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Mathematical Model for Major Mode of HIV/AIDS Transmission

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Abstract: In this study, we developed and examine 3 transmission Dynamics models for 3 major mode of HIV/AIDS transmission. We evaluate the disease-free steady states and examine them for stability. We also compared the 3 mode of transmission to determine which of them is responsible for high level of transmission of the HIV/AIDS.

Key words: Steady state, population dynamics, transmissibility of infection, mortality, life expectancy

INTRODUCTION

HIV/AIDS has now become one of the world pandemic diseases. However, 3 mode of transmission has been recognized to be responsible for the spread of the virus. Others mode of transmission has been reduced to very low level, due to campaigns of governments and non-governmental organization. The three major mode of transmission includes, mother to child transmission, called vertical transmission, heterosexual and homosexual mode, respectively. In trying to address the HIV/AIDS problems and also examine the transmission dynamics of the disease, Hsieh (1996) developed a transmission dynamics model for treatment of HIV/AIDS in a varying population, while Kimbir *et al.* (2003) extended Hsieh model to study effect of prevention of the disease on the population dynamics. Michael study the spread of the virus in a constant population, using both horizontal and vertical transmission. Andrea Puglisi in their research examines vertical transmission of the virus in a varying population, with density depended mortality and disease induced mortality. However, in this research, we examined the 3 major modes of transmission, with the aim of identifying which of them is responsible for high level of transmission of HIV.

MOTHER TO CHILD TRANSMISSION

Model formulation: Transmission of the virus from an infected mother to her new born occurs during pregnancy, birth and breast feeding. Assume that population is homogeneous mixing and divided into compartments namely, susceptible, exposed, infected and AIDS, respectively. The vital rates are assumed same for all compartments and that the exposed mother does not transmit the virus to her new born. Since they are not yet infectious. Suppose a fraction α of the offspring of the infected are born into the exposed population compartment. Then the total number new recruitment into the exposed compartment per unit of time is, $\alpha\Delta I$ and the number of new recruitment into the susceptible population compartment is Δ-$\alpha\Delta I$. Suppose, the population of AIDS compartment are not sexually active and not reproductive. They don't give birth to new born. Using these assumptions we have the following equations describing dynamics of the population compartment.

$$\frac{dS}{dt} = \Delta_1 - \alpha\Delta_1 I - \beta_1 C_1 IS - \mu_1 s$$

$$\frac{dE}{dt} = \beta_1 C_1 SI - (\nu_1 + \mu_1)E$$

$$\frac{dI}{dt} = \nu_1 E - (k_1 + \mu_1)I$$

$$\frac{dA}{dt} = k_1 I - (\mu_1 + \sigma_1)A$$

Where N = S + E + I + A, is the total population size.
At disease-free steady states we have E = I = A = 0 and the disease-free steady is,

$$F_0 = (\frac{\Delta_1}{\mu_1}, 0, 0, 0)$$

Where

$$\frac{\Delta_1}{\mu_1}$$

is the asymptotic carrying capacity of the population. Let us examine the stability of the disease-free steady state, F_0. The Jacobian matrix at F_0 is,

$$J_{F_0} = \begin{pmatrix} -\mu_1 & 0 & -(\alpha + \frac{\beta_1 C_1}{\mu_1})\Delta_1 & 0 \\ 0 & -(\nu_1+\mu_1) & \frac{\Delta_1}{\mu_1}\beta_1 C_1 & 0 \\ 0 & \nu_1 & -(k_1+\mu_1) & 0 \\ 0 & 0 & k_1 & -(\mu_1+\sigma_1) \end{pmatrix}_{(\frac{\Delta}{\mu},0,0,0)} = 0$$

The eigen value of matrix J_{F_0} satisfies,

$$(\mu+\sigma+\lambda)(\mu+\lambda)\det\begin{pmatrix} -(\nu_1+\mu_1+\lambda) & \frac{\Delta_1}{\mu_1}\beta_1 C_1 \\ \nu & -(k_1+\mu_1+\lambda) \end{pmatrix} = 0$$

$$M = \begin{pmatrix} -(\nu_1+\mu_1) & \frac{\Delta_1}{\mu_1}\beta_1 C_1 \\ \nu & -(k_1+\mu_1) \end{pmatrix}$$

