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A Simple Assessment for Nonlinear Control Using Fuzzy Logic Approximators

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Abstract: This study is aimed at looking into the use of fuzzy logic systems for the control of nonlinear problems. Two approaches of control are used. The first approach considers the determination of optimal control strategies. The second one considers the use of variable structure control systems. The aim is the use of fuzzy systems as approximators of nonlinear control laws.

Key words: Nonlinear systems, optimal control, sliding mode control, fuzzy systems

INTRODUCTION

In most if not all cases, real systems are nonlinearand consequently the synthesis of adequate control is difficult, particularly the determination of the analytic expression of such control is practically impossible.

The emergence of automatic control systemsand microprocessor systemsand the introduction of new numerical control strategies have greatly improved the dynamicand the performances of nonlinear system controllers.

In the case of linear systems, the optimal control is easy to obtain using a feedback gain. However, in the case of nonlinear systems, the problem becomes very difficult. The differential dynamic programming can give the solution of the nonlinear problem. It is a combination of the dynamic programmingand Newton's method[1-5]. In fact, it uses a Newton's like quadratic approximation of the Bellman function[6] and combines it with a dynamic programming's like recurrent decomposition. Historically, Mayne[7] was the first to introduce this notionand he named it secondorder method. The term differential dynamic programming was given by Jacobson[8]. The comparison between Newton methodand the differential dynamic programming was presented in many studies[9-10]. During last decades, the differential dynamic programming has been widely used for the determination of optimal control strategies of nonlinear systems. However, it is very important of nonlinear systems. However, it is very importantdepend highly on the initial start state pointand the desired final state. Varying the initial pointand/or the desired final state, inevitably requires a new differential dynamic procedure in order to obtain the new control laws. To overcome such difficulties, authors used the approximation of the optimal control by the use of fuzzy systems and neural networks [11-14].

Sliding Mode Control (SMC), or the paradigm of Variable Structure Systems (VSS), is a popular robust control method among control engineers[15]. It is simple to useand has been quite effective against model uncertainties.The sliding mode control is also well known for its robustness to disturbancesand parameter variations[16]. Conventional sliding mode control that utilizes state feedback has the disadvantage that the state components of the system have to be measurable, which is seldom fulfilled in many applications. Though the situation can be remedied by the use of state observers. The addition of an observer immediately destroy the theoretical assurance of robustness, that defeats the purpose of employing sliding mode method in the first place. Sliding mode methods that involve output feedback are also available[17].

With a conventional switching surface, the sliding mode occurs after the system reaches the switching surface. However, before the system reaches the switching surface, it is sensitive to parameter variations and disturbances. This produces discontinuous trajectories and causes a chattering which is particularly undesirable when the actuator mechanism may be damaged by rapid switching[18]. Therefore, a method (such as Fuzzy Logic: FL) to design control input is required. In recent years, much interest has been focused upon neural networks and fuzzy systems which are generally used to solve highly nonlinear control problems. Their use in the area of process control has been currently considered in the literature. The powerful performances shown by the neural networkand the fuzzy control is due to the distributed information within the network. In fact, neural network or fuzzy system controllers present a strong robustness which is not affected by the process parameter variations, for which conventional controllers fail.

Fuzzy control is a direct method for controlling a system without the need of a mathematical model, in contrast to the classical control which is an indirect method with a mathematical model.

Fuzzy control research based on fuzzy set theory was initiated by Mamdani[19]. Braaeand Rutherford[20] proposed both an algebraic modeland a linguistic model for fuzzy control. The algebraicmodel cannot deal dIrectly with fuzzy controller rules. This limitation of the algebraic model led to the linguistic model that provided a linguistic structure for dealing with fuzzy control. Based on the phase planes, Langari [21] proposed an analytical formulation of fuzzy control that is essentially nonlinear. Harris and Moore[22] proposed a graphical analysis tool for considering overall system performance. However, this method becomes so difficultand impossible for higher order systems. Johansen[23] carried out stabilityand performance analysis of a multiinput multioutput fuzzy model based control system. The proposition of Wang[24] is an adaptive fuzzy system, for which the parameters of the fuzzy system are adjusted by a training algorithm, using numerical inputoutput pairs.

Actually, fuzzy control has been implemented in many industrial applications. However, there are few systematic procedures available for analysisand design of fuzzy control. In order to investigate the local stability of fuzzy systems, Kiszka *et al.*[25] proposed the use of energy for fuzzy relations. These fuzzy relations were exactly known; this yields to a contradiction with the main feature of fuzzy control. Langari and Tomizuka[26] proposed a method for the stability analysis of fuzzy control systems, in which Lyapunov's direct method provided sufficient conditions for the stability.

This study presents a hybrid design which eliminates the limitationsand incorporates the merits of both classical control approachesand the fuzzy logic control.

