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Portfolio Selection Problem: Models Review

Rula Hani Salman AlHalaseh, Md. Aminul Islam and Rosni Bakar
School of Business Innovation and Technopreneurship,
University Malaysia Perlis, Arau, Perlis, Malaysia

Abstract: This study introduces a survey of contributions to dynamic multi-objectives portfolio from finance and operation research to the portfolio selection. This survey includes popular risk-measures and extends to operation research models and mathematical models. In contrast to other survey, this study focuses on highlighting the strengths and weaknesses of different models to choose the most appropriate model achieves the optimality and easy in application by investors. To describe the latest results accompanying each model and the similarity between them. To illustrate the modeling idea and to show the effectiveness of the proposed approach. This paper discusses in brief the most popular mean-risk models then multi-period models from the point view of operation research and stochastic programming. Many researchers conducted portfolio optimization problem by offering new models. These researches success in providing mathematical and theoretical models that enriched the finance literature but few of it satisfies the market application. This study reviewed some of these models relating to single-period, multi-periods models, single-objective and multi-objectives and concluded that SGMIP is the most effective model since it able to deal with real world application considering multi-factors, multi-periods, different risk measures without affecting the computation time which facilitate the mission of decision makers. There is a plenty of models discussing optimizing portfolio for that the writer of this study selects the original models MV and MAV, risk measures models VaR and CVaR and the latest models related to operation research to achieve the study objectives.

Key words: Portfolio selection, Stochastic Mixed Integer Programming (SMIP), Stochastic Goal Mixed Integer Programming (SGMIP), mean variance model, mean absolute deviation model, Value at Risk (VaR), Conditional Value at Risk (CvaR)

INTRODUTION

The problem of portfolio selection is in the scarce of resources it is not just which stocks to own but how to distribute the investor's wealth amongst stocks (Ravipati, 2012). In Finance, portfolio selection is famous as a leading problem; giving that the future return of an asset is unknown when investment decision made therefore, the decision making is under uncertainty: one can evaluate the today decision just in future time once the assets return is revealed (Roman and Mitra, 2009).

Modern Portfolio theory has emerged many different models sought to provide some assistant in decision making environment. Each model is a simplification or simulation of reality (Pastor, 2000). By capturing the real world features, models become more complex, therefore many attempts provided as a simplification. In spite of the majority of models seems worthless to financial decision maker these models valued at least theoretically (Azmi and Tamiz, 2010).

Researchers approached portfolio selection differently; some of them approach mean-variance of Markowitz (1952) focusing on the trade-off between risk and return neglecting other essential factors. Therefore, all models aimed to maximize the return attached with specified level of risk assuming that it will satisfy the investors' interests. The wide applications of the models were neither desirable nor important (Azmi and Tamiz, 2010). Accordingly, the portfolio selection problem gets enlarged and remains unsolved, even after extending to involve other factors such as liquidity, cardinality constraint, transaction cost, short sale and ext., which encourage the researchers to apply either other risk measures or simplify the mathematical models.

Mean-risk models are the common used approach in portfolio selection practice. In these models, the return distribution is distinguished and evaluated using two conflict statistics measures: the expected return value and the determined risk measure. The risk measure chosen by portfolio manager plays a significant role in decision

Corresponding Author: Rula Hani Salman AlHalaseh, School of Business Innovation and Technopreneurship, University Malaysia Perlis, Arau, Perlis, Malaysia

making. In portfolio selection models, a considered risk measure has been a subject of debate. Variance was chosen by Markowitz (1952) as a first measure used in mean-risk models, although it has criticism and many suggestions for new risk measures (Konno and Yamazaki, 1991; Ogryczak and Ruszczynski, 1999; Rockafellar and Uryasev, 2000, 2002), it remains the most commonly used measure in the portfolio selection practices. Monographs of Grinold and Kahn, Litterman and Meucci were detailed the practical applications of the mean-variance framework (Fabozzi *et al*., 2010). VaR has become the industry benchmark for risk management due to its intuitive appeal and acceptance by bank regulators (Jorion, 2006).

Konno and Yamazaki (1991) employed absolute Deviation as a risk measure with retaining some theoretical features of MV model which is comparable to variance and built their commonly used Mean-Absolute-Deviation Model (MAD) (Karacabey, 2006) assuming multivariate normal returns (Jobst *et al*., 2001). Feinstein and Thapa (1993) adjusted Karacabey (2006) portfolio and reformulate MAD model by minimizing the securities' number. The researcher of Worzel offered an overview of the MAD model (Jobst *et al*., 2001). Chance used Feinstein and Thapa's model and built his own by decreasing the number of variable and constraints kept the same (Karacabey, 2006). The proposed model by Karacabey (2006) distinguished between the sign of deviations positive or negative and assumed that an investor prefers a portfolio with higher upside deviations and lower downside deviations (Karacabey, 2006). Liu and Qin (2012) in their study fills the gap by means of defining semi-absolute deviation for uncertain variables and establish the corresponding mean semi-absolute deviation models in uncertain environment.

