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On the Tau Method for a Class of Non-Overdetermined Second Order Differential Equations

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Abstract: This study is concerned with the Tau methods for initial value problems in the class of non-overdetermined second order ordinary differential equations. Three variants namely the differential, the integrated and the recursive formulation are considered. The corresponding error estimates for the 3 various are obtained and some selected examples are provided for illustration. The numerical evidences confirm the order of the Tau approximants so obtained for all the cases.

Key words: Tau method, differential, integrated, recursive, formulation, variant, canonical, polynomials, approximant, error, estimates

INTRODUCTION

Accurate approximate solution of initial value problems and boundary value problems in linear ordinary differential equations with polynomial coefficients can be obtained by the Tau method originally introduced by Lanczos (1938). Techniques based on this method have been reported in literature with application to more general equations including non-linear ones as well as to both partial differential equations and integral equations. We review briefly here some of the variants of the method.

Differential or original form of the Tau method: Consider the m-th order ordinary differential Eq.

$$Ly(x):=\sum_{r=0}^{m}P_r(x)\,y^{(r)}(x)=f(x),\ \ a\le x\le b \tag{1.1a}$$

with associated conditions

$$L^*y(x_{rk}):=\sum_{r=0}^{m-1}a_{rk}\,y^{(r)}(x_{rk}) = \alpha_k\ ,\ k=1(1)m \tag{1.1b}$$

and where, $|\alpha|<\infty$, $|b|<\infty$, α_{rk}, x_{rk}, α_k, $r = 0\ (1)\, m-1$, $k = 1(1\ )m$ are given real numbers, f (x) and $P_r(x)$, $r = 0\ (1)\ m$, are polynomial functions or sufficiently close polynomial approximants of given real function.

For the solution of Eq. (1.1) by the Tau method (Lanczos, 1938, 1956; Adeniyi, 2000; Adeniyi and Edungbola, 2007; Crisci and Russo, 1983; Fox, 1968; Fox and Parker, 1968; Freilich and Ortiz, 1982), we shall seek an approximant of the form:

$$y_n(x)=\sum_{r=0}^{n}a_r\ x^r, n<+\infty \tag{1.2}$$

of y (x) which satisfies exactly the perturbed problem

$$Ly_n(x)=f(x)+\sum_{r=0}^{m+s-1}\tau_{m+s-r}\ \ T_{n-m+r+1}(x) \tag{1.3a}$$

$$L^*y_n(x_{rk})=\alpha_k,\ k=1\ (1)\,m \tag{1.3b}$$

for $a\le x\le b$ and where, τ_r, $r = 1(1)\ m+s$ are parameters to be determined along with a_r , $r = 0\ (1)m$, in Eq. (1.2)

$$T_r(x)=\cos\{r\cos^{-1}[(2x-2a)/(b-a)-1]\}\equiv\sum_{k=0}^{r}C_k^{(r)}\ x^k \tag{1.4}$$

is the r-th degree Chebyshev polynomial valid in the interval (a , b) and

$$s = \max\{N_r-r/0\le r\le m\}$$

is the number of overdetermination of Eq. (1.1a) (Fox and Parker, 1968). We determine α_r, $r = 0\ (1)\ n$ and τ_r, $r = 1\ (1)\ m+s$ by equating corresponding coefficients of powers of x in Eq. (1.3a) together with conditions Eq. (1.3b). Consequently we obtain this desired approximant $y_n(x)$ in Eq. (1.2).

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The integrated formulation of the Tau method: The integrated form of Eq. (1.1a) is given by:

$$I_L(y(x)) = \int\int\int ... \int f(x)\,dx + C_m(x) \qquad (1.6)$$

where, $C_m(x)$ denotes an arbitrary polynomial of degree (m-1), arising from the constants of integration and

$$I_L = \int\int\int ..m \int L(.)\,dx, \qquad (1.7)$$

which is the m times indefinite integration of L(.). The corresponding Tau problem is therefore,

$$I_L(y_n(x)) = \int\int\int ... \int f(x)\,dx + C_m(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-1}\, T_{n+r+1}(x) \qquad (1.8a)$$

$$L^* y_n(x_{rk}) = \alpha_k,\ k = 1\ (1)\ m \qquad (1.8b)$$

where, $y_n(x)$ is here again given by Eq. (1.2). Problem Eq. (1.8) often give a more accurate Tau approximant than Eq. (1.3) does, due to its higher order perturbation term.

The recursive formualtion of the Tau method: The so-called canonical polynomials $\{Q_r(x)\}$, $r \in N_0$ - S associated with operator L of Eq. (1.1) is defined by

$$LQ_r(x) = x^r \qquad (1.9)$$

where, S is a small finite or empty set of indices with cardinality s ($s \le m+h$), h being the maximum difference between the exponent r of x and the leading exponent of the generating polynomial Lx^r, for $r \in N_0$ (Ortiz, 1969, 1974; Adeniyi and Edungbola, 2007). Once, these polynomials are generated, we seek, in this study, an approximant of y (x) of the form:

$$y_n(x) = \sum_{r=0}^{n} a_r\, Q_r(x)\ ,\ n < +\infty \qquad (1.10)$$

which is the exact solution of the perturbed problem

$$y_n(x) = \sum_{r=0}^{F} f_r\, Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)}\, Q_k(x) \qquad (1.11)$$

where, f_r r = 0 (1) F, are the coefficients in f(x).