PROBLEM FORMULATION

Let us consider a nonlinear system described by the following nonlinear differential equation:

$$\dot{x}(t) = f\left[x(t),\ u\ (t),t\right] \tag{1}$$

where: $X \in \Re^n$ the state vectorand $u < \Re^m$ thecontrol vector. In the following, we will consider the class of nonlinear systems for which state equations are modelled by: $\dot{X}$ =f(x)+g(x)u.There are several interesting applications verifying this limitation, for example: electric machinesand robotic systems. In the case of linear systems, the state equation becomes:

$$\dot{X}(t) = A\ (t)\ X(t) + B(t)u(t) \tag{2}$$

OPTIMAL CONTROL

The optimal control (considered in this paper) consists on the determination of control laws which minimize a criterion expressed as:

$$J = \int_0^{+\infty} r(x,u,t)dt \tag{3}$$

Function r(x,u,t) is always a quadratic functionof vectors x and u. In this case, the objective function J becomes:

$$J = \frac{1}{2}\int_0^{+\infty}\left(x^T\ QX + U^T\ Ru\right)dt \tag{4}$$

Q and R represent ponderation matrices of theoptimal control problem. (Q , 0and R > 0).

For nonlinear systems, there is no systematic explicit solutions. However, the case of linear systems have explicit optimal solutions. In the following, we will express the optimal solution of linear systems.

The Hamiltonian of the problem can be expressed by:

$$H = \frac{1}{2}\left(x^TQx + u^T\ Ru\right) + \Psi^T(Ax + Bu) \tag{5}$$

Optimal conditions can be written as:

$$\dot{x} = H_\Psi = Ax + Bu \tag{6}$$

$$\dot{\Psi} = -H_x = -Qx - A^T\ \Psi \tag{7}$$

$$0 = H_u = Ru + B^T\Psi \tag{8}$$

which give: $u = -R^{-1}B'\psi$ and the following system which is called Hamiltonian system:

$$\dot{x} = Ax - BR^{-1}\ B^T\ \Psi \tag{9}$$

$$\dot{\Psi} = -Qx - A^T\ \Psi \tag{10}$$

which will be written in the condensed form as follows

$$\begin{bmatrix}\dot{X}\\ \dot{\Psi}\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}\ B^T\\ -Q & -A^T\end{bmatrix} \times \begin{bmatrix}X\\ \Psi\end{bmatrix} \tag{11}$$

The following matrix is called the Hamiltonian matrix of the problem:

$$A = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \quad (12)$$

Hamiltonian system can be expressed in the condensed form as: Z = AZ, for:

$$Z = \begin{bmatrix} X \\ \Psi \end{bmatrix} \quad (13)$$

Let us consider the following change of variables: Z = M W, where:

$$M = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \quad (14)$$

with I and 0 represent the identityand the null matrices of pertinent dimensionand where:

$$W = \begin{bmatrix} X \\ \xi \end{bmatrix} \quad (15)$$

The variations of vector W can be determined as follows: $\dot{Z} = \dot{M}W + M\dot{W}$, which gives: $AMW = \dot{M}W + M\dot{W}$. Thus:

$$W = M^{-1}\left(AM - \dot{M}\right)W = BW \quad (16)$$

where B is the new Hamiltonian matrix:

$$B = \begin{bmatrix} A-BR^{-1}B^TP & -BR^{-1}B^T \\ B_{21} & \left(A - BR^{-1}B^TP\right)^T \end{bmatrix} \quad (17)$$

and:

$$B_{21} = -\left(\dot{P} + PA + A^TP - PBR^{-1}B^TP + Q\right) \quad (18)$$

Matrix B is a bloctriangular one if $B_{21} = 0$, or:

$$\dot{P} + PA + A^TP - PBR^{-1}B^TP + Q = 0 \quad (19)$$

which represents a differential nonlinear Riccati equation. Matrix P have the following properties:

- P(t) is unique.
- P(t) is a symmetric matrix. In fact, if P(t) is a solution, P T (t) is also a solution.
- P(t) is a positive matrix.
- The optimal control solution can be expressed in a feedback loop (Fig. 1):

$$u(t) = -K(t)x(t) \quad (20)$$

where K(t) the optimal gain expressed by:

$$K(t) = R^{-1}B^TP(t) \quad (21)$$

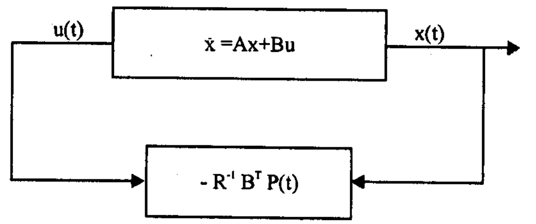

Fig. 1: Closed loop system

- The dynamic behaviour of the closed loop system can be described by:

$$x = (A-BK)x = \left(A-BK^{-1}B^TP\right)x \quad (22)$$

- The optimal criterion can be expressed by:

$$J = \frac{1}{2}x(0)^T P(0)x(0) \quad (23)$$

- In the stationary case:

– matrix P is constant and then:

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \quad (24)$$

which represent a nonlinear algebraic Riccati equation.

- The feedback gain K is constant.
- The closed loop state matrix F =A BK is a Hurwitz matrix (its eigenvalues have negative real parts). In fact, it is easy to verify that:

$$PF + F^TP = \left(PBR^{-1}B^TP + Q\right) < 0 \quad (25)$$

- Eigenvalues of matrix A are λ_I λ_I, where λ_I is an eigenvalue of matrix F.
- In the case where Q = 0, the eigenvalues of matrix F are the stable eigenvalues of matrix Aand the opposite of the unstable eigenvalues of matrix A.