The minimax model proposed by Young (1998) as MAD it is employed in portfolio selection researches. Both models benefit from the drawbacks of MV model and developed models using linear equations which making them more appropriate for practical use. The concern turned to left tails distributions (unfavorable outcome) to find other risk measures for purpose of regulatory and reporting. The most widely measure covers this purpose is Value-at-Risk (VaR). According to Fabozzi *et al.* (2010) VAR model suffer from unfavorable theoretical properties and encountered some criticisms as it lacks the sub additivety and fail to reward diversification in addition to the computation complexity (non-convex NP-hard problem) when conducting the optimality. Because of these difficulties, other risk measure with left tail raised the Conditional Value at Risk (CVaR). It obtained greater acceptance because of the attractive theoretical properties that characterize, such as controlling the size of losses that exceed VaR and it is coherent. Fabozzi *et al.* (2010). Optimizing CVaR for discrete random variable which characterized by different outcomes under several scenarios is leading to linear programming model with finite dimension (Rockafellar and Uryasev, 2002).

The concern was extended to the mathematical programming where the uncertainties area accompanies with the decision making especially when it concern with allocating the scarce of resources and across time. The input of financial decision making for optimization problem is usually asset prices, returns and interest rates which are stochastic in nature. Uncertainties are measured by either sensitivity analyses or executing scenario tests on the parameters to observe how the optimal solutions vary. But both approaches have obstacles made them incomplete because they assumed one scenario of the future and it occurs with certainty.

To overcome these obstacles, probabilistic methods in addition to optimization techniques are needed. The new development from joining later techniques is called stochastic programming. The stochastic programming objective is to provide tools for designing and controlling stochastic system and optimizing its performance. Stochastic Programming (SP) captures the uncertainty in a general framework by assigning an objective or subjective forecast of the scenarios and their probability distribution and formed a mathematical model. Hence, the components of stochastic are the securities return and a time series technique to forecast the stochastic process of the return and estimates the model parameters. The remaining parts of this study are organized as follows. Section two discusses the mean-risk models while section three discusses stochastic models, section four revealed the conclusions and further studies.

Mean-risk models: Many models follow portfolio theory (Markowitz, 1952) to select portfolios under risky environments. Portfolio's mean return was derived from probability distribution concerning the utility function of the investor. The Central Tendency Theory gives the return its standard deviation which is relate to dispersion measures and revealed the distribution of returns around its mean which is considered as a risk measure. The optimum portfolio of MV aims to minimize the variance subject to a given expected return restrictions.

The MAD and Minimax Models are engaged in portfolio optimization studies and take a linear structure and can be solved using linear programming techniques. This represent as an advantage for these models over Mean-Variance model and its quadratic programming solution. Hence, the linear programming reduces the computation time which makes them more practicable for large-scale portfolio selection.

MATERIALS AND METHODS

The Mean-Variance (MV) model: Assume that there are n assets in the market. Let R_i be the random return rate of the asset i, x_i is the amount of money allocate to asset i, i = 1, …, n. The return rate of a portfolio R_p is denoted by $\sum_{i=1}^{n} R_i x_i$ and the expected return rate of asset i, x_i is denoted by r_i. Denoting by r(x) = E(R(x)) = $(r_1, \ldots, r_n)$ then the variance is V (x) = E[R(x)-r(x)]^2 and the standard deviation is: σ (x) = √V (x) = √E[R(x)-r(x)]^2 (Yu *et al.*, 2003). According to Young (1998), the MV model is described as:

$$\underset{i=1}{\text{Min}} \sum_{i=1}^{n}\sum_{i=1}^{n} x_i x_j \rho_{ij} \tag{1}$$

Subject to:

$$\sum_{i=1}^{n} x_i r_j \geq G \tag{2}$$

$$\sum_{i=1}^{n} x_i = X \tag{3}$$

$$0 \leq X \leq M_i, i = 1, \ldots, n \text{ and } j = 1, \ldots, N$$

where, $\rho_{ij} = 1/(T-N) \sum_{i=1}^{n} (r_{it}-r_i)(r_{jt}-r_j)$ for N which is the finite number of financial assets at time T; r_{it} is the return of asset i at time t; r_i denote the average return of the asset i; r_{jt} is the return of asset j at time t; r_j is the mean return of the asset j; $\sum_{i=1}^{n} x_i r_i$ is the portfolio mean return; x_i and x_j are the allocations of the assets i, j and M_i is the maximum budget share that can be invested in the asset i. In the first equation, the portfolio selection of MV model characterizes the portfolio with minimum variance, subject to the constraint that the portfolio expected return overcomes a given level, G (mentioned in Eq. 2), so the total allocations to the portfolio cannot exceed the total budget, X (Eq. 3).

From this analysis a significant result is appeared that the risk level corresponding with the portfolio decrease for a given rate of return, this result stems from the fact that says when the correlation between assets decreases the benefits from portfolio diversification increases. Therefore, the lower the correlation the higher the risk diversification will be (Farias *et al.*, 2004).

Mean-Absolute Deviation (MAD) Model: Konno and Yamazaki (1991) present Mean Absolute Deviation (MAD) Model which is define as follows:

$$l_1(x) = E\sum_{i=1}^{n} R_i x_i - E\sum_{i=1}^{n} [R_i x_i] \tag{4}$$

The risk of this model measures by absolute deviation of the assets rate of return which forms a main characteristic. This model gains much attention from researchers and practitioners since its risk function l_1 can be transformed to parametric linear programming and the implementation of this model to portfolio optimization can be simply attained. The advantages of this model concentrated on its simplicity and computational robustness (Yu *et al.*, 2003).

The research of Rudolf concludes that the minimization of MAD revealed similar results as Markowitz MV model at the same time it is equivalent to expected utility maximization under risk aversion when the distribution of return exhibited normally and multivariatly. In the contrast to MV, MAV model is consistent with the second degree stochastic dominance where the tradeoff between risk and return is bounded by certain constant. Furthermore, the MAD can be extended to frictional case easily because it is continually converted to linear programming problem. This case is not applicable for MV as it appears more difficult (Yu *et al.*, 2003).

