Total Vertex Irregularity Strength of Comb Product Graph of $P_m$ and $C_n$

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Abstract: A vertex irregular total $k$-labeling of a graph $G(V, E)$ with a non-empty set $V$ of vertices and a set $E$ of edges is a labeling $\lambda: V \cup E \rightarrow \{1, 2, ..., k\}$ such that for every two distinct vertices have different weight. The weight of a vertex $v$, under a total labeling $\lambda$, is the sum of the label of vertex $v$ and all labels of edges that incident with $v$. In other word, $w_t(v) = \lambda(v) + \sum_{uv \in E} \lambda(ux)$. The total vertex irregularity strength, denoted by $\text{tvst}(G)$ is the minimum biggest label that use to label graph $G$ with the vertex irregular total labeling. Some classes of graphs have been obtained its total vertex irregularity strength. In this study, researcher observe about the total vertex irregularity strength of comb product graph of $P_m$ and $C_n$, denoted by $TVs(G)$. The result of this research is $\text{tvst}(P_m \times C_n) = \lceil (n-1)m + 2/3 \rceil$ for $m \geq 3$ for odd number $m$.

Key words: Total labeling, comb product, vertex irregularity strength, vertices, graphs, weight, edges

INTRODUCTION

Let $G = (V, E)$ is a simple graph. Graph labeling is a function that carries graph elements to the numbers, usually to the positive or non-negative integers that satisfies a certain requirement. The most common choices of domain are the set of all vertices (vertex labeling), the set of all edges (edge labeling) or the set of all vertices and edges (total labeling). Other domain are possible.

The labeling of graphs was introduced by Sedlacek in 1963. Based on the weight of the graph elements, the labeling divided into several types, they are graceful labeling, magic labeling, antimagic labeling and irregular labeling, etc. Irregular labeling has been introduced by Chartrand. But, their study titled “Irregular Network” was published in 1988.

Baca et al. (2007) introduced two kinds of irregular total labeling, namely, vertex irregular total labeling and edge irregular total labeling. Let $G = (V, E)$ be a graph, function $f: V \cup E \rightarrow \{1, 2, 3, ..., k\}$ is called vertex irregular total $k$-labeling if every two different vertices in $V$ have different weight. The weight of vertex $x$ in $V$ under function $f$ is $w_t(x) = f(x) + \sum_{uv \in E} f(xy)$. The minimum $k$ for which a graph $G$ has a the vertex irregular total $k$-labeling is called total vertex irregularity strength of $G$, denoted by $TVs(G)$. Baca et al. (2007) obtained lower bound and upper bound of total vertex irregularity strength of a graph $G$ such as the following theorem.

Theorem 1: Let $G$ be a graph $(p, q)$ with minimum degree $\delta$ and maximum degree $\Delta$ then (Baca et al., 2007):

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq p+\Delta-2\delta+1$$

MATERIALS AND METHODS

Determination of total vertex irregularity strength of all of graphs has not done completely. Until now, only several of class of graph have been determined its total vertex irregularity strength. Baca et al. (2007) observed about total vertex irregularity strength of cycles, paths, stars, complete graphs and prism graphs. By Baskoro et al. (2010) determined a new upper bound for the total vertex irregularity strength of graphs and found total vertex irregularity strength of forest. Nurdin also investigated the total vertex irregularity strength of trees. And then, Ahmad et al. (2011, 2012, 2014) obtained the total vertex irregularity strength of wheel related graphs, disjoint union of helm graphs and certain classes of unicyclic graph. By Al-Mushayt et al. (2013) also observed about total vertex irregularity strength of convex polytope graphs.

Baca et al. (2007) and Antholzer et al. (2009, 2010) obtained some results about the total vertex irregularity strength and the total edge irregularity strength of cycles and paths as stated in the following theorems.

