(t,x)\}_{i=1}^{\infty} \text{ of } W(D)$$

Let

$$\alpha_j = < u(t,x), \bar{\rho}_j(t,x) >$$

we get

$$u(t,x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < u(t,x), L^*\varphi_k(t,x) >_W \bar{\psi}_i(t,x) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(t,x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < (Lu)(x), \varphi_k(x) >_{W_2^1} \bar{\psi}_i(t,x) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(t,x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < G(x), \varphi_k(x) >_{W_2^1} \bar{\psi}_i(t,x) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(t,x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(s_k) \bar{\psi}_i(t,x) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(t,x)$$

Specially, it holds that

$$u(0,x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(s_k) \bar{\psi}_i(0,x) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(0,x)$$

From the uniqueness of the solution and the initial values (7), α_j can be obtained by solving infinite system of linear Eq. 16.

In calculation, from Lemma (3), α_j can be obtained by solving infinite system of linear equations

$$< f(0,x), \rho_l(0,x) >_{W_2^1} = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(x_k) < \bar{\psi}_i(0,x),$$
$$\rho_l(0,x) >_{W_2^1} + \sum_{j=1}^{\infty} \alpha_j < \bar{\rho}_j(0,x), \rho_l(0,x) >_{W_2^1}$$

From the form (13), we get

$$f(0,\xi_l) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(x_k) \bar{\psi}_i(0,\xi_l) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(0,\xi_l),\ l = 1,2,\cdots,$$

Hence α_j in the approximate solution $u_{n,m}(t, x)$ can be obtained by solving finite system of linear equations

$$f(0,\xi_l) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} G(x_k) \bar{\psi}_i(0,\xi_l) + \sum_{j=1}^{m} \alpha_j \bar{\rho}_j(0,\xi_l)\ l = 1,2,\cdots m,$$

Theorem 2: Assume u(t, x) be the solution of Eq. 6 and $r_{n,m}(t, x)$ are the error in the approximate solution $u_{n,m}(t, x)$, where u (t, x), $u_{n,m}$ are given by 15-17, respectively. Then the following conclusions hold.

- The approximate solution $u_{n,m}(t, x)$ converge to the exact solution in the sense of $\|.\|W$.
- The error $r_{n,m}(t, x)$ is monotone decreasing in the sense of $\|.\|W$.
- The error $r_{n,m}$ possesses convergence order

$$\|r_{n,m}(t,x)\|_W = O(\frac{1}{n} + \frac{1}{m})$$

Proof: From the form (15) and orthonormal basis

$$\{\bar{\psi}_i(t,x)\}_{i=1}^{\infty} \cup \{\bar{\rho}_i(t,x)\}_{i=1}^{\infty}$$

of W(D), it follows that

$$\|u(t,x)\|_W^2 = \left\| \sum_{i=k}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(x_k) \bar{\psi}_i(t,x) + \sum_{j=1}^{\infty} \alpha_j \bar{\rho}_j(t,x) \right\|^2$$
$$= \sum_{i=1}^{\infty} [\sum_{k=1}^{i} \beta_{ik} G(s_k)]^2 + \sum_{j=1}^{\infty} \alpha_j^2$$

Note that if $\|u\|_W < \infty$, then there exist constants a and b such that

$$\sum_{i=1}^{\infty} [\sum_{k=1}^{i} \beta_{ik} G(s_k)]^2 = a, \qquad \sum_{j=1}^{\infty} \alpha_j^2 = b$$

This implies that

$$\sum_{i=1}^{\infty} [\sum_{k=1}^{i} \beta_{ik} G(x_k)]^2 \in l^2$$

and

$$\alpha_j \in l^2, \qquad j = 1,2,\cdots$$

Subsequently, it holds that

$$\left| \sum_{k=1}^{i} \beta_{ik} G(s_k) \right| \le \frac{a_1}{i}, \qquad |\alpha_j| \le \frac{a_2}{j}$$

Since

$$\sum_{i=n+1}^{\infty} \frac{1}{i^2} \le \frac{a_3}{n}, \qquad \sum_{j=m+1}^{\infty} \frac{1}{j^2} \le \frac{a_4}{m}$$

Where α_i, i = 1,2,3,4 are constants. We get

$$\begin{aligned}\|r_{n,m}(t,x)\|_W^2 &= \|u(t,x)-u_{n,m}(t,x)\|_W^2 \\ &= \sum_{i=1}^{\infty}[\sum_{k=1}^{i}\beta_{ik}G(x_k)]^2+\sum_{j=1}^{\infty}\alpha_j^2 \\ &= c_1\sum_{i=n+1}^{\infty}\frac{1}{i^2}+c_2\sum_{j=m+1}^{\infty}\frac{1}{j^2} \\ &\le C(\frac{1}{n}+\frac{1}{m})\end{aligned} \quad (19)$$

Where c_1, c_2 are constants and $C = \max\{c_1, c_2\}$. From the form (17), we get $\|u_{n,m}\|_W \to \|u\|_W$, $(n \to \infty)$.
Suppose that u(t, x) and $u_{n,m}$ (t, x) are given by (15) and (17), respectively. We have

$$\begin{aligned}\|r_{n,m}(t,x)\|_W^2 &= \|u(t,x)-u_{n,m}(t,x)\|_W^2 \\ &= \sum_{i=n+1}^{\infty}[\sum_{k=1}^{i}\beta_{ik}G(s_k)]^2+\sum_{j=m+1}^{\infty}\alpha_j^2\end{aligned} \quad (20)$$

Obviously, the error $r_{n,m}$ (t, x) is monotone decreasing in the sense of norm.

