

Numerical Investigation of Second Order Singular System Using Single-Term Haar Wavelet Series Method

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Abstract: Recently, Single-Term Haar Wavelet series (STHW) is introduced to overcome the difficulties in solving some singular system problems. Preliminary experiments have shown that this method is usually more efficient than the other methods. In this study, STHW is developed to approximate the solution of second order singular system problems in the Ordinary Differential Equations (ODE). These types of problems arisen in science and engineering field frequently. The obtained discrete solutions using the STHW are found to be very accurate and are compared with the exact solutions of the second order singular system problems in the ODE. The results obtained show that the STHW is more useful for solving second order singular system problems in the ODE and the solution can be obtained for any length of time.

Key words: Haar wavelet, single-term Haar wavelet series ordinary differential equations, singular systems, second order singular systems

INTRODUCTION

Many physical processes are most naturally and easily modeled as singular differential equations. In recent years there has been an increased interest in several areas in exploiting the advantages of these implicit models. Considerable amount of research has been done on the design of observers for linear time-invariant singular systems. However, many singular systems of interest are linear time-invariant or linear time-varying and it is difficult to find the solution of time-varying singular systems.

In science and engineering, singular systems often have to be solved. Although, some cases can be solved analytically, the majority of singular systems are too complicated to have analytical solutions. Even when analytical solutions can be found they are not always useful in practice since the computational cost involved is very high. In recent years, there has been an increased interest in several methods were arisen to solve the second order singular system problems. STHW plays an important role in both the analysis and numerical solution of singular systems. STHW can have a significant impact on what is considered a practical approach and on the

types of problems that can be solved. However, working with singular systems places special demands on STHW codes. Nandhakumar *et al.* (2009) introduce Haar Wavelet series to numerical investigation of an industrial robot arm control problem.

Sekar and team of his researchers (Sekar and Manonmani, 2009; Sekar and Balaji, 2010; Sekar and Duraisamy, 2010; Sekar and Jaganathan, 2010; Sekar and Kumar, 2011; Sekar and Paramanathan, 2011; Sekar and Prabakaran, 2010; Sekar and Vijayarakan, 2010) introduced the STHW to study the time-varying nonlinear singular systems, analysis of the differential equations of the sphere to study on CNN based hole-filter template design, analysis of the singular and stiff delay systems and nonlinear singular systems from fluid dynamics, numerical investigation of nonlinear volterra-hammerstein integral equations to study on periodic and oscillatory problems and numerical solution of nonlinear problems in the calculus of variations. Sekar (2006) solving singular systems using RK-butcher algorithm, researchers consider the same system by using the STHW Method. The results are compared with Haar Wavelet Method and with exact solution of the problem.

SINGLE-TERM HAAR WAVELET SERIES METHOD

The orthogonal set of Haar wavelets is a group of square waves with magnitude of in some intervals and zeros elsewhere (Sekar and Manonmani, 2009). In general:

$$h_n(t) = h_1(2^j t - k), n = 2^j + k \left. \begin{array}{l} j \geq 0, 0 \leq k < 2^j, n, j, k \in Z \end{array} \right\}$$

$$h_1(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \end{cases}$$

Namely, each Haar wavelet contains one and just one square wave and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems.

Any function $y(t)$ which is square integrable in the interval $[0,1)$. Can be expanded in a Haar series with an infinite number of terms:

$$y(t) = \sum_{i=0}^{\infty} c_i h_i(t), i = 2^j + k \left. \begin{array}{l} j \geq 0, 0 \leq k < 2^j, n, j, t \in [0,1] \end{array} \right\} \quad (1)$$

where the Haar coefficients:

$$c_i = 2^j \int_0^1 y(t) h_i(t) dt$$

are determined such that the following integral square error ϵ is minimized:

$$\epsilon = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt \left. \begin{array}{l} m = 2^j, j \in \{0\} \cup N \end{array} \right\}$$

usually, the series expansion Eq. 1 contains an infinite number of terms for a smooth $y(t)$. If $y(t)$ is a piecewise constant or may be approximated as a piecewise constant then the sum in Eq. 1 will be terminated after m terms that is:

$$\begin{aligned} y(t) &\approx \sum_{i=0}^{m-1} c_i h_i(t) = c_{(m)}^T h_{(m)}(t), t \in [0,1] \\ c_{(m)}(t) &= [c_0 c_1 \dots c_{m-1}]^T \\ h_{(m)}(t) &= [h_0(t) h_1(t) \dots h_{m-1}(t)]^T \end{aligned} \quad (2)$$

where, T indicates transposition, the subscript m in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix P (Sekar and Manonmani, 2009):

$$\int h_{(m)}(\tau) d\tau \approx P_{(m \times m)} h_{(m)}(t), t \in [0,1]$$

where, the m -square matrix P is called the operational matrix of integration and single-term $P_{(1 \times 1)} = 1/2$. Let us define (12):

$$h_{(m)}(t) h_{(m)}^T(t) \approx M_{(m \times m)}(t) \quad (3)$$

and $M_{(1 \times 1)}(t) = h_0(t)$. Equation 3 satisfies:

$$M_{(m \times m)}(t) c_{(m)} = C_{(m \times m)} h_{(m)}(t)$$

where, $C_{(m)}$ is defined in Eq. 2 and $C_{(1 \times 1)} = c_0$.

