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Some Properties on Verbal and Marginal Subgroups in Varietal Nilpotent Groups

Shokufeh Lotfi and S. Mostafa Taheri
Faculty of Mathematics, Golestan University Gorgan, Iran

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Corresponding Author:
Shokufeh Lotfi
Faculty of Mathematics, Golestan University Gorgan, Iran

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Abstract: In the present research, we study the concept of varietal nilpotent groups and prove some results about their verbal and marginal subgroups. Also, we give some relations between varietal nilpotent groups and cyclic groups. Finally, we study the connection between p-groups and varietal nilpotent groups.

INTRODUCTION

Let F be the free group on the countably infinite set $X = \{x_1, x_2, \dots\}$ and $V \subseteq F$ be a set of words. We assume that the reader is familiar with the notion of the variety of groups defined by a set of words for more information see[1].

Throughout the study, we assume that ϑ is variety of groups defined by the set of words V and the notions are taken from[2] and for more investigation see[3-5]. Also, V(G) denotes the verbal subgroup and V*(G) the marginal subgroup of G with respect to ϑ as follows:

$$V(G) := \left\langle \left\{ v(g_1, g_2, \dots, g_n); v \in V, g_i \in G, 1 \le i \le n, n \in N \right\} \right\rangle,$$

$$V^*(G) := \{ g \in G; v(g_1, g_2, \dots, g_i g, \dots, g_n) = v(g_1, g_2, \dots, g_n); v \in V, g_i \in G, 1 \le i \le n, n \in N \}$$

It is easily to check that V(G) is a fully invariant subgroup and V*(G) is a characteristic subgroup in G. For a group G with a normal subgroup N, [NV*G] is defined to be the subgroup generated by the set:

$$\left\{ v(g_1, g_2, \dots, g_i a, \dots, g_n) v(g_1, g_2, \dots, g_n)^{-1}; v \in V, g_i \in G, 1 \le i \le n, a \in N \right\}$$

One may easily show that [NV*G] is the least normal subgroup of G contained in N, satisfying $N/[NV^*G] \subseteq V^*(G/[NV^*G])$. In the special case, when ϑ is the variety of groups defined by the set of words $V = \{[x_1, x_2]\}$ and G be any group, then clearly ϑ is the variety of abelian groups and V(G) = G', V*(G) = Z(G) and if $N \trianglelefteq G$, then [NV*G] = [N, G]. If $V = \{[x_1, x_2, \dots, x_{c+1}]\}$ is the nilpotent word, then ϑ is the variety of nilpotent groups of class at most c and $V(G) = \gamma_{c+1}(G)$, $V^*(G) = Z_c(G)$ and $[NV^*G] = [N_c, G]$.

The following Lemma give basic properties of verbal and marginal subgroups of a group G with respect to the variety ϑ which is useful in our investigation. See[1-3] for more details.

Lemma 1.1: Let ϑ be a variety of groups and $N \trianglelefteq G$. Then the following properties hold:

- V(V*(G) = 1and V*(G/V(G)) = G/V(G)
- V(G) = 1 iff V*(G) = G iff $G \in \vartheta$

- $[NV^*G] = 1$ iff $N \subseteq V^*(G)$
- $V(G/N) = V(G)N/N$ and $V^*(G/N) \supseteq V^*(G)N/N$
- $V(N) \subseteq [NV^*G] \subseteq N \cap V(G)$. Inparticular $V(G) = [GV^*G]$
- if $N \cap V(G) = 1$, then $N \subseteq V^*(G)$ and $V^*(G/N) = V^*(G)/N$
- If $[G, N] \subseteq V^*(G)$, then $[V(G), N] = 1$. Inparticular $[V(G), V^*(G)] = 1$