The model can be converted to linear programming problem as follows: Let M_0 = the initial wealth held by the investor, ρ = the investor's required rate of return, μ_i = maximum amount of asset i the investor wants to invest in i = 1, …, n. Short selling is not allowed therefore x_i = 0, i = 1, …, n. Denoted by:

$$S = \{x_i = (x_1, \ldots, x_n): \sum_{i=1}^{n} r_i x_i \geq \rho\, M_0, \sum_{i=1}^{n} x_i = M_0, 0 \leq x_i \geq \mu_i, i = 1, \ldots, n\} \tag{5}$$

Konno's Model is:

$$\text{Min } w(x) = E\,|\sum_{i=1}^{n} R_i x_i \text{-} E\sum_{i=1}^{n} [R_i x_i]| \tag{6}$$

$$S \times T;\ x \in S$$

The model will express as the following since the objective function is not linear (Konno and Yamazaki, 1991):

$$\text{Min } w(x) = 1/T \sum^{T} y$$

$$S \times T \quad y_t \geq \sum_{i=1}^{n} (r_{it} - r_i) x_i, t = 1, \ldots, T \tag{7}$$

$$S \times T \quad y_t \geq -\sum_{i=1;\, x \in S}^{n} (r_{it} - r_i) x_i, t = 1, \ldots, T \tag{8}$$

Here, r_{it} is the expected rate of ith stock during period t. For this model no need to estimate the matrix of variance-covariance and the constraints' size can be controlled through the number of periods. Since, there are so many advantages in the MAD Model, it is worth discussing and considering its extension.

Minimax model: Minimax model was applied to portfolio selection by Young (1998). This model is based on game theory, where each player (two or more) knows the goals and has complete information about it. The rational behavior of each payer leads to assure that the player aims to either maximize his expected minimum return (Maximin criterion) or minimize his maximum expected losses (Minimax criterion) through the solution of each player situation. This model is suitable for solving one agent decision-making process under risky environment on contrary to his nature. The formula of Minimax model that applied to portfolio selection by Young (1998) is described as follows: let N = a finite number of financial assets, T = a time horizon:

$$r_i = 1/T \sum_{i=1}^{n} r_{it}$$

$$R_P = \sum_{i=1}^{n} r_i x_i$$

$$R_{Pt} = \sum_{i=1}^{n} x_i r_{it}$$

$$M_p = \min_t R_{Pt} \tag{9}$$

Where:

- r_{it} = Denote the return of the money invested in asset i at time t
- R_P = The expected portfolio retrun
- R_{pt} = The portfolio return at time t
- M_p = The portfolio minimum return for time period

This equation is referred to Maximin portfolio selection, the term Minimax will be used since it is more often mentioned in the specialized literature for this formulation (Farias *et al*., 2004). The minimax model attempts to minimize the maximum expected losses or in other words to maximize the portfolio minimum return for time period M_p for that R_P portfolio expected return exceeds a given value level, G and the total portfolio allocation cannot exceed the total budget X. The optimization problem for the above definition is described as follows. Max M_p; Mp, X, Subject to:

$$R_{Pt} = \sum_{i=1}^{N} x_i r_{it} - M_p \geq 0, \; t = 1, \ldots, T$$

$$\sum_{i=1}^{N} x_i r_i \geq G$$

$$\sum_{i=1}^{N} x_i \leq X$$

$$0 \leq X \leq M_i, i = 1, \ldots, n \tag{10}$$

The objected function is to maximize the minimum portfolio return (Maximin). This means that for every time period the minimum portfolio return M_p will be smaller than or equal to portfolio return (Eq. 2). Therefore, M_p cannot exceed the portfolio return and bounded from upper side by the given level G while the $R_p = \Sigma x_i r_i$ exceeds the G, this prove the attempting of minimax to maximize the minimum portfolio return. From Young (1998) point view this model has the logical advantages for portfolio optimization over other model if assets price follows normal distribution and when they are not.

Performance measures of risk: The key percentile risk measures Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are popular functions in risk management. In academic finance literature on risk management, there is a common problem of choosing between VaR and CvaR. The reasons are related to the differences in mathematical properties, stability of statistical estimation, simplicity of optimization procedures, acceptance by regulators, etc.

Value at risk VaR: VaR is widely used as a performance measure considering the confidence level that accompanies the maximum loss. Various methodologies was used for modeling VaR, most of them rely on linear approximation of risk and assume the joint normal or log-normal distribution of the underlying market parameters (Uryasev, 2000). VaR is one of the most accepted measures of risk at the same time is has unfavorable characteristics (Artzner *et al*., 1999) such as lack of sub-additivity (VaR of the portfolio of two assets can be greater than the sum of individual value at risk of these two assets. Moreover when calculation requires scenario VaR becomes hard to optimize. In this situation, VaR is non-convex, non-smooth as a function of positions and it has a multiple local extremes.