The trace of matrix, $M = -(\Psi_1 + \mu_1)$, $\Psi_1 = (\nu + k + \mu)$ is not negative and the determinant of

$$M = \tau_1 - (\nu\frac{\Delta}{\mu}\beta)_1$$

where $\tau_1 = (\nu_1 + \mu_1)(k_1 + \mu_1)$. Thus, the disease-free steady state is locally asymptotically stable if,

$$(\beta C)_1 \le \frac{\tau_1\mu_1}{(\nu\Delta C)_1}$$

for all, $\mu > 0$, $\nu > 0$, $\beta > 0$, $\Delta > 0$. The probability of transmission of HIV virus from an infectious person to a susceptible individual is defined by

$$\beta \le (\frac{\tau\mu}{\nu\Delta C})_1$$

Hence any infection would not spread in the population but would die out.

HETEROSEXUAL MODE OF TRANSMISSION

The spread of the virus is through sexual contact between the opposite sexes. Usually between male and female, in which one of them is infectious. Let us assume that the population is in a proportionate mixing, with uniform vital rates and per unit contact rate. Assume that transmission occurs only when an infectious individual have sexual contact with a susceptible individual of opposite sex, which is of disease transmitting type and that the exposed compartment is not infectious, while individuals in the infectious compartments are not reproductive. Also suppose HIV virus courses abortion in pregnant infectious women. So that no vertical transmission is involved. Let the new recruitment of susceptible into the susceptible population be denoted by Δ. Dividing the population into three compartments, as in the case of mother to child transmission. The dynamics of the population compartment can be described by the following system of equations.

$$\frac{dS}{dt} = \Delta_2 - C_2\beta_2 SI - \mu_2 S$$

$$\frac{dE}{dt} = C_2\beta_2 SI - (\nu_2 + \mu_2)E$$

$$\frac{dI}{dt} = \nu_2 E - (k_2 + \mu_2)I$$

$$\frac{dA}{dt} = k_2 I - (\mu_2 + \sigma_2)A$$

Where

$$N = S + E + I + A$$

The disease-free steady state, E_0, is obtained as,

$$(\frac{\Delta_2}{\mu_2}, 0, 0, 0)$$

The Jacobian matrix at this state is,

$$J_{E_0} = \begin{pmatrix} -\mu_2 & 0 & -\frac{\Delta_2}{\mu_2}C_2\beta_2 & 0 \\ 0 & -(\nu_2+\mu_2) & \frac{\Delta_2}{\mu_2}C_2\beta_2 & 0 \\ 0 & \nu_2 & -(k_2+\mu_2) & 0 \\ 0 & 0 & k_2 & -(\mu_2+\sigma_2) \end{pmatrix}$$

The eigen-value of matrix J_{E_0} satisfies,

$$(\mu_2+\sigma_2+\lambda)(\mu_2+\lambda)\det\begin{pmatrix} -(\nu_2+\mu_2+\lambda) & \frac{\Delta_2}{\mu_2}C_2\beta_2 \\ \nu_2 & -(k_2+\mu_2+\lambda) \end{pmatrix} = 0$$

$$\text{Let } M = \begin{pmatrix} -(\nu_2+\mu_2) & \frac{\Delta_2}{\mu_2}C_2\beta_2 \\ \nu_2 & -(k_2+\mu_2) \end{pmatrix}$$

The trace of matrix M = ($-\Psi_2 + \mu_2$), where $\Psi_2 = (\nu_2 + k_2 + \mu_2)$. The determinant of

$$M = \tau_2 - \nu_2 \frac{\Delta_2}{\mu_2} C_2\beta_2, \text{ where } \tau_2 = (\nu_2+\mu_2)(k_2+\mu_2)$$

The stability condition at the disease-free steady state holds if

$$\tau_2 \geq (\frac{\Delta}{\mu}\beta C\nu)_2$$

This means that the net transmission, (βC_2) must satisfy

$$\beta C \leq (\frac{\tau\mu}{\nu\Delta})_2$$

and the transmissibility of the disease must also satisfy

$$\beta \leq (\frac{\tau\mu}{\nu\Delta C})_2$$

The disease-free steady state E_0 is locally asymptotically stable. Any infection would die out from the population.