Let ϑ be a variety of groups. Then the lower ϑ-verbal series of G is defined as follows:

$$G = V^0(G) \geq V^1(G) = V(G) \geq, ..., \geq V^n(G) \geq, ...,$$

where, for n>0, $V^n(G) = V(V^{n-1}(G))$. It is easily to check that $V^{n-1}(G)/V^n(G) \in \vartheta$. The upper ϑ-marginal series of G is defined as follows:

$$1 = V_0^*(G) \leq V_1^*(G) \leq, ..., \leq V_n^*(G) \leq, ...,$$

where, for $n > 0, V_n^*(G)/V_{n-1}^*(G) = V^*(G/V_{n-1}^*(G))$. The corresponding lower ϑ-marginal series of G is given by:

$$G = V_0(G) \geq V_1(G) \geq, ..., \geq V_n(G) \geq, ...,$$

where, for n>0, $V_n(G) = [V_{n-1}(G)V^*(G)]$. In the special case, if ϑ is the variety of abelian groups, then $V^n(G) = G^{(n)}$ is the derived series and $V_n^*(G) = Z_n(G)$ is the upper central series and $V_n(G) = \gamma_{n+1}(G)$ is the lower central series of G.

The proof of the following lemma is straightforward by using the above notations and Lemma 1.1.

Lemma 1.2: Let G be an arbitrary group. Then for $i, j \geq 0$:

- $V^i(V^j(G)) = V^{i+j}(G)$
- $V_i^*(G/V_j^*(G)) = V_{i+j}^*(G)/V_j^*(G)$
- $V_i(G)/V_{i+1}(G) \leq V^*(G/V_{i+1}(G))$
- $V^*(V^i(G)/V^{i+1}(G)) = V^i(G)/V^{i+1}(G)$

MATERIALS AND METHODS

Nilpotent group in the variety: A group G is called ϑ-nilpotent if there exists a series $1 = G_0 \leq G_1 \leq, \cdots, \leq G_n = G$ where $G_i \trianglelefteq G$ and $G_{i+1}/G_i \leq V^*(G/G_i)$, for $i = 0,1, \ldots, n-1$. The length of the shortest series is called ϑ-nilpotent class of G. A group G is ϑ-nilpotent if and only if $[G_{i+1}V^*G] \leq G_i$, for $i = 0,1, \ldots, n-1$.

The class of ϑ-nilpotent groups is closed under the formation of subgroups, images and finite direct products.

Theorem 2.1: (Taheri[3], Theorem 1.3) Let $1 = G_0 \leq G_1 \leq, \cdots, \leq G_n = G$ be a series in a ϑ-nilpotent group G, then the following properties hold:

- $V_{n+1}(G) = 1$
- $V_n^*(G) = G$

The following theorem gives a connection between the marginal subgroup and minimal normal subgroup of G.

Theorem 2.2: Let ϑ be a variety of groups and G be a ϑ-nilpotent group and N be a minimal normal subgroup of G. Then $N \subseteq V^*(G)$.

Proof: We define the sequence $\{N_i\}_{i \in I}$ of normal subgroups of G such that $N_0 = N$, $N_{i+1} = [N_i V^*G]$ for $i \geq 0$. By induction on i, it is easily checked that $N_i \leq V_i(G)$. Since, G is a ϑ-nilpotent group, so there exists the series $N_0 = N \geq N_1 \geq, \cdots, \geq N_n \geq N_{n+1} = 1$. By the minimality of N, it follows that $N_0 = N$, $N_1 = 1$. Therefore, $[NV^*G] = 1$ and hence, $N \subseteq V^*(G)$. The following corollary is an immediate consequence of the above theorem.

Corollary 2.3: Let ϑ be a variety of groups and G be a ϑ-nilpotent group and N be a non-trivial normal subgroup of G. Then $N \cap V^*(G) \neq 1$. The following theorem is about ϑ-nilpotent groups and their verbal subgroups.

Theorem 2.4: Let ϑ be a variety of groups and G be a ϑ-nilpotent group of class n and $G = HV(G)$. Then $G = H$.

Proof.: Using the induction hypothesis and Theorem 2.4 of Neumann[1], we have $G = HV_n(G)$. Hence, $V(G) = V(H)[V_n(G)\ V^*G]$ and so, $G = HV_{n+1}(G)$. Since, G is a ϑ-nilpotent group of class n, thus, $V_{n+1}(G) = 1$ which gives the result. The following theorem is about the marginal subgroups of ϑ-nilpotent groups.