Reviewing the methodology by following the work of the researchers with Campbell *et al*. (2001) where they focused on maximizing the expected return subject to down side risk rather than standard deviation which encompass any non-normality in the return distribution of the financial assets. Suppose M_0 = invested amount, T = investment horizon, B when B>0 is borrowing amount, B<0 is lending amount, r_f is interest rate that investor can borrow or lend during T, n = number of available assets, x_i = fraction invested in risky asset i where $\Sigma x_i = 1$, P_{it} is the price of asset i at time t, c is the confidence level. Then:

$$\text{Initial value of portfolio} = M_0 + B = \sum_{i=1}^{n} x_i P_{i0}$$

In t = 0 the investor must choose the value of x_i and B.

$$\text{Down side risk} = P_r\left\{M_0 - M_{tp} \geq VaR^*\right\} \leq (1-c)$$

Let P_r equals expected probability:

$$\text{Portfolio } P = P_r\left\{M_{tp} \leq M_0 - VaR^*\right\} \leq (1-c)$$

For constructing the optimal portfolio, let r_p represents expected total return:

$$\text{Expected wealth from } P = E_0\left(M_{tp}\right)$$
$$= (M_0 + B)\left(1 + r_p\right) - B(1 + r_f)$$

$$P' = \max S_p = \left(r_p - r_f\right) \backslash \left(M_0 r_f - q_{cp}\right)$$

Where:
S_p = Ratio of expected risk premium of P
q_{cp} = Quantil of probability (1-c)

$$\text{Max } S_e = (r_p - r_f) \backslash (M_0\ r_f - VaR_{cp})$$

$(M_0\ r_f - VaR_{cp}) = \varphi_{cp}$ = risk faced by investor:

$$\text{Max } S_p = \left(r_p - r_f\right) \backslash \varphi_{cp}$$

$$B = M_0\left(VaR^* - VaR_{cp'}\right) \backslash \varphi_{cp'}$$

*Means alternative normality distribution.

Conditional value at risk CVaR: CVaR is an alternative measure of loss with attractive properties. In literatures, it may call a mean excess loss, mean shortfall or tail VaR interchangeably. It is overcome the obstacle of VaR since it is sub-additive and convex (Artzner *et al*., 1999). Uryasev (2000) reports that the CVaR can be optimize using linear programming and nonsmooth optimization algorithms which allows handling portfolios with large number of instruments and scenarios. Many experiment studies revealed that the minimization of CVaR manage to reach optimal solution in terms of VaR since the CvaR is always greater than or equal VaR. But when the return-loss follow normal distribution both measures provide same optimal investment portfolio (Rockafellar and Uryasev, 2000).

Other case studies revealed that optimization the risk may do for large portfolio and large quantity of scenarios with relatively small computation time when using CVaR performance function and constraints. To test the minimum CVaR with set of scenarios the following algorithm (LP techniques) can be used. Let the density function P(y) is not available but J of scenario y_j can be offered, j = 1, ..., J, Sampled from P(y). To price the portfolio instruments, one can use historical observation if available or using Monte Carlo simulation. After replacing F_β (x, α) the terms $(f(x, y_j)-\alpha)^+$ by variable z_j, the constraint will be:

$$z_j \geq f\left(x, \mathbf{y}_j\right) - \alpha, z_j \geq 0$$

$$\text{Min } \alpha + \upsilon \sum_{j=1}^{j} z_j \qquad (11)$$

x∈X subject to:

$$z_j \geq f\left(x, \mathbf{y}_j\right) - \alpha,\ z_j \geq 0\ j = 1, \ldots, J$$

where, υ is constant and equals $\upsilon = (1-\beta)\,J)^{-1}$ and + = max (0, t), CVaR replaced by F_β (x, α)≤C_β as a constraint, this constraint can be approximated using F_β (x, α)≤C_β using scenarios, j = 1, ..., J:

$$\alpha + \upsilon \sum_{j=1}^{J} z_j \leq C_\beta$$

Let (x^*, α^*) be the solution of the minimization problem, respecting the two variable optimizing CVaR, then, F_β (x^*, α^*) = the optimal CVaR where the optimal portfolio = x^* and risk measure = α^* = VaR if the last constraint is active.

RESULTS AND DISCUSSION

Stochastic goal mixed integer programming: This model consists of three programming techniques Stochastic Programming (SP) to capture or measure the uncertainty of the security price especially in future time, Goal

Programming (GP) to smooth the using of multi-objectives and mixed-integer programming to guarantee the fraction is not allowed.

Goal programming: The general formulation of a GP problem is as the following:

$$\text{Min } (\varrho^+)^T \gamma^+ + (\varrho^-)^T \gamma^-$$

$$S \times T \;\; f(x) + \gamma^+ - \gamma^- = z$$

$$Ax = b;\; x \geq 0;\; \gamma^+,\; \gamma^- \geq 0$$

where, $x = x_1, \ldots, x_n$ is the vector of decision variables, ϱ^+, ϱ-ϱR^m are goal weighting parameters, $f(x) = \Sigma^n_{j=1} c_{ij} x_j$, $i = 1, \ldots, m$ are the goals associated with objective i and γ^+; $\gamma^- \epsilon R^m$ are the positive and negative deviations from goals $z \epsilon R^m$, respectively (Stoyan and Kwon, 2011).