VARIABLE STRUCTURE CONTROL

The Variable Structure Control (VSC) or the Sliding Mode Control (SMC) forces to the system to follow a subvariety of the state space, whose dimension is n ¡ m. This subvariety is described by the following m dimension equation:

$$s(x) = 0 \quad (26)$$

This variety is called the sliding hypersurface, or surface in the case of three dimension systems.

It becomes a sliding curve in the case of two dimension systems. Always, s(x) is linear in term of state vector x:

$$s(x) = Kx \tag{27}$$

In the case where the control imposes to the system to follow the sliding surface, the system becomes insensitive with respect parameter variationsand external perturbations.

In the following, we will consider the case where state variables can be represented as:

$$\dot{x} = f(x) + g(x)\, u \tag{28}$$

Hypothesis:

- Function f(x) is o(x), (this allows to write that f(x) = 0 for x = 0: the desired position)
- Function s(x) is o(x), (leading s(0) = 0).
- Matrices g(x)and sx are full rank m in aneighbor of the desired position x = 0, with

$$s(x) = \frac{ds}{dx} \tag{29}$$

There are some configurations while expressing the adequate sliding mode control which can be resumed in the three configurations.

First configuration: The first configuration change the structure bya variable state feedback commutation such as:

$$u_i = \begin{vmatrix} -K_{1i}x & \text{if } s_i(x) > 0 \\ -K_{2i}x & \text{if } s_i(x) < 0 \end{vmatrix} \tag{30}$$

for i = 1; m.

Second configuration: The change of the structure will be by a commutation the identity matrix of order of the control actuator:

$$u_i = \begin{vmatrix} U_i \max = U_{i0} \text{ if } s_i(x) > 0 \\ U_i \min = U_{i0} \text{ if } s_i(x) < 0 \end{vmatrix} \tag{31}$$

Third configuration: In the case of the second configuration, the system oscillates around the variety s(x) = 0 many times before the system reach this varietyand slide on it. To ensure the sliding on s(x) = 0 in the first time when the system reaches the variety s(x) = 0, magnitudes Ui0 should be increased. This is not usually possible, resulting from the limitation of the actuatorsand the high dissipated energy. To resolve such problem, the control expression becomes:

$$u = u_{eq} + \Delta u \tag{32}$$

where u_{eq} is the socalled equivalent control. ueq is the required control which forces to the system to follow the variety s(x) = 0. Δu is expressed by:

$$\Delta u_i = \begin{vmatrix} \Delta u_i \max = u_0 \text{ if } s_i(x) > 0 \\ \Delta u_i\, , \min = -u_0\ s_i(x) < 0 \end{vmatrix} \tag{33}$$

In the following, we will consider the third configuration.

Equivalent control: If the system slides on the variety s(x) = 0, we can write that s (x) = 0, so:

$$\frac{ds}{dx} \times \frac{dx}{dt} = 0 \tag{34}$$

or

$$\frac{ds}{dx} \times \left[f(x) + g(x)u \right] = 0 \tag{35}$$

If matrix $s_x g(x)$ is regular (the required condition for ensuring sliding mode control), the expression of u_{eq} can be written as [4]:

$$u_{eq} = -\left(\frac{ds}{dx}\, g(x) \right)^{-1} \frac{ds}{dx}\, f(x) \tag{36}$$

In the case of linear systems, the expression u_{eq} becomes easier:

$$u_{eq} = -\,(KB)^{-1}\ KAx \tag{37}$$

State equation on the sliding surface: The state equation on the sliding surface can be represented as:[27]

$$\frac{dx}{dt} = \left[I - g(x) \left(\frac{ds}{dx}\, g(x) \right)^{-1} \frac{ds}{dx} \right] f(x) \tag{38}$$

where I is the identity matrix of order n. In the case of linear systems, state equations becomes:

$$\frac{dx}{dt} = A^*\, x = \left[I_n - B\ (KB)^{-1}\ K \right] Ax \tag{39}$$

Theorem: For any initial condition x(t0) such as s(x(t0)) = 0, the system remains on s(x) = 0, for all t≥to.

Proof: Let's denote $x_o = x(t_o)$ such as $s(x(t_o)) = 0$.We will show that s(x) is constant. Then, the derivative of s(x) with respect to time should be equal to zero.

$$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$$
$$= \frac{ds}{dx}\left[I_n - g(x)\left(\frac{ds}{dx} g(x)\right)^{-1} \frac{ds}{dx}\right] f(x)$$
$$= \left[\frac{ds}{dx} - \frac{ds}{dx} g(x)\left(\frac{ds}{dx} g(x)\right)^{-1} \frac{ds}{dx}\right] f(x) \quad (40)$$

Thus:

$$s(x(t)) = s(x(t_o)) = 0 \quad (41)$$

Comments: The last result is very important. In fact, it is clear that the first contact of the state trajectory with the variety s(x) = 0 forces the system toremain on this variety.

The equivalent control corresponds to the required control for ensuring the sliding on s(x)=0. The control term Δu corresponds to the the correction of the control when there are any parameter variations, or any external perturbationand or any error on the considered model.