Their use is advantageous as they neither depend on the boundary conditions nor on the interval of solution. Furthermore, they are re-useable for approximants of higher degrees.

ERROR ESTIMATION OF THE TAU METHOD

We review briefly here error estimation of the Tau method for the 3 variants of the preceding section and which we had earlier reported in Andeniyi *et al.* (1990) Andeniyi and Onumanyi (1991).

Error estimation for the differential form: While, the error function

$$e_n(x) = y(x) - y_n(x) \qquad (2.1)$$

Satisfies the error problem

$$Le_n(x) = -\sum_{r=0}^{m+s-1} \tau_{m+s-r}\, T_{n-m+r+1}(x) \qquad (2.2a)$$

$$L^* e_n(x_{rk}) = 0,\ k = 1(1)m \qquad (2.2b)$$

$$\left(e_n(x)\right)_{n+1} = \mu_m(x)\, \varphi_n\, T_{n-m+1}(x) / C_{n-m+1}^{(n-m+1)} \qquad (2.3)$$

Satisfies the perturbed error problem

$$L\left(e_n(x)\right)_{n+1} = -\sum_{r=0}^{m+s-1} \tau_{m+s-r}\, T_{n-m+r+1}(x) + \sum_{r=0}^{m+s-1} \bar{\tau}_{m+s-r}\, T_{n-m+r+2}(x) \qquad (2.4a)$$

$$L^*\left(e_n(x)\right)_{n+1} = 0 \qquad (2.4b)$$

where, the extra parameters $\bar{\tau}_r$, r = 1 (1) m + s and φ_n are to be determined and $\mu_m(x)$ is a specified polynomial of degree m which ensures that $(e_n(x))_{n+1}$ satisfies the homogeneous condition in Eq. (2.2b).

We insert Eq. (2.3) in Eq. (2.4a) and then equate corresponding coefficients of x^{n+s+1}, x^{n+s}, ... x^{n-m+1} and the resulting linear system is solved for only φ_n by forward elimination, since we do not need the $\bar{\tau}$'s in Eq. (2.3). Consequently we obtain

$$\bar{\varepsilon}_1 = \max_{a \le x \le b} \left|e_n(x)_{n+1}\right| = \left|\varphi_n\right| / \left|C_{n-m+1}^{(n-m+1)}\right| \cong \max_{a \le x \le b} \left|e_n(x)\right| = \varepsilon_1 \qquad (2.5)$$

Error estimation for the integrated form: The error polynomial Eq. (2.3) satisfies the perturbed problem

$$I_L\left(e_n(x)\right)_{n+1} = -\int\int m\ldots\int\left(\sum_{r=0}^{m+s-1}\tau_{m+s-r}\ T_{n-m+r+1}(x)\right) dx + C_m(x) + \sum_{r=0}^{m+s-1}\bar{\tau}_{m+s-r}\ T_{n-m+r+3}(x) \qquad (2.6)$$

We insert Eq. (2.3) in Eq. (2.6) and then equate coefficients of $x^{n+s+m+1}$, x^{n+s+m},..., x n-m for the determination of the parameter φ_n of $(e_n(x))_{n+1}$. Subsequent procedures follows suit as described above in the study Eq. (2.1) in order to obtain the error estimate $\bar{\varepsilon}_2$.

Error estimation for the recursive form: Once the canonical polynomials of the study 1 are generated, they can be used for an error estimation of the Tau method (Crisci and Ortiz, 1981; Lanczos, 1956; Namasivayam and Ortiz, 1981; Onumanyi and Ortiz, 1982). Here we consider a slight perturbation of the given boundary condition Eq. (1.1b) by $\bar{\varepsilon}_1$ to obtain an estimate of the Tau parameter τ_{m+s}, in terms of canonical polynomials, which is then substituted back into expression for $\bar{\varepsilon}_1$ in (25) for a new estimate $\bar{\varepsilon}_3$.

A CLASS OF NON-OVERDETERMINED SECOND ORDER DIFFERENTIAL EQUATIONS

We consider here the 3 variants of Tau method of the preceding the study for the problem Eq. (1.1) when m = 2 and s = 0, that is, the class

$$Ly(x) = \left(P_{20} + P_{21}x + P_{22}x^2\right)y''(x) + \left(P_{10} + P_{11}x\right)y'(x) + P_{00}\ y(x) = \sum_{r=0}^{F} f_r\ x^r,\ a \le x \le b \qquad (3.1a)$$

$$y(a) = \alpha_0,\ y'(a) = \alpha_1 \qquad (3.1b)$$

Without loss of generality we shall assume that a = 0 and b = 1 since the transformation

$$v = (x-a)/(b-a),\ a \le x \le b \qquad (3.2)$$

takes the problem Eq. (3.1) into the interval [0, 1].