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Theorem 2: Let $P_n$ and $C_n$ be a path and a cycle, respectively with $n \geq 1$ vertices \cite{Baca et al., 2007}. Then:

$$\text{tes}(P_n) = \left\lceil \frac{n+1}{3} \right\rceil$$

and:

$$\text{tes}(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$$

Theorem 3: Let $P_n$ be a path on $n$ vertices. Then:

$$\text{tvs}(P_n) = \left\lceil \frac{n+1}{3} \right\rceil$$

Corollary 4: Let be an $n$-regular graph on $n$ vertices \cite{Baca et al., 2007}. Then:

$$\left\lceil \frac{n+r}{1+r} \right\rceil \leq \text{tvs}(G) \leq n-r+1$$

Hence, we have:

$$\text{tvs}(C_n) = \text{tes}(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$$

RESULTS AND DISCUSSION

We will determine total vertex irregularity strength of comb product graph of $P_n$ and $C_n$ denoted by $\text{tvs}(P_n \circ C_n)$ for $m \geq 3$ and $n \geq 2$ such as the following theorem.

Theorem 3: For $m \geq 3$ and $n \geq 2$ then:

$$\text{tvs}(P_m \circ C_n) = \left\lceil \frac{(n-1)m+2}{3} \right\rceil$$

Proof: First, we will show $\text{tvs}(P_m \circ C_n) = \left\lceil \frac{(n-1)m+2}{3} \right\rceil$. Note that the smallest degree of vertices of graph $P_m \circ C_n$ is 2 and sum of vertices whose degree 2 in $P_m \circ C_n$ is $(n-1)m$. To get optimal labeling, the weight of each vertices whose degree 2 are consecutive integer from 3, 4, 5, ..., $(n-1)m+2$. While the weight of graph $P_m \circ C_n$ whose degree 2 is the sum of 3 positive integers which is called label, that is a label of that vertex and two labels of edges which is associated with that vertex. Therefore, we get the largest minimum label which is used is $\left\lceil \frac{(n-1)m+2}{3} \right\rceil$ and it isn't possible to have been smaller than $\left\lceil \frac{(n-1)m+2}{3} \right\rceil$. So, we conclude that:

$$\text{tvs}(P_m \circ C_n) \geq \left\lceil \frac{(n-1)m+2}{3} \right\rceil$$

Next, we will prove that:

$$\text{tvs}(P_m \circ C_n) \leq \left\lceil \frac{(n-1)m+2}{3} \right\rceil$$

By showing there exist a vertex irregular total $\left\lceil \frac{(n-1)m+2}{3} \right\rceil$ labeling of graph $P_m \circ C_n$. Let the set of vertices of graph is:

$$V = \{P_m \circ C_n\} = \{x_i, y_i | 1 \leq j \leq m dan 1 \leq i \leq n \}$$

And the set of edges of graph $P_m \circ C_n$ is:

$$E(P_m \circ C_n) = \{x_i y_j | 1 \leq j \leq m \} \cup \{y_i x_{i+1} | 1 \leq j \leq m \} \cup \{y_i x_{i+1} | 1 \leq j \leq m \} \cup \{x_i x_{i+1} | 1 \leq j \leq m \} \cup \{x_i y_j | 1 \leq j \leq m \}$$

Construct a vertex irregular total $\left\lceil \frac{(n-1)m+2}{3} \right\rceil$ labeling as stated in the following. Let:

$$r_j = \left\lceil \frac{(n-1)j+2}{3} \right\rceil \text{ untuk } 1 \leq j \leq m$$

$$\lambda(x_i) = \left\lceil \frac{(n-1)m+j+4-2}{3} \right\rceil \text{ if } j \text{ is positive and } 1 \leq j \leq m-2$$

$$1 \text{, if } m(n-1)+j+4-2 \leq \left\lceil \frac{(n-1)m+j+2}{3} \right\rceil$$

$$\lambda(y_j) = \left\lceil \frac{(n-1)m+n+2}{3} \right\rceil \text{ if } j = m-1$$