Stochastic goal programming: The combination between GP and SP is as follows: Let $w \epsilon \Omega$ random events and y (w) is the second stage decision variable, then the general Stochastic-Goal Programming (SGP) problem is:

$$\text{Min } \left(\varrho^+\right)^T \gamma^+ + \left(\varrho^-\right)^T \gamma^- + \left(\varsigma^+\right)^{T\delta+} + \left(\varsigma^-\right)^{T\delta-}$$

$$S \times T \; f(x) + \gamma^+ - \gamma^- = z$$

$$Q(y) + \delta^+ - \delta^- = z(w)$$

Where:

- $\varsigma^+, \varsigma^- \epsilon R^p$ = Are goal weighting parameters
- $Q(y)$ = $E[\min \{q_v (w)^T y(w) \backslash W (w) = h (w) - T (w) x, y(w) \geq 0\}]$ $v = 1, \ldots, p$ are the goals associated with the second-stage objective v
- γ^+, γ^- = Relaxation variables (γ^+; $\gamma^- \epsilon R^m$ are the positive and negative deviations from goals $z \epsilon R^m$)
- $\delta^+, \delta^- \epsilon R^p$ = The positive and negative deviations from second-stage goals $z(w) \epsilon R^p$, respectively

Stochastic goal mixed integer programming

Problem model: The formulation of SGMIP (incorporating between three members of programming techniques) began by defining the decision variables related to different assets in portfolio which are securities, bonds and treasury bills riskless assets by maximizing the following objective function:

$$\max_{F^T_{obg}} \sum_{1=1}^{L} \sum_{i=1}^{n} \varnothing^1_{il} x_i + \sum_{t=1}^{T} \sum_{1=1}^{L} \sum_{i=1}^{n} pl\, U^{tj}_{1}\; z_{jl}^{t-h^*_j} + \sum_{t=1}^{T} \sum_{1=1}^{L} \sum pl\, \varsigma B^t_1 + \varsigma B^0 \quad (12)$$

Where:

- x_i = The fraction of the portfolio invested in security i that is purchased in the first-stage (t = 0)
- y^t_{il} = The fraction of the portfolio invested in security I that is purchased in the second-stage (t>0)
- m = Total of time period, for the simplicity T = m-1
- L = Total number of scenarios where l = 1, 2, 3, ..., L because the model will capture the future market uncertainty by expressing the outcomes of future economic as scenarios
- $\varnothing^t_{il}$ = The unit price of security i at time t = 0,1,..., m under scenario l = 1, 2, ..., L, i = 1, 2, ..., n Where $x_i \epsilon R$ and $y^t_{il} \epsilon R$, note that the security price is known at t = 0 and there is only one scenario in the first stage
- z^t_{il} = The fraction of the portfolio invested in bond j to purchase at time t under scenario l, hence $z^t_{il} \epsilon R$
- h = The total of different types of bond j = 1, 2, ..., h with respect to coupons and maturities embed
- h^*_j = The time to maturity for each bond j
- φ^t_{jl} = The price of bond j at time t under scenario l
- U^t_{jl} = Bond return at maturity
- ς = Cost of riskless asset
- B^t_1 = Amount invested in riskless investment that incurs small cost of ς percent over a specified time period. $B^t_1 \epsilon R$. There is only one scenario in stage 1 l=1 for B^0_1
- B = Initial wealth of the portfolio
- F^t_{obg} = Represent the liabilities involved in the model that require the portfolio to meet a terminal financial obligation, $F^T_{obg} = 0$. Used with debt case given $F^T_{obg} > 0$

The designed portfolio takes a passive investment strategy. One of the objectives of dynamic portfolio is minimizing transaction costs, which is minimizing the number of transactions between time periods. Therefore, w^t_{il} defines as the following:

$$\ddot{w}^t_{il} = y^t_{il} - y_{il}^{t-1} \; i = 1, 2, \ldots, n \;\; t = 2, \ldots, T, l = 1, \ldots, L$$

For t = 1:

$$\ddot{w}^1_{il} = y^t_{il} - x_i \; i = 1, 2, \ldots, n, \; l = 1, \ldots, L$$

where, $w^0_{il} = 0$, w^t_{il} represents the fraction of a security that is bought or sold between time periods. This fraction subject to be minimized in objective function to keep the portfolio's cost to minimum. Other object in this model is to achieve the portfolio diversity by distributed the portfolio elements throughout all financial sectors. The variable Q (i, s) determine the security i to which sector is belongs where S is the total of sectors Q (i, s)ϵB. Let the f^t_s is the fraction of the portfolio that invested in sector s at time t, since it is a fraction then $\Sigma^s_{s=1} f^t_s = 1$ t = 0, …, T and $f^t_s \epsilon [0, 1]$. In addition, f^0_s is a first stage parameter whereas f^t_s is a second stage parameter when t>0. The form of sector exposure element will be as follows:

$$\sum_{i=1}^{n} Q(i,s) Ø^t_{il} y^t_{il} = f^t_s \sum_{i=1}^{n} Ø^t_{il} y^t_{il} + \xi^t_{sl}$$

$$s = 1, .., S; t = 1, \dots, T; l = 1, \dots, L$$

where, ξ^t_{sl} is the sector relaxation variable that compatibles to f^t_s under scenario l = 1, …, L. ξ^t_{sl} assists the model to find other f^t_s if the feasible solution cannot be found with the current used variable. The above constraint can be used for variable x_i instead of the variable y^t_{il}.