SOLUTION OF THE TWO POINT BOUNDARY VALUE PROBLEM BY STHWS

Consider the two point boundary value problem:

$$y''(t) = Ay'(t) + By(t) + Cu(t) \quad (4)$$

with $y(0) = y_0$ and $y(1) = y_1$, where A and B is an $n \times m$ matrix, C is an $n \times r$ matrix, $y(t)$ is an n state vector. With the STHWS approach, the given function is expanded into Single term Haar wavelet series in the normalized interval $[0, 1)$ which corresponds to $t \in [0, 1/m)$ by defining $mt = \tau$, m being any integer. The Eq. 4 becomes in the normalized interval:

$$\begin{aligned} y''(\tau) &= \frac{A(\tau)}{m} y'(\tau) + \frac{B(\tau)}{m^2} y(\tau) + \frac{C(\tau)}{m^2} u(\tau) \\ \text{with } y(0) &= y_0, y(1) = y_1, \tau \in [0,1] \end{aligned} \quad (5)$$

Let $y''(\tau)$, $y'(\tau)$ and $y(\tau)$ be expanded by STHWS in the n th interval as:

$$\begin{aligned} y''(\tau) &= C_n h_0(\tau) \\ y'(\tau) &= A_n h_0(\tau) \\ y(\tau) &= B_n h_0(\tau) \\ u(\tau) &= H_n h_0(\tau) \end{aligned} \quad (6)$$

the following set of recursive relations has been obtained to determine the discrete values of $y'(\tau)$ and $y(\tau)$ in Eq. 4 with $y(0) = y_0$ and $y(1) = y_1/m$:

$$C_n h_0(\tau) = \left(M_0 - \frac{M_1}{2m} - \frac{M_2}{4m^2} - \dots \right)^{-1} G_n \quad (7)$$

$$\begin{aligned} A_n &= \frac{1}{2} C_n + y'(n-1) \\ B_n &= \frac{1}{2} A_n + y'(n-1) \\ y'(n) &= C_n + y'(n-1) \\ y(n) &= A_n + y(n-1) \end{aligned} \quad (8)$$

Where:

$$G_n = \left[\frac{A}{m} + \frac{B}{2m^2} \right] y'(n-1) + \frac{B}{m^2} y(n-1) + \frac{C}{m^2} H_n$$

and $n = 1, 2, 3, \dots$, the interval number, substituting Eq. 6-8 into Eq. 5. Researchers obtain linear algebraic equations and they may be solved to obtain $y(n)$. The single-term Haar wavelet series with piecewise constant orthogonal functions is an extension of single-term Haar wavelet series algorithm (Eq. 4) that avoid the inverse of the big matrix induced by the kronecker product. This approach is applicable for any transform with piecewise constant basis.

The above equations give solutions for all integer values of m , not necessarily restricted to 2^k as proposed in the original Haar wavelet analysis. Higher accuracy can be obtained for increased value of m . Using the above recursive relations, piecewise constant solutions of second order rate vector \ddot{y} rate vector \dot{y} and state vector y can be obtained. In Eq. 8 even though the matrix is singular, the difference:

$$\left[M_0 - \frac{M_1}{2m} - \frac{M_2}{4m^2} \right]$$

turns out to be non-singular which is an added advantage of this method.

GENERAL FORM OF SECOND ORDER SINGULAR SYSTEMS

A time-invariant second order singular system of the form:

$$K\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t) \quad (9)$$

with initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ is considered where K is an $n \times n$ singular matrix A and B are $n \times n$ and $n \times p$ constant matrices, respectively. The $x(t)$ is an n -state vector and $u(t)$ is the p -input control vector and C is an $n \times p$ matrix. Also, a time-varying second order singular system is represented by:

$$K(t)\ddot{x}(t) = A(t)\dot{x}(t) + B(t)x(t) + C(t)u(t) \quad (10)$$

With initial condition

$$x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0$$

Where:

- $K(t)$ = An $n \times n$ singular matrix
- $A(t)$ and $B(t)$ = $n \times n$ and $n \times p$ matrices, respectively
- $C(t)$ = An $n \times p$ matrix

The elements (not necessarily all the elements) of the matrices $K(t)$, $A(t)$ and $B(t)$ are time dependent.