From the form (20), it holds that

$$\|r_{n,m}(t,x)\|_W = O(\frac{1}{n}+\frac{1}{m})$$

NUMERICAL EXPERIMENT

In the study, we will give some examples to use the method given in the study. All computations are performed by the Mathematica 4.0 software package. We calculate the approximate solution $u_{n,m}$ (t, x) by (17), (18) and absolute error.We present the numerical results in Table 1 and 2.

Example 1: We first consider the linear Volterra-Fredholm integral equation

$$u(t,x) = f(t,x)+\int_0^t\int_0^1 F(t,x,\tau,\xi,u(\tau,\xi))d\xi d\tau, \quad t\in[0,1] \quad (21)$$

Where

$$F(t,x,\tau,\xi,u(\tau,\zeta)) = \tau\cos(x-\xi).$$

Table 1: The results of example 1

Node	u	ũ	|u-ũ|	Node	u	ũ	|u-ũ|
$(\frac{1}{10},\frac{1}{15})$	0.816912	0.818221	1.30E-3	$(\frac{7}{10},\frac{1}{15})$	0.246049	0.248285	2.23E-3
$(\frac{1}{10},\frac{2}{5})$	0.754101	0.755338	1.23E-3	$(\frac{7}{10},\frac{2}{5})$	0.227131	0.227568	4.37E-4
$(\frac{1}{10},\frac{11}{15})$	0.608274	0.609004	7.30E-3	$(\frac{7}{10},\frac{11}{15})$	0.183209	0.169654	1.35E-2
$(\frac{3}{10},\frac{1}{15})$	0.547593	0.554697	7.10E-3	$(\frac{9}{10},\frac{1}{15})$	0.164932	0.14778	1.71E-3
$(\frac{3}{10},\frac{2}{5})$	0.505489	0.512153	6.66E-3	$(\frac{9}{10},\frac{2}{5})$	0.15225	0.15013	2.12E-3
$(\frac{3}{10},\frac{11}{15})$	0.407738	0.410699	2.96E-3	$(\frac{9}{10},\frac{11}{15})$	0.122808	0.111906	1.0E-2

Table 2: The results of example 2

Node	u	ũ	|u-ũ|	Node	u	ũ	|u-ũ|
$(\frac{1}{10},\frac{1}{15})$	0.408456	0.409073	6.16E-4	$(\frac{1}{2},\frac{11}{15})$	0.13448	0.132859	1.62E-3
$(\frac{1}{10},\frac{2}{5})$	0.376616	0.377193	5.77E-4	$(\frac{7}{10},\frac{1}{15})$	0.123024	0.121958	1.06E-3
$(\frac{1}{10},\frac{11}{15})$	0.299292	0.299591	2.99E-4	$(\frac{7}{10},\frac{2}{5})$	0.113435	0.111281	2.15E-3
$(\frac{3}{10},\frac{1}{15})$	0.273796	0.276999	3.20E-3	$(\frac{7}{10},\frac{11}{15})$	0.183531	0.187178	3.64E-3
$(\frac{3}{10},\frac{2}{5})$	0.252453	0.255402	2.94E-3	$(\frac{9}{10},\frac{1}{15})$	0.0824658	0.0799499	2.51E-3\
$(\frac{3}{10},\frac{11}{15})$	0.200621	0.20153	9.0E-4	$(\frac{9}{10},\frac{2}{5})$	0.0760375	0.0615089	1.45E-2
$(\frac{3}{10},\frac{11}{15})$	0.183531	0.187178	3.64E-3	$(\frac{9}{10},\frac{11}{15})$	0.0604259	0.0582806	2.14E-2

$$f(t,x)=\frac{1}{8}e^{-2t}(8\cos x+e^{-2t}-2t-1)(\cos x+\sin 1\cos(1-x))$$

Then $u(t,x)=\cos xe^{-2t}$ is the true solution of the Eq. 21. The approximate solution ũ wuth n = 150, m = 80 with in the form (16) and error are shown in Table 1.

Example 2: We first consider the linear Volterra-Fredholm integral equation

$$u(t,x)=f(t,x)+\int_0^t\int_0^1 F(x,\xi,\tau,)(1+u^2(\xi,\tau))d\xi d\tau,\ t\in[0,1] \tag{22}$$

Where

$$F(t,\xi,\tau)=-e^{\tau}\cos(x-\xi)$$

$$f(t,x)=e^{-2t}(1-\frac{x^2t^2}{2})-\frac{1}{12500}(e^{-5t}(20(-44+33e^{5t}-220t-550t^2$$
$$-500t^3-625t^3)\cos(1-x)+(3012+12500e^{4t}-15512e^{5t}$$
$$+2560t+6400t^2+6500t^3+8125t^4)\sin(1-x)-4(-869$$
$$-3125e^{4t}+3994e^{5t}-1220t-3050t^2-3000t^3-3750t^4)\sin x)).$$

Then

$$u(t,x)=(1-\frac{x^2t^2}{2})e^{-2t}$$

is the true solution of the Eq. 22. The approximate solution ũ with n = 150, m = 100 with in the form (16) and error are shown in Table 2.

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