Theorem 2.5: Let ϑ be a variety of groups and G be a finite group. Then there exists a subgroup H of G such that $G = HV^*(G)$ and $H \cap V^*(G)$ is a ϑ-nilpotent group.

Proof: By induction on the order of G, if $V^*(G) \subseteq \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G, then trivially $H = G$ and since, $V(V^*(G)) = 1$. Thus, $V^*(G)$ is a ϑ-nilpotent group and hence, $H \cap V^*(G) = V^*(G)$ which gives the result. Now suppose that $V^*(G) \not\subseteq \Phi(G)$, then there is a maximal subgroup M of G such that $V^*(G) \not\subseteq M$. By induction hypothesis, there is a subgroup H of M such that $M = H(V^*(G) \cap M)$. But $H \cap (V^*(G) \cap M) = H \cap V^*(G)$ is a ϑ-nilpotent group, thus, $G = HV^*(G)$ and hence, the assertion hold. The following corollary gives a connection between ϑ-nilpotent groups and p-groups where p is a prime number.

Corollary 2.6: Let G be a ϑ-nilpotent group with a normal subgroup N such that $|N| = p^n$. Then $N \leq V_n^*(G)$.

Proof: We use induction on $n\geq 1$. If $n = 1$, then $1\neq N\cap V^*(G)\leq N$. Thus, $|N\cap V^*(G)| = p = |N|$ implies that $N\cap V^*(G) = N$ and hence, $N\leq V^*(G)$. Suppose it has been proved for all $i<n$, $|N| = p^n$ and $M = N\cap V^*(G)\neq 1$. So, $|N/M| = p^m$ where $m<n$. Since:

$$\frac{N}{M}=\frac{N}{N\cap V^*(G)}\cong\frac{NV^*(G)}{V^*(G)}$$

the induction hypothesis implies that:

$$\frac{NV^*(G)}{V^*(G)}\leq V_m^*\left(\frac{G}{V^*(G)}\right)=\frac{V_{m+1}^*(G)}{V^*(G)}$$

so, $N\subseteq V_{m+1}^*(G)\leq V_n^*(G)$.

RESULTS AND DISCUSSION

Main results: If ϑ is the variety of abelian groups, then Theorem 2.1, implies that there exists a nilpotent group G of class n if and only if $\gamma_{n+1}(G) = 1$ or $Z_n(G) = G$. Also, by corollary 2.3, if N is a non-trivial normal subgroup of nilpotent group G, then $N\cap Z(G)\neq 1$ and hence by corollary 2.6, if $|N| = p^n$ (where p is prime number), then $N\subseteq Z_n(G)$.

In this final section, we suppose that $V = \{[x_1, x_2, \ldots, x_n]\}$ is the nilpotent word and ϑ is the variety of groups, which defined by these words. Using the discussion of the previous section, we give and prove our main results. The following lemma gives a necessary condition for an arbitrary group to be equal with its marginal subgroup.

Lemma 3.1: Let $V = \{[x_1, x_2, \ldots, x_n]\}$ and ϑ be a variety of groups which defined by V. If G is an arbitrary group such that $G/V^*(G)$ is a cyclic group, then $V^*(G) = G$.

Proof: If there exists $g\in G$ such that $G/V^*(G) = \langle gV^*(G)\rangle$, then for any generator $v(g_1, g_2, \ldots, g_n)\in V(G)$ there exists $x_1, x_2, \ldots, x_n\in V^*(G)$ such that $v(g_1, g_2, \ldots, g_n) = v\left(g^{l_1}x_1, g^{l_2}x_2, \ldots, g^{l_n}x_n\right)=v\left(g^{l_1},\ldots,g^{l_n}\right)=1$ where $l_i\in Z$, $1\leq i\leq n$. So, $V(G) = 1$ and hence, $G = V^*(G)$. The following corollary is an immediate consequence of the above lemma.