Tau approximant by the diffrential form: By inserting Eq. (1.2) into the perturbed form of Eq. (3.1a) we have

$$\left(P_{20} + P_{21}x + P_{22}x^2\right)\sum_{r=0}^{n} r(r-1)\,a_r\ x^{r-2} + \left(P_{10} + P_{11}x\right)\sum_{r=0}^{n} r\,a_r\,x^{r-1} + P_{00}\sum_{r=0}^{F} a_r\ x^r = \sum_{r=0}^{F} f_r\ x^r + \tau_1\ T_n(x) + \tau_2\ T_{n-1}(x)$$

This leads to

$$\sum_{r=0}^{n-2}(r+1)(r+2)\ P_{20}\ a_{r+2}\ x^r + \sum_{r=0}^{n-1}\left(rP_{21} + P_{10}\right)(r+1)\ a_{r+1}\ x^r + \sum_{r=0}^{n}\left[(r-1)\ r\,P_{22} + rP_{11} + P_{00}\right]a_r x^r = \sum_{r=0}^{F} f_r\ x^r + \tau_1\sum_{r=0}^{n} C_r^{(n)}\ x^r + \tau_2\sum_{r=0}^{n-1} C_r^{(n-1)}\ x^r$$

Hence,

$$\begin{aligned}&\Big\{\left[(n-1)n\ P_{21} + nP_{10}\right]a_n + \left[(n-2)(n-1)\ P_{22} + (n-1)P_{11} + P_{00}\right]a_{n-1} - \tau_1\ C_{n-1}^{(n)} - \tau_2\ C_{n-1}^{(n-1)} - f_{n-1}\Big\}\ x^{n-1}\\ &+ \Big\{\left[(n-1)n\ P_{22} + nP_{11} + P_{00}\right]a_n - \tau_1 C_n^{(n)} - f_n\Big\}\ x^n\\ &+ \sum_{r=0}^{n-2}\Big\{(r+1)(r+2)\alpha_0\ a_{r+2} + \left[r(r+1)\ P_{21} + (r+1)\ P_{10}\right]a_{r+1}\\ &+ \left[(r-1)r\ P_{22} + rP_{11} + P_{00}\right]a_r - f_r - \tau_1\ C_r^{(n)} - \tau_2\ C_r^{(n-1)}\Big\}\ x^r = 0\end{aligned} \qquad (3.3)$$

We now equate coefficients to have the linear system of (n + 1) equation

$$(r+1)(r+2)\ P_{20}\,a_{r+2} + \left(rP_{21} + P_{10}\right)(r+1)\ a_{r+2} + \left[(r-1)r\,P_{22} + rP_{11} + P_{00}\right]a_r - f_r - \tau_1\ C_r^{(n)} - \tau_2\ C_r^{(n-1)} = 0\ ,\ r = 0(1)n-2$$

$$\left[(n-1)n\,P_{21} + nP_{10}\right]a_n + \left[(n-2)(n-1)\ P_{22} + (n-1)P_{11} + P_{00}\right]a_{n-1} - \tau_1\ C_{n-1}^{(n)} - \tau_2\ C_{n-1}^{(n-1)} - f_{n-1} = 0$$

$$\left[(n-1)n\ P_{22} + nP_{11} + P_{00}\right] a_n - \tau_1\ C_n^{(n)} - f_n = 0$$

We solve this system together with 2 other equations arising from the conditions Eq. (3.1b) for the determination of the (n + 3) parameters a_r , r = 0 (1) n and τ_r, r = 1, 2. Consequently, we obtain from Eq. (1.2) our desired approximant $y_{n,1}$ (x).

Error estimation for the differential form: For problem Eq. (3.1) we have from Eq. (2.4a) that

$$L(e_n(x)) = \bar{\tau}_1\ T_{n+1}(x) + (\bar{\tau}_2 - \tau_1)\ T_n(x) - \tau_2\ T_{n-1}(x) \quad (3.5)$$

where,

$$L = (P_{20} + P_{21}x + P_{22}x^2)\frac{d^2}{dx^2} + (P_{10} + P_{11}x)\frac{d}{dx} + P_{00} \quad (3.6)$$

$$\begin{aligned}(e_n(x))_{n+1} &= x^2\varphi_n\ T_{n-1}(x)/C_{n-1}^{(n-1)}\\ &= \varphi_n\left(\sum_{r=0}^{n-1} C_r^{(n-1)}\ x^{r+2}\right)\Big/C_{n-1}^{(n-1)}\end{aligned} \quad (3.7)$$

We equate coefficients of x^{n+1}, x^{n-1} from Eq. (3.5) to have the system

$$\begin{aligned}&\theta\left[P_{00} + (n+1)\ P_{11} + n(n+1)P_{22}\right] C_{n-1}^{(n-1)} = \bar{\tau}_1\ C_{n-1}^{(n-1)}\\ &\theta\left[C_{n-1}^{(n-1)}\ P_{00} + nC_{n-2}^{(n-1)}\ P_{11} + (n-1)n\ C_{n-2}^{(n-1)}\ P_{22} + (n+1)\ C_{n-2}^{(n-1)}\ P_{10}\right.\\ &\left.+ n(n+1)\ C_{n-1}^{(n-1)}\ P_{21}\right] = \bar{\tau}_1\ C_n^{(n+1)} + (\bar{\tau}_2 - \tau_1)\ C_n^{(n)}\end{aligned} \quad (3.8)$$

$$\begin{aligned}&\theta\left[C_{n-3}^{(n-1)}\ P_{00} + (n-1)\ C_{n-3}^{(n-1)}\ P_{11} + (n-2)(n-1)C_{n-3}^{(n-1)}\ P_{22}\right.\\ &\left.+ nC_{n-2}^{(n-1)}\ P_{10} + (n-1)n\ C_{n-2}^{(n-1)}\ P_{21} + n(n+1)\ C_{n-1}^{(n-1)}\ P_{20}\right]\\ &= \bar{\tau}_1\ C_{n-1}^{(n+1)} + (\bar{\tau}_2 - \tau_1)\ C_{n-1}^{(n)} - \tau_2\ C_{n-1}^{(n-1)}\end{aligned}$$