To bound the number of portfolio instruments (securities and bonds), let g^t_{il} is number of securities i invested in the portfolio at time t under scenario l, $g^t_{il}\epsilon$B where there is one scenario in the first stage l = 1 for g^0_{il}, then $g^t_{il} = 1$, if security i is used in the portfolio at time t under scenario l (i.e., if xi, y^t_i>0) and 0, otherwise. Also, the upper bound on the number of stocks to hold in the portfolio is G^t and in order to constraint the number of security to hold, the cardinality constraint will be:

$$\sum_{i=1}^{n} g^t_{il} \le G^t$$

$$t = 0, \dots, T, l = 1, \dots, L$$

The number of different bond to hold in portfolio follows the same steps of securities where, $g^{\sim t}_{jl} = 1$, if z^t_i>0 and zero otherwise. The corresponding constraint will be: $\Sigma^n_{j=1} g^{\sim t}_{il} = G^{\sim t}$ t = 0, …, T, l = 1, …, L. Where, $G^{\sim t}$ is the upper bound on the number of bonds to hold in the portfolio $g^{\sim t}_{jl}\epsilon$B. It is necessary to add portfolio accounting constraints to the model as follows:

$$B' = \sum_{i=1}^{n} Ø^0_i x_i + \sum_{j=1}^{h} \varphi^0_{j1} z^0_j + B^0$$

$$B^1_1 = \sum_{i=1}^{n} Ø^1_{il} x_i - \sum_{i=1}^{n} Ø^1_{il} y^1_{il} + \sum_{i=1}^{h} U^1_{jl} z_j^{1-h*j} - \sum_{j=1}^{h} \varphi^1_{jl} z^1_j - \sum_{i}^{n} \tau^1_1 \ddot{w}^1_{il} + \varsigma B^0$$

$$B^t_1 = \sum_{i=1}^{n} Ø^t_{il} y^{t-1}_{il} - \sum_{i=1}^{n} Ø^t_{il} y^t_{il} + \sum_{j=1}^{h} U^t_{jl} z_j^{t-h*j} - \sum_{j=1}^{h} \varphi^t_{jl} z^t_j - \sum_{i}^{n} \tau^t_1 \ddot{w}^t_{il} + \varsigma B^{t-1}_l \quad (14)$$

where, τ^t_l is the relative cost of security transaction, other symbols are defined earlier. The last three equations guarantee that the all wealth of the portfolio including dividends is invested at each time stage. In subsection 4.2 GP approach added to the problem for considering the various portfolio goals in the model.

SGMIP Model: The portfolio of this model follows the passive investment strategy (p≤1) for that performance measure will be the first goal to ensure that the model cannot outperform. To do so, let R^t_l as a maximum benchmark the investment not allowed to outperform at time t and under scenario l. Index market may consider as a benchmark only if the original investment in the portfolio is greater than or equal the present index value (Stoyan and Kwon, 2011). The value of R^t_l derived in a separate sub-problem. It constraints the model as follows:

$$\text{For first stage} \sum_{i=1}^{n} Ø_{il}^{t+1} x^t_{il} \le R^t_1 + \chi^t_1 \quad t = 0, \dots, T, l = 1, \dots, L$$

$$\text{For second stage} \sum_{i=1}^{n} Ø_{il}^{t+1} y^t_{il} \le R^t_1 + \chi^t_1 \quad t = 1, \dots, T \; l = 1, \dots, L$$

where, $\chi^t_l \ge 0$ is a relaxation element that satisfies the GP model, $\chi^t_l \epsilon$r and l = 1 for R^0_l as noticed from last two equations, the performance of the securities is only constrained which permits the portfolio to invest in bond when the investment in securities is not favorable. This constraint benefited from the historically inverse relationship between securities and bonds respecting their index values (Konno and Kobayashi, 1997). The second portfolio goal, considered risk related to individual stock and bond investments. Beta coefficient used to minimize security risk. Beta defined as a measure of the volatility or systematic risk of a security or a portfolio in comparison to the market as a whole market index portfolio I^t_M at time t. The following constraint added to the model considering β accompanying with securities:

$$\sum_{i=1}^{n} \beta_i g^0_i \le \beta^* + \delta^0$$

where, δ^0 is a penalty variable for the optimal β^*. The optimal value of risk β^* solved in separate sub-problem and minimized in objective function by multiplying penalty variable with penalty parameter shown in the object function final model page 14. The optimum value obtained by β^* is based on market history price, since β_i is computed using historical stock price. For time t>0 uncertainty must add to the optimal risk value by including scenarios. Then for time t>0, the optimal security risk β_l^* becomes β^*_{il} as shown in Eq. 27.

Although, bonds are less risky than securities they are subject to many sources of risks namely, variability in return rates, issuer default on his obligation and it may call before maturity and offer less interest (callable bond). Therefore, the quality of bond is constructed as a numerical values from bond rating (AAA) as high to low (D) according to SEC. This implies that bond quality rate α_j must be involved in the equation as a constraint when $0 \leq \alpha_j \leq 1$. After solving sub-problem as to what done for β, the maximum α^* value is produced and the constraint becomes:

$$\sum_{j=1}^{h} \alpha^t_{jl} g^{\sim}_{jl}{}^t \geq \alpha_l^* - \delta_l^{\sim t}, t = 0, \ldots, T, l = 1, \ldots, L$$

where, $\delta^{\sim 0}$ is a penalty variable for the optimal α^* value. At higher time t>0, the future uncertainty must be used and α_l^*, α^t_{il} will be employed, hence the constraint will be as following $\delta^{\sim t}$ is a penalty variable combined with penalty parameter which is minimized in the objective function. Liquidity is other element to be included and measured in the portfolio selection model. Portfolio investment of this study is dealing with many instruments where securities are the most liquid one. The liquidity level varies between investing instruments. According to number of shares, the security that has million share or more is assumed to be more liquid comparing with security has only thousand shares. According to this logic the liquidity is defined for each security and bond in the portfolio. Let E (i, t, l) is a liquidity value for security i at time t under scenario l and E~ (j, t, l) is a liquidity value for bond j at time t under scenario l. On the first stage at time t = 0, there is only one scenario therefore l is omitted. The portfolio aims to invest in high liquid instrument for that the optimal liquidity value E_l* under each l is assigned and solved. The following constraint is added to the model as:

$$\sum_{i=1}^{n} (i, t, l) g_{il}{}^t + \sum_{j=1}^{h} E(j, t, l) g^{\sim}_{jl}{}^t \geq_l {}^* - \lambda^t, t = 0, \ldots, T, l = 1, \ldots, L$$

where, $\lambda^t > 0$ is a penalty variable combined with penalty parameter which is minimized in the objective function. As well in first stage there is only one scenario, therefore $l = 1$ is not written with respect to*. After defining the set of securities Υ: = {i: i∈[1, n]}, the set of bonds Ξ: = {j: j∈[1, h]}, the set of sectors S:= {s: s∈[1, S]}, the set of scenarios Ω: = {l: l∈[1, L]} and the set of time periods Ť: = {t: t Є [0; T]}; T¯: = {t: tЄ[1; T]} and T: = {t: tЄ[2; T]}, the SGMIP model with recourse becomes: the SGMIP model with recourse:

$$\underset{+}{\text{Min}} - \mu_1 \left(\begin{array}{l} \sum_{l=1}^{L} \sum_{i=1}^{n} \varnothing^1_{il} x_i + \sum_{t=1}^{T} \sum_{l=1}^{L} \sum_{i=1}^{n} pl \varnothing_{il}{}^{t+1} y^t_{il} + \\ \sum_{t=1}^{T} \sum_{l=1}^{L} \sum_{i=1}^{h} pl U^{tj}{}_1 z_{jl}{}^{t-h}{}_j{}^* + \sum_{t=1}^{T} \sum_{l=1}^{L} pl \varsigma B^t{}_1 + \varsigma B^0 \end{array} \right) \tag{14}$$

$$\mu_2 \left(\sum_{t=1}^{T} \sum_{l=1}^{L} \sum_{i=1}^{n} pl \ddot{w}^t_{il} \right) + \mu_3 \left(\sum_{S=1}^{S} \xi_s{}^0 + \sum_{t=1}^{T} \sum_{l=1}^{L} \sum_{S=1}^{S} pl \xi_{sl}{}^t \right) + \mu_4 \left(\delta^0 + m \sum_{t=1}^{T} pl \delta^t_1 \right) + \mu_5 \left(\delta^0 + \sum_{t=1}^{T} \sum_{l=1}^{L} pl \delta_l{}^t \right) + \mu_6 \left(\lambda^0 + \sum_{t=1}^{T} \sum_{l=1}^{L} pl \lambda^t{}_1 \right) + \mu_7 \left(\chi^0 + \sum_{t=1}^{T} \sum_{l=1}^{L} \chi^t{}_1 \right) \tag{15}$$

$$S \times T \sum_{i=1}^{n} \varnothing^0_i x_i + \sum_{j=1}^{h} \varphi^0_{jl} z^0{}_j + B^0 = B^{\wedge}$$

$$\sum_{i=1}^{n} \varnothing^1_{il} x_i - \sum_{i=1}^{n} \varnothing^1_{il} y^1_{il} + \sum_{j=1}^{h} U^1{}_{jl} z_j{}^{1-hj} - \sum_{j=1}^{h} \varphi^1{}_{jl} z^1{}_j - \tau^1{}_1 \sum_{il}^{1} + \varsigma B^0 = B_1, l \in \Omega \tag{16}$$

$$\sum_{i=1}^{n} \varnothing^t_{il} y_{il}{}^{t-1} - \sum_{i=1}^{n} \varnothing^t_{il} y^t_{il} + \sum_{j=1}^{h} U^t{}_{jl} z_j{}^{t-h*j} - \sum_{j=1}^{h} \varphi^t{}_{jl} z^t{}_{jl} - \sum_{i}^{n} \tau^t{}_1 \sum_{il}^{t} + \varsigma B_l{}^{t-1} = B^t{}_1, l \in \Omega, t \in T \tag{17}$$

$$\sum_{i=1}^{n} \varnothing^1_{il} x_i \leq R^0 + \chi^0$$

$$\sum_{i=1}^{n} \varnothing^t{}_{il} y^t{}_{il} \leq R^t{}_1 + \chi^t{}_1, \sum_{i=1}^{n} Q(i, s) \varnothing^0{}_i x_i = f_s{}^0 \sum_{i=1}^{n} \varnothing^0{}_i x^t{}_{il} + \xi^0{}_s, s \in S \tag{18}$$

$$\sum_{i=1}^{n} Q(i, s) \varnothing^t_{il} y^t_{il} = f_s{}^t \sum_{i=1}^{n} \varnothing^t_{il} y^t_{il} + \xi^t{}_{sl} \ l \in \Omega, s \in S, t \in T, l \in \Omega, t \in T \tag{19}$$

$$\sum_{i=1}^{n} g^0{}_i \leq G^0 \tag{20}$$

$$\sum_{i=1}^{n} g^t{}_{il} \leq G^t l \in \Omega, t \in T \tag{21}$$

$$\sum_{j=1}^{n} \tilde{g}^t_i \leq \tilde{G}^0 \tag{22}$$

$$\sum_{j=1}^{n} \tilde{g}^t_{il} \leq \tilde{G}^t l \in \Omega, t \in T \tag{23}$$

