Neirameh, 2014. ***Sci. Tech. and Dev.*** **33 (3): 132-136**

NEW OPTICAL SOLITARY WAVE SOLUTIONS TO THE NLSE WITH A COMBINED DISPERSION TERM

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Abstract

In the present study new exact solitary wave solutions are obtained for nonlinear Schrödinger equation with a combined dispersion term, using Kudryashov method. The exact solutions showed that the developed scheme provide an efficient and reliable way for computing long-range solitary solutions given by coupled nonlinear Schrödinger equations.

Keywords: Nonlinear Schrödinger equation with a combined dispersion term**,** Kudryashov method, Exact solutions.

Introduction

The Nonlinear Schrodinger equations are highly used in modelling various phenomena in nonlinear fiber optics, like propagation of pulses. The NLSE plays a vital role in various areas of STEM disciplines (Atif et al., 2010; Banaee and Young, 2006; Biswas, 2009; Biswas et al., 2011; Ebadi et al., 2012; Ganji et al., 2009; Hadzievski et al., 2002; Johnpillai et al., 2012; Kudryashov, 2004, 1990, 2011, 1991; Ryabov, 2010; Ma and Fuchssteiner, 1996). It appears in the study of nonlinear optics, plasma physics, mathematical biosciences, quantum mechanics, fluid dynamics and several other disciplines. The main feature of NLSE is that it supports soliton solution which makes it very widely applicable. Solitons are stable nonlinear waves or pulses and is the outcome of a delicate balance between dispersion and nonlinearity. Therefore, these solitons are the essential fabrics that dictate our daily lives.

One of the most famous model equations in nonlinear science is the nonlinear Schrödinger equation with a combined dispersion term

$$iu_t + u_{xx} + a\left(u\,|u|\right)_{xx} + \mu u\,|u|^2 = 0, \qquad (1)$$

where the parameter a is some constant, one can find that Eq. 1 becomes the cubic nonlinear Schrödinger when $a = 0$, which arises from the research of nonlinear wave propagation in dispersive and inhomogeneous media. It also describes the evolution of the slowly varying envelope of an optical pulse (De Bouard and Debussche, 2010; Nakkeeran and Wai, 2005; Ndzana et al., 2007).

The NLSE equation plays an important role in understanding pulse propagation in optical fibers, which is of critical importance to the field of fiber-based telecommunications (Hoseini and Marchant, 2010).

The results in the next sections show that the nonlinear Schrödinger equation with a combined dispersion term does have some new solutions.

Method applied

The aim of this section is to present the algorithm of the modified Kudryashov method for finding exact solutions of the nonlinear evolution equations.

Let us consider the nonlinear partial differential equation in the form

$$E\left(u_t, u_x, ..., x, t\right) = 0. \qquad (2)$$

We use the following ansatz

$$u = U(\xi)\,\mathrm{e}^{i(\alpha x + \beta t)}, \qquad \xi = x - ct, \qquad (3)$$

From Eq. (2) we obtain the ordinary nonlinear differential equation

$$\begin{aligned}&\varphi(-cU'(\xi)\mathrm{e}^{i(\alpha x+\beta t)} + i\beta U\,\mathrm{e}^{i(\alpha x+\beta t)},\\ &U'(\xi)\mathrm{e}^{i(\alpha x+\beta t)} + i\alpha U\,\mathrm{e}^{i(\alpha x+\beta t)}, ...) = 0.\end{aligned} \qquad (4)$$

Now we show how one could obtain the exact solution of Eq. 4, using the approach by modified Kudryashov (Kudryashov, 2004, 2011). This method consists of the following steps.

Determination of the dominant term

To find the dominant terms, we substitute

$$U = \xi^p, \qquad (5)$$

all terms of Eq. 4. Then we compare degrees of all terms in Eq. (4) and choose two or more with the smallest degree. The minimum value of P defines the role of solution for Eq. 4. We have to point out that the method can be applied when N is an integer. If the value for N is noninteger one can transform the equation studied and repeat the procedure.

The solution structure

We look for exact solution of Eq. 4 in the form

$$U = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) + ... + a_N Q^N(\xi), \qquad (6)$$

where a_i are unknown constants to be determined later, such that $a_N \neq 0$, while $Q(\xi)$ has the form

$$Q(\xi) = \frac{1}{1+e^{\xi}}, \qquad (7)$$

These functions satisfy the first order ordinary differential equations (Riccati equations)

$$Q'(\xi) = Q^2(\xi) - \mathrm{Q}(\xi), \qquad (8)$$

Eq. 8 is necessary to calculate the derivatives of functions $Q(\xi)$.