where,

$$\theta = \varphi_n\left(C_{n-1}^{(n-1)}\right)^{-1}$$

From this system and by using the well-known relations:

$$C_n^{(n)} = 2^{2n-1},\quad C_{n-1}^{(n)} = -\frac{1}{2}n\ C_n^{(n)} \quad (3.9)$$

we obtain for φ_n the expression

$$\varphi_n = \frac{2^{4n-2}\ \tau^2}{R_1} \quad (3.10)$$

where,

$$\begin{aligned}R_1 &= (P_{00} + P_{11} + nP_{22} + n^2P_{22})\ C_{n-1}^{(n+1)} +\\ &2^{2n-1}\left(2nP_{00} + nP_{11} + 4nP_{10} + 3n^2P_{11} + 6n^2P_{21}\right.\\ &\left.+ 4n^3P_{22} + 4n^2P_{20} + 4n^2P_{20} + 4nP_{20} - 2nP_{21}\right) -\\ &\left(16P_{00} - 16P_{11} + 16nP_{11} + 16n^2P_{22} - 48nP_{22} + 32P_{22}\right)C_{n-3}^{(n-1)}\end{aligned} \quad (3.11)$$

From Eq. (2.5), we obtain the error estimate

$$\bar{\varepsilon}_1 = \frac{2^{2n}\ |\tau_2|}{|R_1|} \quad (3.12)$$

Tau approximant by the integrated form: From Eq. (1.8) we get

$$\begin{aligned}&\int_0^x\int_0^u (P_{20} + P_{21}t + P_{22}t^2)\ y''(t)\ dt\,du\\ &+ \int_0^x\int_0^u (P_{10} + P_{11}t)\ y'(t)\ dt\,du + \int_0^x\int_0^u P_{00}y(t)\ dt\,du\\ &= \int_0^x\int_0^u\left(\sum_{r=0}^{F} f_r\ t^r\right) dt\,du + \tau_1\ T_{n+2}(x) + \tau_2\ T_{n+2}(x)\end{aligned}$$

This leads to

$$\begin{aligned}&P_{20}y(x) + P_{21}\left[xy(x) - 2\int_0^x y(u)\ du\right]\\ &+ P_{22}\left[x^2y(x) - 4\int_0^x uy(u)du + 2\int_0^x\int_0^u y(t)\ dt\,du\right]\\ &+ P_{11}\left[\int_0^x uy(u)\ du - \int_0^x\int_0^u y(t)\,dt\,du\right]\\ &+ P_{00}\int_0^x\int_0^u y(t)\,dt\,du + P_{10}\int_0^x y(u)\ du\\ &+ \alpha_0\ P_{20} + (\alpha_1\ P_{20} + \alpha_0\ P_{10} + \alpha_0\ P_{21} + \alpha_0\ P_{10})x\end{aligned} \quad (3.13)$$

With Eq. (1.2) this gives

$$\begin{aligned}&\sum_{r=0}^{n} P_{20}\ a_r\ x^r + \sum_{r=1}^{n+1}\left(\frac{(r-2)\ P_{21} + P_{10}}{r}\right) a_{r-1}\ x^r\\ &+ \sum_{r=2}^{n+2}\left(\frac{P_{00} + (r-2)\ P_{11} + (r-2)\ (r-3)P_{22}}{(r-1)r^0}\right) a_{r-1}\ x^r\\ &= \sum_{r=2}^{n+2}\frac{f_{r-2}\ x^r}{(r-1)r} + \tau_1\sum_{r=0}^{n+2} C_r^{(n+2)}\ x^r + \tau_2\sum_{r=0}^{n+1} C_r^{(n+1)}\ x^r\end{aligned} \quad (3.14)$$

This gives

$$\begin{aligned}&\left(-P_{20}\ \alpha_0 + P_{20}\ a_0 - \tau_1\ C_0^{(n+2)} - \tau_2\ C_0^{(n+1)}\right)\\ &+ \left[(-P_{20}\ \alpha_1 + P_{21}\ \alpha_0 - P_{10}\ \alpha_0 + P_{20}\ a_1\right.\\ &\left.+ (-P_{21} + P_{10})a_0 - \tau_1\ C_1^{(n+2)} - \tau_2\ C_1^{(n+1)}\right] x\end{aligned}$$

$$
\begin{aligned}
&+\sum_{2}^{n}\left\{P_{20}\ a_r+\left(\frac{(r-2)\ P_{21}+P_{10}}{r}\right)a_{r-1}+\left(\frac{P_{00}+(r-2)\,P_{11}+(r-2)\ (r-3)P_{22}}{(r-1)r}\right)a_{r-2}\right.\\
&\left.-\frac{f_{r-2}}{(r-1)r}-\tau_1\,C_r^{(n+2)}-\tau_2\,C_r^{(n+1)}\right\}x^r+\left\{\left(\frac{(n-1)\ P_{21}+P_{10}}{n+1}\right)a_n\right.\\
&+\left(\frac{P_{00}+(n-1)\ P_{11}+(n-1)\ (n-2)\ P_{22}}{n(n+1)}\right)a_{n-1}-\left.\frac{f_{n-1}}{n(n+1)}-\tau_1\ C_{n+1}^{(n+2)}-\tau_2\ C_{n+1}^{(n+1)}\right\}x^{n+1}\\
&+\left\{\left(\frac{P_{00}+n\,P_{11}+n(n-1)\ P_{22}}{(n+1)(n+2)}\right)a_n-\frac{f_n}{(n+1)(n+2)}-\tau_1\ C_{n+2}^{(n+2)}\right\}x^{n+2}=0
\end{aligned}
\tag{3.15}
$$