Stability: To ensure the stability of the system, we will consider two phases. The first one consists to ensure the convergence of the system to the sliding varietyand the second one consists to the convergence to the desired state when the system remains on the sliding surface.

Outlet of the variety s(x) = 0: Consider the Lyapunov function expressed by:

$$V(x) = \frac{1}{2}|s(x)|^2 = \frac{1}{2} s^T s \quad (42)$$

This function is strictly positive for s(x) = 0. Then, it is definite positive in the outlet of the sliding surface. Its differential with respect to time is expressed by Zhang [28]:

$$\frac{dV}{dt} = s^T\dot{s} = s^T s_x \dot{x} \quad (43)$$

Or:

$$\frac{dx}{dt} = \left[I_n - g(x)\left(\frac{ds}{dx} g(x)\right)^{-1} \frac{ds}{dx}\right] f(x) + g(x)\,\Delta u \quad (44)$$

This yields:

$$\frac{dV}{dt} = s^T s_x\, g(x)\, \Delta u \quad (45)$$

If:

$$\Delta u = -\mathrm{diag}(u_0)\left[s_x\, g(x)\right]^{-1} \times \mathrm{sign}\,(s) \quad (46)$$

one can write:

$$\frac{dV}{dt} = -s^T\, \mathrm{diag}(u_0)\, \mathrm{sign}(s) = -\sum_{i=1}^{m} u_{i0}|s_i| < 0 \quad (47)$$

Note that the choice where the matrix sxg(x) is definite positive, the control expression can be simplified as:

$$\Delta u = -\mathrm{diag}(u_0)\, \mathrm{sign}(s) \quad (48)$$

Inlet of the variety s(x) = 0: To guarantee the stabilityand the convergence of the state to its desired position, the choice of the variety is very important. In fact, a bad choice leads to unstable cases. If the state representation of the system is given in the controllability canonical form (for the case where m = 1), we have:

$$\frac{dx_i}{dt} = x_{i+1} \quad (49)$$

for i = 1; 2..., n − 1. The choice of s (x) = Kx yields:

$$s(x) = \sum_{i=1}^{n} k_i\, x_i = \sum_{i=1}^{n} k_i + 1\, \frac{d^i x_1}{dt^i} \quad (50)$$

When the system remains on the sliding surface, we can write:

$$\sum_{i=1}^{n} k_i + 1\, \frac{d^i x_1}{dt^i} = 0 \quad (51)$$

Thus, it is clear that the system is stable if the following polynom is a Hurwitz one (that is to say that its roots have negative real parts):

$$\sum_{i=0}^{n-1} k_i + 1 p^i = 0 \quad (52)$$

In the general case, if there exists a matrix P such as the derivative of $V(x) = x^T Px$ is definite negative on s(x) = 0, the system is stable.

In a neighbor of the desired position x = 0and remaining on s(x) = 0, we have:

$$s(x) = s_x\, x + o(x)^2 \quad (53)$$
$$f(x) = f_x\, x + o(x)^2 \quad (54)$$
$$\dot{x} = \left[\mathrm{In}\, g(x)\left(\frac{ds}{dx} g(x)\right)^{-1} \frac{ds}{dx}\right] \frac{df}{dx} x + o(x)^2 \quad (55)$$
$$\dot{V} = \left(\dot{x}^T Px + x^T P\dot{x}\right) = -x^T Qx + o(x)^3 \quad (56)$$
$$Q = -P\left[I_n - g(x)\left(\frac{ds}{dx} g(x)\right)^{-1} \frac{ds}{dx}\right] f_x$$