$$\left\lceil \frac{(n-1)m+n+2}{3} \right\rceil \text{ if } j = m$$

$$\left\lceil \frac{(n-1)m+2}{3} \right\rceil \text{ if } i = 1 \text{ and } j = 1$$

$$1 \text{, if } i = 1 \text{ and } 2 \leq j \leq m$$

$$\left\lceil \frac{i+n+1}{3} \right\rceil \text{ if } 2 \leq i \leq n-1 \text{ and } 1 \leq j \leq m$$
\[ \lambda(y_{n-1}^i, x_i) = \left[ \frac{2n-1+(j-2)(n-1)}{3} \right] \]

\[ \lambda(y_{n}^i, x_i) = \left[ \frac{2n+(j-2)(n-1)}{3} \right] \]

\[ \lambda(y_{n}^i, x_{n-i}) = \left[ \frac{i+(n-n+1)(j-2)(n-1)}{3} \right] \] for \( 1 \leq i \leq n-3 \) and \( 1 \leq j \leq m \)

\[ \lambda(y_{n}^i, y_{n-i}) = \left[ \frac{2n+(j-3)(n-1)}{3} \right] \] for \( 1 \leq j \leq m \)

\[ \lambda(x_{n}, x_{n-i}) = r_n \] for \( 1 \leq j \leq m-2 \)

\[ \lambda(x_{n}, x_{n-i}) = r_n \] for \( 1 \leq j \leq m \) and \( 3 \leq i \leq n-1 \)

Based on this labeling, we have the weight of the vertices of graph such as following:

\[ \text{wt}(y_{n}^i) = \lambda(y_{n}^i) + \lambda(y_{n-1}^i, y_{n}^i) + \lambda(y_{n}^i, y_{n-i}) \]

\[ = 1 + \left[ \frac{1+n+1+(j-2)(n-1)}{3} \right] + \left[ \frac{2n+(j-2)(n-1)}{3} \right] \]

\[ = 1 + \frac{\frac{3}{3}}{3} \cdot \left[ \frac{3}{3} + \frac{2}{3} \right] \]

\[ = 3 \]

\[ \text{wt}(y_{n}^i) = \lambda(y_{n}^i) + \lambda(y_{n}^i, y_{n-i}) + \lambda(y_{n}^i, y_{n-i}) \]

\[ = \left[ \frac{2n-2+(j-3)(n-1)}{3} \right] + \left[ \frac{1+n+1+(j-2)(n-1)}{3} \right] + \left[ \frac{2n+(j-3)(n-1)}{3} \right] \]

\[ = \left[ \frac{2n-2+jn-j-3n+3}{3} \right] + \left[ \frac{n+1+n+1+(j-2)(n-1)}{3} \right] + \left[ \frac{2n+jn-j-3n+3}{3} \right] \]

\[ = \left[ \frac{-n+1+(j-1)(n-1)}{3} \right] + \left[ \frac{(-n+4)(j-1)(n-1)}{3} \right] + \left[ \frac{n+1+jn-j-3n+3}{3} \right] \]

\[ = \left[ \frac{(-n+1+jn-j)(n-1)}{3} \right] + \left[ \frac{-n+1+jn-j-3n+3}{3} \right] + \left[ \frac{(-n+1+jn-j-2)(n-1)}{3} \right] \]

\[ = \left[ \frac{(-n+1)(n-1)}{3} \right] + \left[ \frac{(-n+1)(n-1)+3}{3} \right] + \left[ \frac{(-n+1)(n-1)+2}{3} \right] \] for \( 2 \leq j \leq m \)

\[ \text{wt}(y_{n}^i) = \lambda(y_{n}^i) + \lambda(y_{n}^i, y_{n-i}) + \lambda(y_{n}^i, y_{n-i}) \]

\[ = \left[ \frac{2n+2+jn-j-3n+3}{3} \right] + \left[ \frac{1+n+1+(j-2)(n-1)}{3} \right] + \left[ \frac{2n+(j-3)(n-1)}{3} \right] \]