Remark 1: This Riccati equation also admits the following exact solutions:

$$\begin{aligned} Q_1(\xi) &= \frac{1}{2}\left(1 - \tanh\left[\frac{\xi}{2} - \frac{\varepsilon \ln \xi_0}{2}\right]\right), \quad \xi_0 > 0, \\ Q_2(\xi) &= \frac{1}{2}\left(1 - \coth\left[\frac{\xi}{2} - \frac{\varepsilon \ln \xi_0}{2}\right]\right), \quad \xi_0 < 0, \end{aligned} \qquad (9)$$

Derivatives calculation

We should calculate all derivatives of functions $Q(\xi)$. One can do it, using the computer algebra systems Maple or Mathematica. As an example, we consider the general case when N is arbitrary. Differentiating the expressions (7) with respect to ξ taking into account (8) we have

$$\begin{aligned} Q'(\xi) &= \sum_{i=1}^{N} a_i\, i\,(Q-1)Q^i, \\ Q''(\xi) &= \sum_{i=1}^{N} a_i\, i\,((i+1)Q^2 - (2i+1)Q + i)Q^i, \end{aligned} \qquad (10)$$

The high order derivatives of functions $Q(\xi)$ can be calculated (Kudryashov, 1990, 1991).

Defining the values of unknown parameters

We substitute expressions (10) in Eq. (6). After it we take $Q(\xi)$ from (10) into account. Thus Eq. (6) takes the form

$$P\left[Q(\xi)\right]$$

where $P\left[Q(\xi)\right]$ is a polynomial of functions $Q(\xi)$. Then we collect all terms with the same powers of functions $Q(\xi)$ and make these expressions equal to zero. As a result we obtain system of algebraic equations. Solving this system, we get the values of unknown parameters.

Application to the Nonlinear Schrödinger equation with a combined dispersion term

We study Eq. 1, considering the following transformation:

$$u = U(\xi)\,\mathrm{e}^{i(\alpha x + \beta t)}, \qquad \xi = x - ct, \qquad (11)$$

Inserting the expression (11) into (1), we obtain the following equation:

$$\begin{aligned} &\mu U^3 - a\alpha^2 U^2 + U'' + 2a\left(U'\right)^2 - \\ &i\,(c + 2\alpha)U' - (\beta + \alpha^2)U + \\ &2aUU'' + 4ia\alpha UU' = 0 \end{aligned} \qquad (12)$$

The pole of the Eq. 12 is equal to $N = 1$, thus we look for exact solution in the forms

$$U = a_0 + a_1 Q(\xi), \qquad (13)$$

Substituting Eq. 13 in Eq. 12 and taking Eq. 10 into account, we obtain the polynomial of functions $Q(\xi)$. Collecting all terms with the same power of functions $Q(\xi)$ and equating these expressions to zero we obtain the system of algebraic equations. Solving this system we find the following values of parameters:

Case 1: we have

$$\begin{aligned} a_0 &= \frac{a\alpha^2 + \sqrt{a^2\alpha^4 + 4\mu(\beta + \alpha^2)}}{-2a\alpha^2}, \\ a_1 &= \frac{10a\alpha - 4ia\alpha^2 \pm \sqrt{\alpha^2(10a - 4ia\alpha)^2 - 4\mu(2\alpha^2 - 2(a\alpha^2 \pm \sqrt{a^2\alpha^4 + 4\mu(\beta + \alpha^2)}))}}{2\alpha\mu}, \\ c &= \left(a\alpha^2 + \sqrt{a^2\alpha^4 + 4\mu(\beta + \alpha^2)}\right)\left(3i\,\mu a_0 - 2ia\alpha^2 + 2ia + 4a\alpha\right) - 2ia\alpha^2(1 - \beta - \alpha^2) + 4a\alpha^3, \end{aligned} \qquad (14)$$