This yields the system

$$
\begin{aligned}
&P_{20}\ a_0-\tau_1\ C_0^{(n+2)}-\tau_2\ C_0^{(n+2)}=\alpha_0\ P_{20}\\
&(-P_{21}+P_{10})\,a_0+P_{20}a_1-\tau_1\ C_1^{(n+2)}-\tau_2\ C_1^{(n+2)}=\alpha_1\ P_{20}-\alpha_0\ P_{21}+\alpha_0\ P_{10}\\
&P_{20}\ a_r+\left(\frac{(r-2)\ P_{21}+P_{10}}{r}\right)a_{r-1}+\left(\frac{P_{00}+(r-2)\ P_{11}+(r-2)(r-3)P_{22}}{(r-1)r}\right)a_{r-2}-\\
&\qquad-\tau_1\ C_r^{(n+2)}-\tau_2\ C_r^{(n+1)}=\frac{f_{r-2}}{(r-1)r}\ ,\ r=2(1)n\\
&\left(\frac{(n-1)\ P_{21}+P_{10}}{n+1}\right)a_n+\left(\frac{P_{00}+(n-1)\ P_{11}+(n-1)\,(n-2)P_{22}}{n(n+1)}\right)a_{n-1}\\
&\qquad-\tau_1\ C_{n+1}^{(n+2)}-\tau_2\ C_{n+1}^{(n+1)}=\frac{f_{n-1}}{n(n+1)}\\
&\left(\frac{P_{00}+nP_{11}+(n-1)n\ P_{22}}{(n+1)\ (n+2)}\right)a_n-\tau_1\ C_{n+2}^{(n+2)}=\frac{f_n}{(n+1)(n+2)}
\end{aligned}
\tag{3.16}
$$

We solve this system for a_r , r = 0(1)n and τ_1 , τ_2 to subsequently obtain from Eq. (1.2) the approximant $y_{n,2}(x)$ of y(x).

Error estimation for the integrated form: From Eq. (2.6), we have for problem Eq. (3.1)

$$
\begin{aligned}
&\int_0^x\int_0^u\left(P_{20}+P_{21}t+P_{22}t^2\right)\left(e_n''(t)\right)_{n+1}\ dt\ du\\
&+\int_0^x\int_0^u\left(P_{10}+P_{11}t\right)\left(e_n'(t)\right)_{n+1}\ dt\ du+\int_0^x\int_0^u P_{00}\left(e_n(t)\right)_{n+1}\ dt\ du\\
&=-\int_0^x\int_0^u\left(\tau_1\ \sum_{r=0}^{n}C_r^{(n)}\ t^r+\tau_2\ \sum_{r=0}^{n-1}C_r^{(n-1)}\ t^r\right)dt\ du\\
&+\ \bar{\tau}_1\ \sum_{r=0}^{n+3}C_r^{(n+3)}\ x^r+\bar{\tau}_2\ \sum_{r=0}^{n+2}C_r^{(n+2)}\ x^r
\end{aligned}
\tag{3.17}
$$

where, $(e_n(x))_{n+1}$ is given by Eq. (3.7). This leads to the Eq.

$$
\begin{aligned}
&\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left\{P_{20}\ \sum_{r=0}^{n-12}C_r^{(n-1)}\ x^{r+2}+P_{21}\left(\sum_{r=0}^{n-1}C_r^{(n-1)}\ x^{r+3}-2\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}\ x^{r+3}}{(r+3)}\right)\right.\\
&+P_{22}\left(\sum_{r=0}^{n-1}C_r^{(n-1)}\ x^{r+4}\ \sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+4)}-2\ \sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+3)\,(r+4)}\right)\\
&+P_{10}\ \sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+3}}{(r+3)}\Bigg)+P_{11}\left(\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+4)}-\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+3)\,(r+4)}\right)\\
&\left.+P_{00}\ \sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+3}}{(r+3)\ (r+4)}\right\}=\tau_1\ \sum_{r=0}^{n}\frac{C_r^{(n)}x^{r+2}}{(r+1)\,(r+2)}-\tau_2\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+2}}{(r+1)\,(r+2)}\\
&+\bar{\tau}_1\ \sum_{r=0}^{n+3}C_r^{(n+3)}x^r+\bar{\tau}_2\sum_{r=0}^{n+2}C_r^{(n+2)}x^r
\end{aligned}
\tag{3.18}
$$

We equate coefficients of x^{n+3}, x^{n+2} and x^{n+1} to have the system

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left[\frac{P_{00}+(n+1)P_{11}+n(n+1)P_{22}}{(n+2)(n+3)}\right]C_{n-1}^{(n-1)} = \bar{\tau}_1 C_{n+3}^{(n+3)}$$