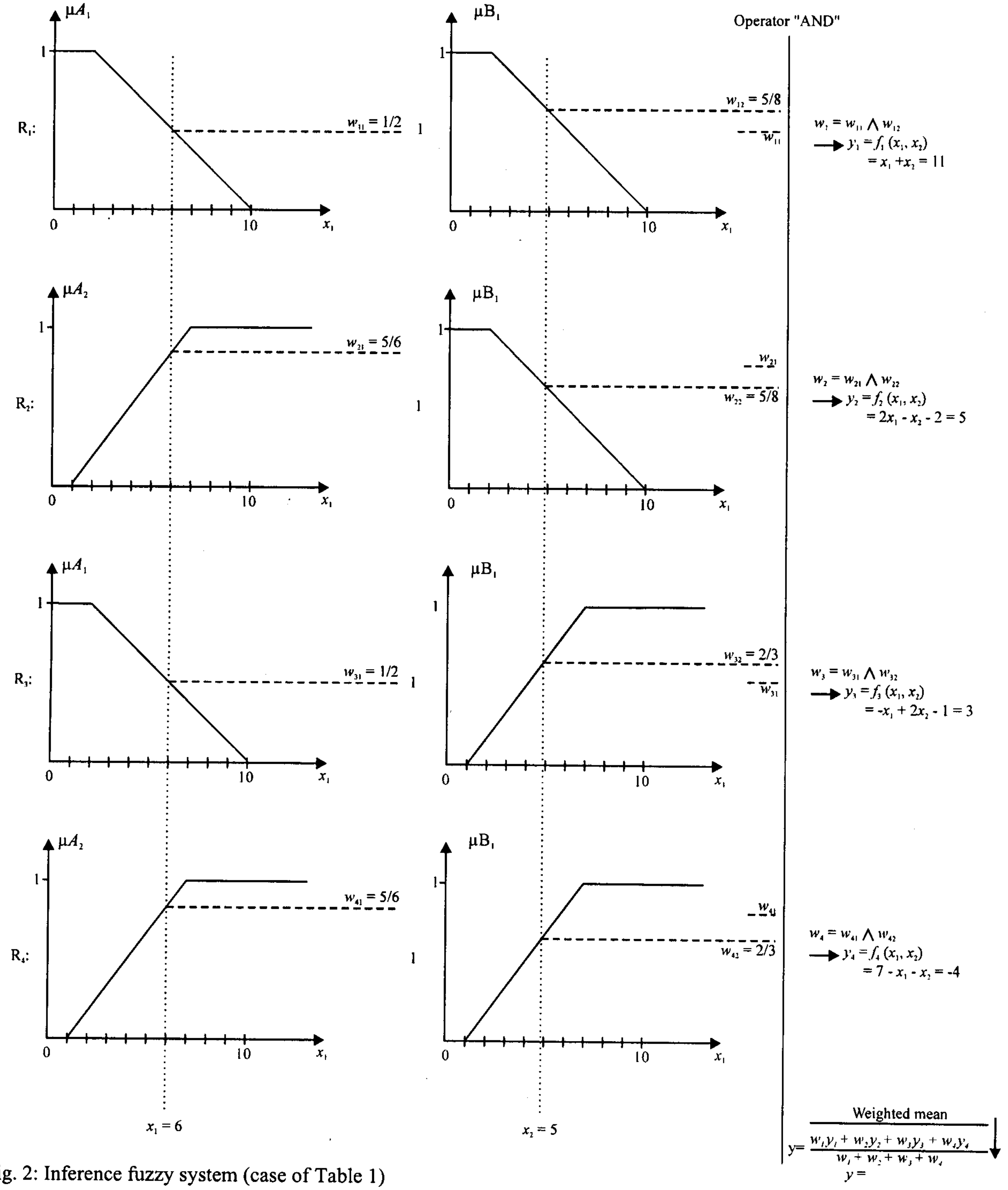

Fig. 2: Inference fuzzy system (case of Table 1)

$$-f_x^T\left[I_n - g(x)\left(\frac{ds}{dx}\, g(x)\right)^{-1} \frac{ds}{dx}\right]^T P \qquad (57)$$

It is evident that the system is stable if the roots !i of the following polynom of ! are all positive:

$$\begin{vmatrix} wI - Q & s_x^T \\ s_x & 0 \end{vmatrix} = 0 \qquad (58)$$

Comments: It is evident that in the case where the considered system is nonlinear, the formulation of theequivalent control becomes complexand sometimes implicit. In order to simplify these computations,function s(x) is chosen linear. In some papers, state equations are presented in the controllability canonical form[29,30], that is to say:

$$\frac{d^n y}{dt^n} = f(x) + g(x)u \quad (59)$$

where y the output of the system, x is its state vectorand with single input u:

$$x = \left[y \; \dot{y} \; .... \; y^{(n-1)} \right]^T \quad (60)$$

with y (i) = d^i y=dt^i

Moreover, it is usually preferred that the system output follows reference trajectory yr. Then, the error will be defined as:

$$e = y - y_r \quad (61)$$

Function s(x) will be defined as:

$$s(e) = \left(\frac{d}{dt} + \lambda \right)^{(n-1)} e \quad (62)$$

This leads to the characteristic equation of the imposed dynamic in the sliding mode:

$$(p + \lambda)^{n-1} = 0 \quad (63)$$

Mathematical tools presented above are valuable if we replace vector x by the state error x xr, where xr is the state of the desired trajectory. To conclude, it is to be noted that the sliding mode control is a robust approach. In fact, in the sliding phase, the controlled system becomes insensitiveto parametric variationsand to external perturbations. However, this approach presents some drawbacks:

- This approach needs the use of state vectoror the derivatives of the output. Such disadvantage is overcome by the construction of an observer.
- Actuators are highly solicited. This is yielded by the control term (¢ui = ui0 sign [si(x)]) which oscillates with an infinite frequency, around the surface s(x) = 0. These variations generates the chattering phenomenon.To solve this problem, function sign will be replaced by a saturation function, or by the hyperbolic tangent function, or well by a fuzzy system [11,31-34].

GENERALITIES ON FUZZY SYSTEMS

Sugeno fuzzy systems: Sugeno fuzzy models were proposed in 1988 by Takagi, Sugenoand and Kang. The aim of these models is to express the output of fuzzy rules by script values or script expressions in terms of the antecedent vector of the rules. The output of a Sugeno fuzzy model can be expressed by the average of all fuzzy rule outputs weighted by the firing rule terms. This phase can be considered as the defuzzification step of the fuzzy model. In the case of two inputs x1and x2and one output y, typical fuzzy rules are defined by:

If "(x_1 is A)and (x_2 is B)" Then
y = f(x_1; x_2)"

where f(x_1; x_2) is a known function, which is generallya polynom of x1and x2.