Corollary 3.2: Let $V = \{[x_1, x_2]\}$ and ϑ be a variety of abelian groups. If G is an arbitrary group such that $G/Z(G)$ is a cyclic group, then G is an abelian group. The following theorem gives a connection between ϑ-nilpotent groups and their marginal and verbal subgroups.

Theorem 3.3: Let $V = \{[x_1, x_2, \ldots, x_n]\}$ and ϑ be a variety of groups which defined by V. If G is a ϑ-nilpotent group such that $G/V(G)$ is a cyclic group, then $G = V^*(G)$.

Proof: By the definition, we have $V_n(G) = [V_{n-1}(G)V^*G]$ for all $n\geq 1$. Then $V(G)/V_2(G)\leq V^*(G/V_2(G))$. But:

$$\frac{G/V_2(G)}{V(G)/V_2(G)}\cong\frac{G}{V(G)}$$

is a cyclic group and by Lemma 3.1, $V^*(G/V_2(G)) = G/V_2(G)$. Therefore, $V_1(G) = [GV^*G]\leq V_2(G)$ and hence, $V_1(G) = V(G) = V_2(G)$. Similarly, $V_3(G) = [V_2(G)V^*G] = [V(G)V^*G] = V_2(G)$ and so, $V_i(G) = V(G)$ for $i\geq 2$. Since, G is a ϑ-nilpotent group, thus, $V(G) = 1$ and hence, $G = V^*(G)$. The following corollary is an immediate consequence of the above theorem and Lemma 3.1.

Corollary 3.4: Let $V = \{[x_1, x_2, \ldots, x_n]\}$ and ϑ be a variety of groups which defined by V. If G is a ϑ-nilpotent group of class $c>1$, then $G/V_{c-1}^*(G)$ is not cyclic.

Proof: Let $G/V_{c-1}^*(G)$ be cyclic. Then:

$$\frac{G/V_{c-2}^*(G)}{V^*\left(G/V_{c-2}^*(G)\right)}=\frac{G/V_{c-2}^*(G)}{V_{c-1}^*(G)/V_{c-2}^*(G)}\cong\frac{G}{V_{c-1}^*(G)}$$

By Lemma 3.1, we have:

$$\frac{V_{c-1}^*(G)}{V_{c-2}^*(G)}=V^*\left(\frac{G}{V_{c-2}^*(G)}\right)=\frac{G}{V_{c-2}^*(G)}$$

So, $G = V_{c-1}^*(G)$ but this is contradiction with assumption. The following theorem gives a connection between ϑ-nilpotent groups and p-groups where p is prime number.

Theorem. 3.5: Let $V = \{[x_1, x_2, \ldots, x_n]\}$ and ϑ be a variety of groups defined by V. If G is a finite p-group, then G is a ϑ-nilpotent group.

Proof: We prove by induction on $|G|$. If $|G| = p$ and $|V^*(G)| = 1$, then $|G/V^*(G)| = p$. By Lemma 3.1, it follows that $|V^*(G)| = p$ and this is contradiction with assumption. Therefore, $|V^*(G)| = p$ and hence, G is a ϑ-nilpotent group of class one. Now assume that $|G| = p^m$ and $m>1$. If $G = V^*(G)$, then G is a ϑ-nilpotent group. Otherwise, the order of $G/V^*(G)$ is less than $|G| = p^m$, so, the induction assumption yields that $G/V^*(G)$ is a ϑ-nilpotent group and has the upper ϑ-marginal series as follows:

$$1=\frac{G_0}{V^*(G)}\leq\frac{G_1}{V^*(G)}\leq,\ldots,\frac{G_k}{V^*(G)}=\frac{G}{V^*(G)'}$$

by isomorphic theorems, we have $G_i/G_{i-1}\leq V^*(G/G_{i-1})$, so, it follows that there exist an upper ϑ-marginal series for G as follows:

$$1=G_{-1}\leq G_0=V^*(G)\leq G_1\leq,\ldots,\leq G_k=G$$

therefore, G is a ϑ-nilpotent group.

CONCLUSION

The following results give basic properties of the verbal and the marginal subgroups in varietal nilpotent groups which is useful in our investigations, Neumann[1] and Hekster[2] for more information.

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