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left[\left(\frac{P_{10}+nP_{21}}{(n+2)}\right)C_{n-1}^{(n-1)} + \left(\frac{P_{10}+nP_{11}+(n-1)nP_{22}}{(n+1)(n+2)}\right)C_{n-1}^{(n-1)}\right] = \bar{\tau}_1 C_{n+2}^{(n+3)} + \bar{\tau}_2 C_{n+2}^{(n+2)} - \frac{\tau_1 C_n^{(n)}}{(n+1)(n+2)} \tag{3.19}$$

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left[P_{20}C_{n-1}^{(n-1)} + \left(\frac{P_{10}+(n-1)P_{21}}{(n+1)}\right)C_{n-2}^{(n-1)} + \left(P_{00}+(n-1)P_{11}+(n-2)(n-1)P_{22}\right)C_{n-3}^{(n-1)}\right] = \bar{\tau}_1 C_{n+1}^{(n+3)} + \bar{\tau}_2 C_{n+1}^{(n+2)} - \frac{\tau_1 C_{n-1}^{(n-1)}}{n(n+1)} - \frac{\tau_2 C_{n-1}^{(n-1)}}{n(n+1)}$$

We solve this by forward elimination for φ_n to get

$$\varphi_n = \frac{2^{2n-3}\tau_2}{n(n+1)R_3} \tag{3.20}$$

where,

$$R_3 = \frac{C_{n+1}^{(n-3)}}{2^{2n+5}}\left[\frac{P_{00}+(n+1)P_{11}+n(n+1)P_{22})}{(n+2)(n+3)}\right]$$
$$-\left\{\left[(n+2)(n+3)\left[2(n+1)(P_{10}+nP_{21}) - (n-1)\left(P_{00}+nP_{11}+n^2-nP_{22}\right)\right]\right. + \frac{(n+1)(n+2)(n+3)\left[P_{00}+(n+1)P_{11}+n(n+1)P_{22}\right]\}}{4(n+1)(n+2)(n+3)}$$

$$+\frac{(n-3)(P_{10}+nP_{21}-P_{21})}{2(n+1)} + \frac{\left(P_{00} - P_{11}+nP_{11}\ n^2P_{22} - 3nP_{22} + 2P_{22}\right)C_{n-3}^{(n-1)}}{2^{2n-3}\ n(n+1)} - P_{20} \tag{3.21}$$

Hence, we obtain the error estimate

$$\bar{\varepsilon}_2 = |\varphi_n| \big/ \left|C_{n-1}^{(n-1)}\right| = \frac{\tau_2}{n(n+1)|R_3|} \tag{3.22}$$

A Tau approximant by the recursive form:

$$y_n(x) = \sum_{r=0}^{F} {}_r\, Q_r(x) + \tau_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(x) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(x) \tag{3.23}$$

If $F < n$, then this becomes

$$y_n(x) = \sum_{r=0}^{n-1}\left[f_r + \tau_1 C_r^{(n)} + \tau_2 C_r^{(n-1)}\right] Q_r(x) + \left[f_n + \tau_1 C_n^{(n)}\right]Q_n(x) \tag{3.24}$$

where, the sequence $\{Q_r(x)\}$, $r \in N_0 - S$ is generated thus: From Eq. (1.9) and Eq. (3.6) and by the linearity of L,

$$Lx^r = \left(P_{20} + P_{21}x + P_{22}x^2\right) r(r-1)\, x^{r-2} + \left(P_{10} + P_{11}x\right) r x^{r-1} + P_{00}x^r = L\left\{r(r-1)P_{20}Q_{r-2}(x) + \left(rP_{10} + r(r-1)P_{21}\right) Q_{r-1}(x) + \left(P_{00} + rP_{11} + r(r-1)P_{22}\right)Q_r(x)\right\}$$

By assuming the existence of L^{-1}, we obtain

$$Q_r(x) = \frac{x^r - r(r-1)P_{20}Q_{r-2}(x) - \left(rP_{10} + (r-1)P_{21}\right)Q_{r-1}(x)}{P_{00} + rP_{11} + r(r-1)P_{22}} \tag{3.25}$$

provided that $P_{00} + rP_{11} + r(r\text{-}1)P_{2.2} \neq 0$ and for $r = 0, 1, 2, \ldots$

Now from Eq. (1.10) and (1.11) we get for problem Eq. (3.1):

$$\sum_{r=0}^{n-1} a_r\; Q_r(x) + a_n\; Q_n(x) = \sum_{r=0}^{n-1}\left[f_r + \tau_1\; C_r^{(n)} + \tau_2\; C_r^{(n-1)}\right] Q_r(x) + \left[f_n + \tau_1\; C_n^{(n)}\right] Q_n(x)$$

giving us

$$\begin{aligned} a_r &= f_r + \tau_1\; C_r^{(n)} + \tau_2\; C_r^{(n-1)},\; r = 0(1)\; n-1 \\ a_n &= f_n + \tau_1\; C_n^{(n)} \end{aligned} \tag{3.26}$$

The values of τ_1 and τ_2 are obtained by applying the conditions Eq. (1.3b-2.23) and this gives the system

$$\begin{bmatrix} y_n(0) \\ y_n'(0) \end{bmatrix} = \begin{bmatrix} \sum_{r=0}^{n} C_r^{(n)}\; Q_r(0) & \sum_{r=0}^{n-1} C_r^{(n-1)}\; Q_r(0) \\ \sum_{r=0}^{n} C_r^{(n)}\; Q_r'(0) & \sum_{r=0}^{n-1} C_r^{(n-1)}\; Q_r'(0) \end{bmatrix} \tag{3.27}$$
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \alpha_0 - \sum_{r=0}^{F} f_r\; Q_r(0) \\ \alpha_1 - \sum_{r=0}^{F} f_r\; Q_r'(0) \end{bmatrix}$$

We solve the Tau system Eq. (3.27) for τ_1 and τ_2, insert them in Eq. (3.26) to determine a_r, r = 0 (1) n and then obtain the desired approximant $y_{n,3}(x)$ from Eq. (1.10).