- If f(x_1; x_2) is a first order polynom (f(x_1; x_2) = $ax_1 + bx_2 + c$), then the fuzzy system is called as a first order one.
- If f(x_1; x_2) is constant, then the fuzzy system is named as a zero'th order one. Figure 2 showes a fuzzy system with four rules described in Table 1. The output, which is a continuous function, is an overlapping of rule consequences weighted by firing rule terms.

Table 1: Inference table of an example with two inputs x1and x2and one output y (for x1 2 2 [0 ; 10]and x2 2 [0 ; 10])

	Antecedents		
Rule	x_1	x_2	y
R_1	A_1	B_1	x_1+x_2
R_2	A_2	B_1	$2x_1-x_2-2$
R_3	A_1	B_2	$-x_1+2x_2-1$
R_4	A_2	B_2	$-x_1-x_2+7$

It is remarkable that quantity $w_1+w_2+w_3+w_4$ oscillates around the value 1. Then Jangand Sun[35] proposed to simplify the com putations by replacing the output expression by: $y = w_1y_1 + w_2y_2 + w_3y_3 + w_4y_4$.

However, such simplification implies a lose of the signification of membership functions. In the case of Rrules, the output of the fuzzy system can be generalized as:

$$y = \frac{\sum_{r=1}^{R} y_r \wedge_{n=1} \mu_{A_i^r}(x_i)}{\sum_{l=1}^{R} \wedge_{n=1} \mu_{A_j^l}(x_j)}$$

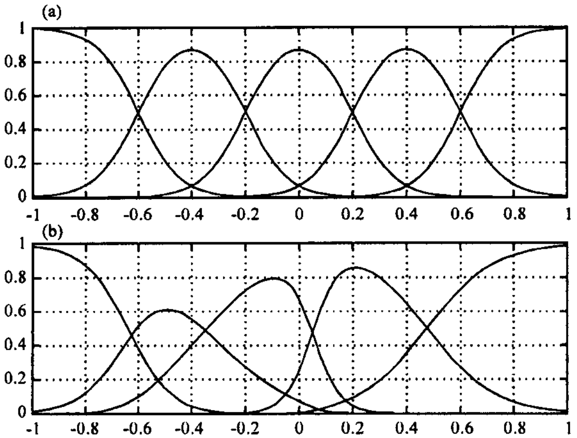

Fig.3: Basis fuzzy functions (gaussian membership functions). a; Symmetrical and equidistant disposition. b; Dissymmetrical

$$= \frac{\sum_{r=1}^{R} w_r\, y_r}{\sum_{l=1}^{R} w_l}$$

$$= \sum_{r=1}^{R} \frac{w_r}{\sum_{l=1}^{R} w_l}\, y_r = \sum_{r=1}^{R} \lambda_r y_r \qquad (64)$$

where y_r represents the output of the rule r, with:

$$W_r = \overset{n}{\underset{i=1}{\wedge}} \mu\, A_i^{\,r}(x_i)$$

$$\lambda_r = \frac{\wedge_{n=1}\, \mu_{A_i^r}(x_i)}{\sum_{l=1}^{R} \wedge^{n}{}_{j=1}\, \mu_{A_j^l}(x_i)}$$

$$= \frac{w_r}{\sum_{l=1}^{R} w_l} \qquad (65)$$

Quantities ,rrepresent the firing rules. ^ rep resents a tnorm for theand inference operator (for example min, product, etc.).

Quantities ,r, for r = 1; R are functions of the input vector x. They represent the fuzzy basis functions [36,37]. Mendel[36] showed that these functions represent universal approximator of all continuous functions defined on a compact. For more comparisons with other basis functions, see the reference[38].To approximate nonlinear functions, these func tions are better than the radial basis functions[39]. Moreover, fuzzy systems can be considered as a special neural network[39] The denominator of functions ,r normalizes these functions. In the case where numerators are these functions.symmetric, these functions can be considered as normalized radial basis function[40].

For the case where the input vector x $\in$ [1; 1] (n = 1), with R = 5 rulesand for a gaussian mem bership func tions: gauss(x; σ ,m_r)= exp [(x - m_r)2 / σ^2], (here σ = 0:25), curves of λ_r are presented in figure 3a, for m_1 = 0:8, m_2 = 0:4, m_3 = 0, m_4 = 0:4and m_5 = 0:8 (input fuzzy sys tems are equidistantand symmetrically disposed). Moreover, curves of ,r are presented inn 3b, for m1 = 0:8, m2 = 0:45, m3 = 0:25, m4 = 0:35 ¡ ¡ ¡and m5 = 0:6 (input fuzzy systems are dissymmet rically disposed). These functions combine the ad vantage of radial basis functions which have good local propertiesand sigmoidal neural networks which have good global properties.

It is suitable to combine rules from the infor mation given by expertsand rules resulting from numerical data[36]:

$$\lambda_r = \alpha\lambda_{Er} + (1 - \alpha)\lambda_{Nr} \qquad (66)$$

with $\alpha \in [0,1]$, λ_{Er} are fuzzy basis functions given by expertsand λ_{Nr} are fuzzy basis functions re sulting from numerical data. It is to be noted that classical basis functions (Laguerre polynom, Legendre Polynom, Trigono metric functions, etc.) are orthogonal. However, fuzzy basis functions are not orthogonal. In the following, we will show that fuzzy basis functions are universal approximators[36].