$$\sum_{i=1}^{n} \beta_i g^0{}_i \leq \beta^* + \delta^0$$

$$\sum_{i=1}^{n}\beta^t_{il}g^t_{il} \le \beta_l^* + \delta^t_l \; l\in \Omega, t\in T \quad (24)$$

$$\sum_{j=1}^{h}\alpha_j g^{\sim}{}_j{}^0 \ge \alpha^* + \delta^{\sim 0} \quad (25)$$

$$\sum_{j=1}^{h}\alpha^t{}_{jl}g^{\sim}{}_{jl}{}^t \ge \alpha_l^* - \delta_l{}^{\sim t}, l\in \Omega, t\in T \quad (26)$$

$$\sum_{i=1}^{n}(i,\ 0)g_i + \sum_{j=1}^{h}{}^{\sim}(j,\ 0)g^{\sim}{}_j \ge^* - \lambda^0 \quad (27)$$

$$\sum_{i=1}^{n}(i,\ t,\ l)g_{il}{}^t + \sum_{j=1}^{h}{}^{\sim}(j,\ t,\ l)g^{\sim}{}_{jl}{}^t \ge_l{}^* - \lambda_l{}^t, l\in \Omega, t\in T \quad (28)$$

$$\sum_{il}^{1} = y^t{}_{il} - x_i \; i \in \Upsilon, l\in \Omega$$

$$\sum_{il}^{t} = y^t{}_{il} - y_{il}{}^{t-1} \; i\in \Upsilon, l\in \Omega, t\in T$$

$$x_i \le Cg^0{}_i \; i\in \Upsilon \quad (29)$$

$$y^t{}_{il} \le Cg^t{}_{il} \; i\in \Upsilon, l\in \Omega,\ t\in T$$

$$z^0{}_j \le Cg^0{}_i g^t{}_{il} \; j\in \Xi$$

$$z^t{}_{jl} \le C\, g^{\sim}{}_{jl}{}^t \; j\in \Xi, l\in \Omega, t\in T$$

$$x_i \le d_i \; i\in \Upsilon \quad (30)$$

$$y^t{}_{il} \le d_i \; i\in \Upsilon, l\in \Omega, t\in T$$

$$z^0{}_j \le d_j \; j\in \Xi$$

$$z^t{}_{jl} \le d_j \; j\in \Xi, l\in \Omega, t\in T$$

$$B^0 \ge 0, B^0 \in R \quad (31)$$

$$B^t{}_l \ge 0,\ B^t{}_l \in R \; l\in \Omega, t\in T$$

$$x_i \ge 0,\ x_i \in R \; i\in \Upsilon$$

$$y^t{}_{il} \ge 0, y^t{}_i \in R\ \ i\in \Upsilon, l\in \Omega, t\in T \quad (32)$$

$$z^0{}_j \ge 0, z^0{}_j \in R \; j\in \Xi \quad (33)$$

$$z^t{}_{jl} \ge 0, z^t{}_{jl} \in R \; j\in \Xi, l\in \Omega, t\in T$$

$$\sum_{il}^{t} \ge\ 0, \sum_{il}^{t} \in R\ \ i\in \Upsilon, l\in \Omega, t\in T \quad (34)$$

$$\delta^0 \ge 0,\ \delta^0 \in R$$

$$\delta^t{}_l \ge 0, \delta^t{}_l \in R \; l\in \Omega, t\in T \quad (35)$$

$$\delta^{\sim 0} \ge 0, \delta^{\sim 0} \in R$$

$$\delta_l{}^{\sim t} \ge 0, \delta_l{}^{\sim t} \in R \; l\in \Omega, t\in T$$

$$\lambda^0 \ge 0, \lambda^0 \in R$$

$$\lambda^t{}_l \ge 0, \lambda^t{}_l \in R \; l\in \Omega, t\in T$$

$$g^0{}_i \in B, g^0{}_i \in B \; i\in \Upsilon, j\in \Xi \quad (36)$$

$$g^t{}_{il} \in B, g^t{}_{il} \in B \; i\in \Upsilon, j\in \Xi, t\in T$$

$$\xi^0{}_s \varepsilon R, \xi^t{}_{sl} \varepsilon R\ \ s\varepsilon \hat{\hat{S}}, l\varepsilon\Omega,\ t\varepsilon T \quad (37)$$

(Stoyan and Kwon, 2011). Where C is a large constant and Eq. 34-37 set up the binary decision variables.

CONCLUSION

This study reviews the modeling of portfolio selection problem which are subject to risk and uncertainty. The researchers differentiate in handling the risk associated with the portfolio returns. Some models used up-side risk such as MV and MAV models which called deviation measures, other models used down-side risk such as VaR and CVaR as which called risk measures, the remaining models corporate the operation research mathematical programming such as stochastic, Goal and integer programming. The first two types of models assumed certainty environment while the last one dealt with uncertainty as real world environment. The strength and weaknesses of each model were given. As a conclusion of this study, SGMIP Model revealed to be the most proper model in terms of (since it) considering multi-objectives (multi-factors), multi-periods, different risk measures by using goal programming and measures the aspects of real world environment representing in uncertainty of securities' and portfolio's returns using stochastic programming. This study recommends more empirical studies on how the models work with real world applications and taking the consideration of the economic that derive the data process, especially for portfolios from operation research.

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