Error estimation for the recursive form: From the conditions Eq. (1.3b) we get

$$\sum_{r=0}^{F} f_r\; Q_r(0) + \tau_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0) = \alpha_0 \tag{3.28}$$

$$\sum_{r=0}^{F} f_r\; Q_r'(0) + \tau_1 \sum_{r=0}^{n} C_r^{(n)} Q_r'(0) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0) = \alpha_1 \tag{3.29}$$

From Eq. (3.28) we have

$$\tau_1 = \left(\alpha_0 - \sum_{r=0}^{n} f_r\; Q_r(0) - \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right)^{-1}$$

We insert this in Eq. (3.29) to obtain

$$\tau_2\left[\left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right]$$
$$= \alpha_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) - \left(\alpha_0 - \sum_{r=0}^{F} f_r\, Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)$$
$$|\tau_2|\left|\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0) - \left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|$$
$$\le \left|\alpha_1\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right)\left(\alpha_0 - \sum_{r=0}^{F} f_r\, Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| + \bar{\varepsilon}_1$$

since, $\bar{\varepsilon}_1 > 0$, given by Eq. (3.12). This leads to

$$|\tau_2| \le \frac{|R_1|\left|\alpha_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) - \left(\alpha_0 - \sum_{r=0}^{F} f_r\, Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|}{|R_1|\left|\left(\alpha_1 \sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| - 2^{2n}} \tag{3.30}$$

Hence,

$$\bar{\varepsilon}_1 = \frac{2^{2n}|\tau_2|}{|R_1|} \le \frac{\left|\alpha_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) - \left(\alpha_0 - \sum_{r=0}^{F} f_r\, Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|}{2^{-2n}\left|\left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| - 1} = \bar{\varepsilon}_1 \tag{3.31}$$

where, R_1 is given in section by Eq. (3.11). Thus, our new error estimate is

$$\bar{\varepsilon}_3 = \frac{\left|\alpha_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) - \left(\alpha_0 - \sum_{r=0}^{F} f_r Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q'_r(0)\right)\right|}{2^{-2n}\left|\left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q'_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q'_r(0)\right)\right| - 1} = \bar{\varepsilon}_1 \tag{3.32}$$

A striking interest in respect of Eq. (3.32) is that an estimate is possible prior to the computation of $y_n(x)$ once the canonical polynomials are known.

NUMERICAL EXAMPLES

We consider here 5 selected examples for experimentation with our results of the preceding the study. The exact errors are defined as:

$$\varepsilon_\ell = \max_{a \le x \le b} \left\{\left|y(x_k) - y_{n,\ell}(x_k)\right|\right\}, \ell = 1, 2, 3$$

where, $\{x_k\} = \{0.01k\}$, for $k = 0(1) \le 100$.

The numerical results are presented in the Table 1-5 the examples:

Example 4.1:

$$y''(x) + y(x) = x^2,\ y(0) = 0,\ y'(0) = 3,\ 0 \le x \le 1 \tag{4.1}$$

Analytical solution $y(x) = 2\cos x + 3\sin x + x^2 - 2$

Example 4.2:

$$y''(x) + 25y(x) = 5x^2 + x,\ y(0) = 0.2,\ y'(0) = 0,\ 0 \le x \le 1 \tag{4.2}$$

Analytical solution $y(x) = (27\cos 5x - \sin 5x + 25x^2 + 5x - 2)/125$

Example 4.3:

$$y''(x) - y'(x) - 2y(x) = 8,\ y(0) = 0,\ y'(0) = 10,\ 0 \le x \le 1 \tag{4.3}$$

Exact solution,

$$y(x) = \frac{1}{2}e^{2x} - \frac{3}{2}e^{2x} + 2e^{3x}$$

Example 4.4:

$$y''(x) + 5y'(x) + 6y(x) = 0,\ y(0) = 1,\ y'(0) = -1,\ 0 \le x \le 1 \tag{4.4}$$

Table 1: Error and error estimates for problems 4.1

Method	Error	Degree (n) 2	3	4	5
Differential	$\bar{\varepsilon}_1$	5.93×10^{-2}	7.59×10^{-3}	2.07×10^{-4}	1.68×10^{-5}
Form	ε_1	8.88×10^{-2}	2.65×10^{-3}	9.03×10^{-4}	4.15×10^{-5}
Interpolated	$\bar{\varepsilon}_2$	$6.46X\ 10^{-4}$	4.23×10^{-5}	7.00×10^{-7}	3.85×10^{-8}
Form	ε_2	8.84×10^{-3}	1.06×10^{-3}	2.86×10^{-5}	2.14×10^{-6}
Recursive	$\bar{\varepsilon}_3$	8.62×10^{-2}	1.82×10^{-3}	5.53×10^{-4}	1.99×10^{-6}
Form	ε_3	9.21×10^{-1}	2.86×10^{-2}	1.54×10^{-3}	9.96×10^{-5}