The homosexual population compartment: Sexual contact per unit of time is between two males, so that there is no vertical transmission of the virus to the new born. Assume proportionate mixing of the males. Let C be the contact rate per unit of time, between an infectious men and a susceptible men. The change of the population compartment per unit of time can be described by the following equations

$$\frac{dS}{dt} = \Delta_3 - \beta_3 C_3 IS - \mu_3 S$$

$$\frac{dE}{dt} = \beta_3 C_3 IS - (\nu+\mu)_3 E$$

$$\frac{dI}{dt} = \nu_3 E - (k+\mu)_3 I$$

$$\frac{dA}{dt} = k_3 - (\mu+\sigma)_3 A$$

Where

$$N = S + E + I + A$$

Also,

$$\frac{dN}{dt} = \frac{d}{dt}(S+E+I+A) = -\sigma A$$

The disease-free steady state,

$$A_0 = (\frac{\Delta_3}{\mu_3}, 000)$$

We examine the stability of this state.

The Jacobian matrix at this state is obtained as,

$$J_{A_0} = \begin{pmatrix} -\mu_3 & 0 & -(\frac{\Delta}{\mu}\beta)_3 & 0 \\ 0 & -(\nu+\mu)_3 & (\frac{\Delta}{\mu}\beta)_3 & 0 \\ 0 & \nu_3 & -(k+\mu)_3 & 0 \\ 0 & 0 & k_3 & -(\sigma+\mu)_3 \end{pmatrix}$$

The eigen value of the matrix J_{A_0} satisfies,

$$(\mu+\sigma+\lambda)_3(\mu+\lambda)_3 \det \begin{pmatrix} -(\nu+\mu+\lambda)_3 & (\frac{\Delta}{\mu}\beta C)_3 \\ \nu_3 & -(k+\mu)_3 \end{pmatrix} = 0$$

$$\text{Let } M = \begin{pmatrix} -(\nu+\mu)_3 & \frac{\Delta}{\mu}(\beta C)_3 \\ \nu_3 & -(k+\mu)_3 \end{pmatrix}$$

Then the trace of M = $-(\Psi_3 + \mu)$, where $\Psi_3 = (\nu + k + \mu)_3$. This is in line with the condition for stability of A_0, Murray (1989) and Leah Edelstein (2005). Also, the determinant of

$$M = \tau_3 - (\frac{\Delta}{\mu}\beta C\nu)_3$$

For stability of A_0, we must have eshet (Kimber *et al.*, 2003). This means that the net transmission of the virus (βC_3) must satisfy,

$$\beta C \leq (\frac{\tau\mu}{\nu\Delta})_3$$

The probability of transmission of the virus from an infectious individual to a susceptible can be defined as,

$$\beta \leq (\frac{\tau\mu}{\nu\Delta C})_3$$

CONCLUSION

The three mode of transmission examined, clearly show the influence of the recruitment rate and the per capital contact rates on the transmissibility of the disease. It is observed from the three models that, stability of the disease-free steady state holds if

$$\beta_j \leq \left(\frac{\tau\mu}{\nu\Delta C}\right)_j \quad j = 1, 2, 3$$

Where j corresponds to the mode of transmission, μ, τ, Δ, ν, C are as defined above.

Thus, we expect, recruitment of new born in MTC transmission, to be higher compared to the other two mode of transmission, since recruitment in the heterosexual mode, is mainly due to migration of those who has completed the developmental stage to adulthood, into its susceptible population. This may not be as large as total population of new born per unit of time. Thus $\Delta_1 > \Delta_2$. The homosexual group is a small population, with small recruitment. In infact this group is not noticed in some part of the world. Using our assumption that the homosexual only have sexual contact with his male counterpart, we would have the per capital contact rate C, also satisfying, $C_1 > C_2 > C_3$. It follows that $\beta_1 < \beta_2 < \beta_3$. The modes responsible for high transmission of HIV are the heterosexual and homosexual modes. In African and some part of the developing world, were homosexual practice is not allowed. The major mode is the heterosexual mode. It then means that all preventive strategies must be directed towards these two groups, so as to control the spread of the virus. Condom and other device should be developed to achieve this.

Definition of variables and parameters:

S (t) : Population Compartment of the Susceptibles
I (t) : Population Compartment of the infected
E (t) : Population Compartment of the exposed
A (t) : Population Compartment of AIDS
Δ : Number of recruitment per unit of time
ν : Progression rate from Exposed to HIV.
k : Progression rate from HIV to AIDS
σ : Disease-induced mortality rate
μ : Natural mortality rate
C : Per capital contact rate

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