Universal Approximators: Let's consider F the set of piecewise continuous scalar functions defined in a compact D $\subset \Re^n$ Let's ε a subset of F .

Definition 1: The subset is dense in F if:

$\forall f \in F, \exists \varepsilon > 0, \exists g \in \varepsilon$, with $\| f - g \| < \varepsilon$
($\|$. is$\|$ a norm defined on F , for example $\| f \| = \max_x \in D$ $|f(x)|$). In this case,ε represents an universal approximator of functions in F .[41-42].

Definition 2: If verifies the following properties:

- $\varepsilon \neq \phi$. ε contains the constant function f(x) = 1,$\forall_x$ $\in$ D
- For all scalars α and β and for all functions f and g $\in$ ε the function $\alpha f + \beta g \in \varepsilon$
- For all functions fand g belong in , function fg belongs in ε .

Then **ε** is dense in (F [41,42]).

Theorem: On the basis of StoneWeierstrass theorem [41-42],the set of fuzzy systems is dense in the set f piecewise continuous functions defined in the compact D [24].

Proof : Let's fand g two fuzzy systems and $\lambda \in \Re$

$$f(x) = \frac{\sum_{i=1}^{p} \mu_i(x)\, f_i}{\sum_{j=1}^{p} \mu_j(x)} \tag{67}$$

$$g(x) = \frac{\sum_{i=1}^{q} v_i(x)\, g_i}{\sum_{j=1}^{q} v_j(x)} \tag{68}$$

- The constant function can be represented by a fuzzy system, for which all rules have the same consequence (fi = 1):

$$f(x) = \frac{\sum_{i=1}^{p} \mu_i(x)\, f_i}{\sum_{j=1}^{p} \mu_j(x)}$$

$$= \frac{\sum_{i=1}^{p} \mu_i(x)}{\sum_{j=1}^{p} \mu_j(x)} = 1$$

- It is clear that the sum of two fuzzy systems is a fuzzy system:

$$f(x) + g(x) = \frac{\sum_{k=1}^{p} \mu_k(x)\, f_k}{\sum_{i=1}^{p} \mu_i(x)} + \frac{\sum_{l=1}^{q} v_l(x) g_l}{\sum_{j=1}^{q} v_j(x)}$$

$$= \frac{\sum_{k=1} \mu_k(x) v_l(x)(f_k + g_l)}{\sum_{i,j} \mu_i(x)\, v_j(x)}$$

$$= \frac{\sum_{k=1}^{p+q} \eta_r(x) h_r}{\sum_{s=1}^{p+q} \eta_s(x)} \tag{69}$$

where:

$$\eta_r(x) = \mu_k(x)\, v_l(x) \tag{70}$$

$$h_r = f_k + g_l \tag{71}$$

Moreover, the multiplication by a scalar of a fuzzy system is also a fuzzy system:

$$\lambda f(x) = \frac{\sum_{i=1}^{p} \mu_i(x)\, f_i\, \lambda_j}{\sum_{j=1}^{p} \mu_j(x)} \tag{72}$$

- The product of two fuzzy systems is a fuzzy system:

$$f(x) g(x) = \frac{\sum_{i,j} \mu_i(x)\, v_j(x)\, f_i\, g_j}{\sum_{k,l} \mu_k(x)\, v_l(x)}$$

$$= \frac{\sum_{r=1}^{p+q} \eta_r(x)\, h_r}{\sum_{s=1}^{p+q} \eta_s(x)} \tag{73}$$

where:

$$\eta_r(x) = \mu_i(x) v_j(x) \tag{74}$$

$$h_r = f_i\, g_j \tag{75}$$

Definition 3: The set ε verifies the property of the best approximation, if for all function $f \in$ Fand for all finite database $\{ (x_i, f(x_i)), x_i \in X \subset D\}$, it exists a function $g \in \varepsilon$ which verifies: $\forall\, x_i \in X,\ f(x_i) = g(x_i)$. That is to say the error between f and g on the database is equal to zero.

Theorem: The set of fuzzy systems verifies the property of the best approximation.

Proof: For simplicity, we will consider the monovariable case. The proof in the multivariable case can be easily deduced. Suppose that scalars x_i verify: $x_1 < x_2 ... < x_p$.

Let's consider the fuzzy system defined by triangular membership functions ¹iand the rule consequences $f_i(x) = f(x_i)$. Membership functions verify:

$$\begin{aligned} &\mu_i(x) = 0 \text{ for } x < x_i - 1 \\ &\mu_i(x) = 1 \\ &\mu_i(x) = 0 \quad \text{for } x \ge x_i + 1 \end{aligned} \tag{76}$$

$x_0$and x_p+1 defined the extremities of D Thus, we can deduce:

$$\mu_i(x_k) = \delta_{ik} = \begin{vmatrix} 1 & \text{if } i = k \\ 0 & \text{if } i \ne k \end{vmatrix} \tag{77}$$

where δ_{ik} is the Kronecker symbol.