Table 2: Error and error estimates for problems 4.2

Method	Error	Degree (n) 2	3	4	5
Differential	$\bar{\varepsilon}_1$	1.74×10^{-1}	5.18×10^{-2}	2.15×10^{-2}	6.22×10^{-3}
Form	ε_1	3.24×10^{-1}	2.28×10^{-1}	3.45×10^{-3}	5.21×10^{-4}
Interpolated	$\bar{\varepsilon}_2$	1.36×10^{-3}	7.77×10^{-4}	1.67×10^{-4}	7.38×10^{-6}
Form	ε_2	1.01×10^{-1}	3.76×10^{-3}	1.34×10^{-4}	5.46×10^{-5}
Recursive	$\bar{\varepsilon}_3$	1.05×10^{-1}	1.03×10^{-1}	1.36×10^{-2}	7.18×10^{-3}
Form	$\bar{\varepsilon}_3$	5.78×10^{-1}	1.54×10^{-2}	1.52×10^{-2}	2.35×10^{-3}

Table 3: Error and Error estimates for problems 4

Method	Error	Degree (n) 2	3	4	5
Differential	$\bar{\varepsilon}_1$	4.00×10^{-1}	1.10×10^{-0}	8.18×10^{-2}	6.31×10^{-3}
Form	ε_1	8.97×10^{-1}	3.36×10^{-0}	7.12×10^{-2}	8.60×10^{-3}
Interpolated	$\bar{\varepsilon}_2$	9.42×10^{-2}	4.27×10^{-3}	2.12×10^{-4}	5.07×10^{-3}
Form	ε_2	2.85×10^{-1}	2.95×10^{-2}	5.40×10^{-3}	4.56×10^{-4}
Recursive	$\bar{\varepsilon}_3$	9.36×10^{-1}	4.04×10^{-1}	8.12×10^{-2}	2.14×10^{-3}
Form	$\bar{\varepsilon}_3$	4.01×10^{-1}	3.33×10^{-1}	5.36×10^{-2}	6.69×10^{3}

Table 4: Error and error estimates for problems 4.4

Method	Error	Degree (n) 2	3	4	5
Differential	$\bar{\varepsilon}_1$	3.43×10^{-2}	1.38×10^{-2}	2.18×10^{-3}	4.24×10^{-4}
Form	ε_1	4.17×10^{-2}	4.63×10^{-2}	8.03×10^{-3}	8.97×10^{-4}
Interpolated	$\bar{\varepsilon}_2$	1.38×10^{-4}	5.38×10^{-3}	8.84×10^{-6}	1.10×10^{-6}
Form	ε_2	4.77×10^{-3}	2.89×10^{-3}	6.63×10^{-4}	9.98×10^{-5}
Recursive	$\bar{\varepsilon}_3$	3.40×10^{-2}	3.47×10^{-3}	6.70×10^{-4}	9.58×10^{-5}
Form	$\bar{\varepsilon}_3$	4.49×10^{-1}	7.98×10^{-2}	4.03×10^{-3}	3.02×10^{-4}

Table 5: Error and error estimates for problems 4.5

Method	Error	Degree (n) 2	3	4	5
Differential	$\bar{\varepsilon}_1$	1.89×10^{-1}	1.40×10^{-0}	5.60×10^{-2}	3.92×10^{-3}
Form	ε_1	2.26×10^{-0}	6.47×10^{-0}	7.46×10^{-2}	5.04×10^{-3}
Interpolated	$\bar{\varepsilon}_2$	1.63×10^{-1}	4.85×10^{-3}	2.35×10^{-4}	1.17×10^{-5}
Form	ε_2	6.93×10^{-1}	2.21×10^{-2}	3.43×10^{-3}	2.28×10^{-4}
Recursive	$\bar{\varepsilon}_3$	2.80×10^{-1}	1.93×10^{-1}	1.97×10^{-2}	2.62×10^{-3}
Form	$\bar{\varepsilon}_3$	1.95×10^{-2}	6.73×10^{-1}	5.22×10^{-2}	3.33×10^{-3}

True solution, $y(x) = 2e^{-2x} - e^{-3x}$.

Examples 4.5:

$$y''(x) - 3y'(x) + 2y(x) = x^2, \quad y(0) = 3/4,\ y'(0) = 5/2,\ 0 \le x \le 1 \tag{4.5}$$

Closed form solution,

$$y(x) = 2e^{2x} - 3e^{x} + \frac{1}{4}(2x^2 + 6x + 7)$$

CONCLUSION

The Tau method for solution of initial value problems in a class of second order ordinary differential equations with non-overdetermination has been presented. Three variants of the method were considered for the corresponding Tau approximants of their desired analytic solutions and the associated error estimates were also obtained.

For all the numerical examples considered the error estimates closely approximate the exact error. The difficulty in the generation of the so-called canonical polynomials for high degree Tau approximants limited the scope of the research to approximations of maximum 5°. While, the differential form may easily be generalized for all classes of differential equations which lies within the scope of the Tau methods as we had reported in Adeniyi *et al.* (1990), integrated the interpolated form has the advantage of higher order accuracy than the other 2 variants due to the higher order of perturbation term it involves and the recursive form, though very cumbersome for high degree approximants, has the advantages of minimum order Tau system, non-dependence of its canonical polynomials on the boundary conditions, as well as the re-usability of these polynomials for approximants of higher degree. The error estimate, in the latter case, may also be determined even prior to the solution of its corresponding Tau problem.

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