\[ = \left[ \frac{2n+2+jn-j-3n+3}{3} \right] + \left[ \frac{n+1+n+1+(j-2)(n-1)}{3} \right] + \left[ \frac{2n+jn-j-2n+3}{3} \right] \]

\[ = \left[ \frac{n+4+jn-j-3n+3}{3} \right] + \left[ \frac{(-n+5)(j-1)(n-1)}{3} \right] + \left[ \frac{n+3+jn-j-3n+3}{3} \right] \]

\[ = \left[ \frac{-n+1+jn-j+3}{3} \right] + \left[ \frac{(-n+1)(j-1)(n-1)+3}{3} \right] + \left[ \frac{(-n+1)(j-1)(n-1)+2}{3} \right] \]

\[ = \left[ \frac{(-n+1)(j-1)(n-1)+3}{3} \right] + \left[ \frac{(-n+1)(j-1)(n-1)+4}{3} \right] + \left[ \frac{(-n+1)(j-1)(n-1)+2}{3} \right] \]

\[ = \left[ \frac{-n+1+jn-j+3}{3} \right] + \left[ \frac{(-n+1)(j-1)(n-1)+4}{3} \right] + \left[ \frac{(-n+1)(j-1)(n-1)+2}{3} \right] \] for \( 2 \leq j \leq m \)
$$wt(x_i) = \lambda(x_1) + \lambda(x_2) + \lambda(x_3) + \lambda(y_1 \cdot x_1) + \lambda(y_2 \cdot x_1)$$

$$= (n-1)m + s - 2 + \frac{(n-1)m + 2}{3} \cdot \frac{n+1}{3} \cdot \frac{n}{3} + \frac{(n-1)m + 2}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{(n-1)m + 2}{3} + \frac{2n+1}{3} \cdot \frac{n+1}{3} + \frac{2n+1}{3} + \frac{n}{3}$$

$$= (n-1)m + s - 2 + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{n}{3}$$

$$= (n-1)m + s - 2 + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{n}{3}$$

If $\frac{(n-1)m + j + 4}{3} > 0$ then:

$$wt(x_j) = \lambda(x_1) + \lambda(x_2) + \lambda(x_3) + \lambda(y_1 \cdot x_1) + \lambda(y_2 \cdot x_1)$$

$$= (n-1)m + j + 4 + \frac{(n-1)m + 2}{3} + \frac{(n-1)m + 2}{3} + \frac{(n-1)m + 2}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{n}{3}$$

$$= \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{n}{3}$$

While if $\frac{(n-1)m + j + 4}{3} \leq 0$ then:

$$wt(x_j) = \lambda(x_1) + \lambda(x_2) + \lambda(x_3) + \lambda(y_1 \cdot x_1) + \lambda(y_2 \cdot x_1)$$

$$= \frac{(n-1)m + 2}{3} + \frac{(n-1)m + 2}{3} + \frac{(n-1)m + 2}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{2n+1}{3} + \frac{n}{3}$$

$$\geq (n-1)m + j + 4 \text{ for } 2 \leq j \leq m-2$$

Note that of function $\lambda$ is mapping from \{V(P_{m}^{n},C_n)\} into \{1, 2, ..., \frac{(n-1)m + 2}{3}\}. The weight of the vertices of graph $P_{m}^{n},C_n$ which is denoted by $wt(x_i)$ is consecutive positive integer from 3 until $(n-1)m + 2$. While the weight of the vertices of graph $P_{m}^{n},C_n$ which is denoted by $wt(x_i)$ is different positive integers starting from $(n-1)m + 3$. This shows that $\lambda$ is a vertex irregular total labeling. Therefore, we conclude that $Tvs(P_{m}^{n},C_n) \in ([n,m] + 2/3)\}.

**CONCLUSION**

In this study, we find that:

$$tvs(P_{m}^{n},C_n) = \left\lfloor \frac{(n-1)m + 2}{3} \right\rfloor$$

for $m \geq 3$ and $n \geq 2$

**REFERENCES**