Substituting Eq. 14 into the Eq. 13, then the solitary wave solution to the Eq. 1 is

$$U=-\frac{a\alpha^2\pm\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}}{2a\alpha^2}+$$
$$\left(\frac{10a\alpha-4ia\alpha^2\pm\sqrt{\alpha^2(10a-4ia\alpha)^2-4\mu(2\alpha^2-2(a\alpha^2\pm\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}))}}{2\alpha\mu}\right)\times \qquad (15)$$
$$\left(1+e^{x-\left(\left(a\alpha^2\pm\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3\right)t}\right)^{-1},$$

and,

$$u=-\frac{a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}}{2a\alpha^2}+$$
$$\left(\frac{10a\alpha-4ia\alpha^2\pm\sqrt{\alpha^2(10a-4ia\alpha)^2-4\mu(2\alpha^2-2(a\alpha^2\pm\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}))}}{2\alpha\mu}\right)\times \qquad (16)$$
$$\left(1+e^{x-\left(\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3\right)t}\right)^{-1}\mathrm{e}^{i(\alpha x+\beta t)}.$$

Case 2: For

$$a_0=\frac{\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}-a\alpha^2}{2a\alpha^2},$$
$$a_1=\frac{10a\alpha-4ia\alpha^2\pm\sqrt{\alpha^2(10a-4ia\alpha)^2-4\mu(2\alpha^2-2(a\alpha^2\pm\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}))}}{2\alpha\mu}, \qquad (17)$$
$$c=\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3,$$

the solitary wave solution to the Eq. 1 is

$$u=\frac{\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}-a\alpha^2}{2a\alpha^2}+$$
$$\left(\frac{10a\alpha-4ia\alpha^2\pm\sqrt{\alpha^2(10a-4ia\alpha)^2-4\mu(2\alpha^2-2(a\alpha^2\pm\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}))}}{2\alpha\mu}\right)\times \qquad (18)$$
$$\left(1+e^{x-\left(\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3\right)t}\right)^{-1}\mathrm{e}^{i(\alpha x+\beta t)},$$

Case 3: For

$$a_1=\frac{3\mu\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2a^2\alpha^2\left(\alpha^2+4-4i\,\alpha\right)}{\left(4ia\alpha-6a\right)\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2ia\alpha^2(\mathrm{c}+2\alpha)+6a\alpha^2},$$

$$a_0=\frac{a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}}{-2a\alpha^2}, \tag{19}$$

$$c=\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3,$$

the solitary wave solution to the Eq. 1 is

$$u=-\frac{a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}}{2a\alpha^2}+$$
$$\left(\frac{3\mu\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2a^2\alpha^2\left(\alpha^2+4-4i\,\alpha\right)}{\left(4ia\alpha-6a\right)\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2ia\alpha^2(\mathrm{c}+2\alpha)+6a\alpha^2}\right)\times \tag{20}$$
$$\left(1+e^{x-\left(\left(a\alpha^2+\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3\right)t}\right)^{-1}\mathrm{e}^{i\,(\alpha\,\mathrm{x}+\beta\,\mathrm{t})},$$

Case 4: For

$$a_1=\frac{3\mu\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2a^2\alpha^2\left(\alpha^2+4-4i\,\alpha\right)}{\left(4ia\alpha-6a\right)\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2ia\alpha^2(\mathrm{c}+2\alpha)+6a\alpha^2},$$

$$a_0=-\frac{a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}}{2a\alpha^2}, \tag{21}$$

$$c=\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3,$$

the solitary wave solution to the Eq. (1) is

$$u=-\frac{a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}}{2a\alpha^2}+$$
$$\left(\frac{3\mu\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2a^2\alpha^2\left(\alpha^2+4-4i\,\alpha\right)}{\left(4ia\alpha-6a\right)\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)+2ia\alpha^2(\mathrm{c}+2\alpha)+6a\alpha^2}\right)\times \tag{22}$$
$$\left(1+e^{x-\left(\left(a\alpha^2-\sqrt{a^2\alpha^4+4\mu(\beta+\alpha^2)}\right)\left(3i\,\mu a_0-2ia\alpha^2+2ia+4a\alpha\right)-2ia\alpha^2(1-\beta-\alpha^2)+4a\alpha^3\right)t}\right)^{-1}\mathrm{e}^{i\,(\alpha\,\mathrm{x}+\beta\,\mathrm{t})}.$$

Conclusions

Exact solutions of NLEEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by

creating appropriate natural conditions, to determine these parameters or functions.

Therefore, investigation of exact travelling wave solutions is becoming successively attractive in nonlinear sciences day by day.

By this method, we have obtained solitary wave solutions to the nonlinear Schrödinger equation with a combined dispersion term. This paper encompassed several studies that were conducted in the past. Our method can also be applied to other models where the location of extreme value points is determined. Based on the study, it might be concluded that the improved method is useful and efficient. It can be widely applied to other nonlinear wave equations. Our study may be useful to further understand the role that the nonlinearly dispersive terms play on the optical wave solutions.

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