EXAMPLE OF SECOND ORDER SINGULAR SYSTEM OF TIME-INVARIANT CASE

The second order linear time-invariant singular system of the form (9) and $B = C = 0$, becomes $\ddot{x}(t) = A\dot{x}(t)$ with initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \ddot{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \dot{x}$$

with initial conditions:

$$x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \dot{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the exact solution is:

$$\begin{aligned} x_1(t) &= 2.0 \\ x_2(t) &= -1 + 2 \exp(t/2) \end{aligned}$$

Using RK-Butcher algorithm and STHW Method to solve the problem 5, the approximate and exact solutions for x_1 and x_2 are calculated for different values of time t and the error between them are shown in the Fig. 1 along with exact solutions. Error graphs are also shown in Fig. 1 shows the efficiency of STHW.

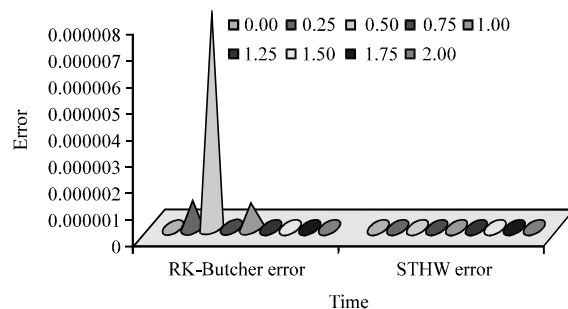


Fig. 1: Error graph for problem 5 at x_2

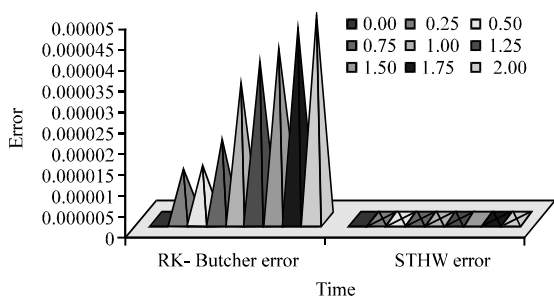


Fig. 2: Error graph for problem 6 at x_1

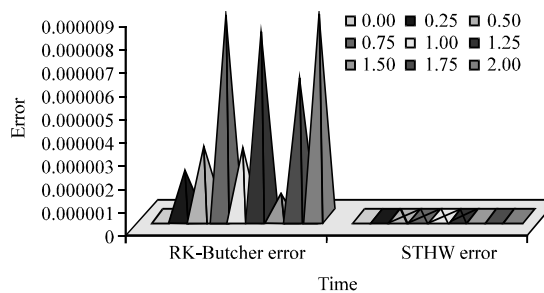


Fig. 3: Error graph for problem 6 at x_2

EXAMPLE OF SECOND ORDER SINGULAR SYSTEM OF TIME VARYING CASE

The following second order linear time-varying singular system of the form (10):

$$\begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -t \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t^2 + 2t & 0 \end{bmatrix} \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

is considered with initial conditions:

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And:

$$\dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This problem can be rearranged as:

$$\begin{aligned} \ddot{x}_1 &= \dot{x}_1 - 3\dot{x}_2 \\ \ddot{x}_2 &= \dot{x}_2 + 2e^t(t+1) \end{aligned}$$

The exact solution is:

$$\begin{aligned} x_1(t) &= 1 - t^3 e^t \\ x_2(t) &= t^2 e^t \\ \dot{x}_1(t) &= t^2 e^t (-t-3) \\ \dot{x}_2(t) &= t e^t (t+2) \end{aligned}$$

Using STHW and RK-Butcher Algorithm Method to solve the problem 6, the approximate and exact solutions for and are calculated for different values of time t and the error between them are shown in the Fig. 2 and 3 along with exact solutions. Also, error graphs are shown in Fig. 2 and 3.

CONCLUSION

The STHW is applied to solve second order singular system problems in ordinary differential equations. The obtained results of the second order singular system problems 5 and 6 using STHW is very closer to these exact solutions of the problem when compared to the RK-Butcher Algorithm Method. This method is simple to use and STHW Method is very practical for computational purpose since considerable computational effort is required to improve accuracy. From the Fig. 1-3 shows STHW fit well to these types of problems compared to the RK-Butcher Algorithm Method. This STHW provided a momentum for advancing numerical methods for solving second order singular system problems in ordinary differential equations.

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