The output of the fuzzy system is expressed by:

$$g(x) = \frac{\sum_{i=1}^{p} \mu_i(x) f_i(x)}{\sum_{j=1}^{p} \mu_j(x)} \tag{78}$$

The expression of $g(x_k)$ is:

$$g(x_k) = \frac{\sum_{i=1}^{p} \delta_{ik} f_i(x_k)}{\sum_{j=1}^{p} \delta_{jk}(x)} = f(x_k) \tag{79}$$

This yields to conclude Pthat fuzzy systems verify the property of the best approximation.

Fuzzy exact modelling of nonlinear functions

Case of two rules: Let us consider a scalar function f(x) defined on a compact $D \subset \Re^n$. f(x) is a bounded and a picewise continued function. Then, it exists two scalars α and β, in such away $\forall x \in D$:

$$\alpha \preceq f(x) \preceq \beta$$

(for example $\alpha = \min_D f(x)$ and $\beta = \max_D f(x)$). Consider the two following functions $g_1$and g_2 defined on D :

$$g_1(x) = \frac{\beta - f(x)}{\beta - \alpha}$$

$$g_2(x) = \frac{f(x) - \alpha}{\beta - \alpha}$$

It is clear that: $\forall x \in D$, $g_1(x) + g_2(x) = 1$ and:

$$f(x) = \frac{\alpha\, g_1(x) + \beta\, g_2(x)}{g_1(x) + g_2(x)}$$

Then, we can conclude that f(x) is represented by a fuzzy system with two rules[43], for which fuzzy basis functions are $g_1$and g_2. Fuzzy rules are defined by: is clear that

- If (x is g_1) Then f(x) = α
- If (x is g_2) Then f(x) = β

It is remarkable that for n = 1, if f is monoton on D, then $g_1$and g_2 are also monoton functions.

Case of multiple rule: Consider the case n=1 and suppose that f is a picewisemonoton function on . It is monoton D on each of the following intervals: I1 = [a c1[, I_2 = [c_1 c_2[,... I_n+1 = [c_n b[. Thus, c_1, c_2 ... c_n are then extremums of f. Let's note: °i = f (c_i). It is obvious that function f can be represented by a fuzzy system with n + 2 rules.

For such proposition, let's define the following fuzzy subsets described by: C^i: near ci, for i = 0 to n + 1, with c_0 = a and c n +1 = b.

Membership functions of these fuzzy subsets are expressed by:

$$\mu c_0(x) = \begin{cases} \dfrac{\gamma_1 - f(x)}{\gamma_1 - \gamma_0} & \text{for } x \in [\alpha, c_1[\\ 0 & \text{else where} \end{cases}$$

$$\mu c_{n+1}(x) = \begin{cases} \dfrac{f(x) - \gamma_n}{\gamma_{n+1} - \gamma_n} & \text{for } x \in [c_n, b[\\ 0 & \text{else where} \end{cases}$$

and for i = 1 to n:

$$\mu c_i(x) = \begin{cases} \dfrac{f(x) - \gamma_{i-1}}{\gamma_i - \gamma_{i-1}} & \text{for } x \in\ ]c_i -1, [\\ \dfrac{\gamma_{i+1} - f(x)}{\gamma_{i+1} - \gamma_i} & \text{for } x \in [c_i, c_i+1[\\ 0 & \text{else where} \end{cases}$$

with $\gamma_0 = f(a)$ and $\gamma_n+1 = f(b)$.

Then, let's define the following fuzzy rules:

If (x is C_i) Then f(x) = γ_i

Thus, it is well obvious that $\forall x \in [c_i ; c_i+_1]$:

$\mu c_i(x) + \mu c_{i+1}(x) = 1$

and for $j \neq i$ and $j \neq i + 1$:

$\mu c_j(x) = 0$

Then,$\forall x \in D$:

$$\sum_{i=0}^{n+1} \mu c_i(x) = 1$$

Finally, itis clear that $\forall x \in D$:

$$f(x)\ \frac{\sum_{i=0}^{n+1} \mu c_i(x)\gamma_i}{\sum_{j=0}^{n+1} \mu c_j(x)} = \sum_{i=0}^{n+1} \mu c_i(x)\gamma_i$$

which represents a zero order Sugeno Fuzzy Sys

Example: *Consider the function h defined on D[- 2π ; 2π] by:*

$$h(x) = \left(\cos x + \frac{1}{2\pi}\sin x\right) e^{-\frac{x}{2\pi}} \qquad (80)$$

Function h(x) possesses three extremums. Fig ure 4 shows the evolution of the membership func tions of subsets Ci, for i=1 to 5. Fuzzy rules are given in Table 2.

Remarks: Figure 4.a Shows the evolution of membership functions of subsets Ciand their approxima tion by gaussian membership functions. Figure 4.b represents the evolution of h(x), its approxi mator by a fuzzy system, with gaussian member ship functions and the fuzzy ap P P proximator non normalized system with gaussian membership functions ($\Sigma\ \mu_i\gamma_i$). These approxi P mations can be improved by the use of any opti misation procedure[44,45].

CONTROL BY FUZZY SYSTEMS

In the following, we propose to approximatenonlinear systems by Sugeno fuzzy systems. We will consider the case of nonlinear systems described by the following equation:

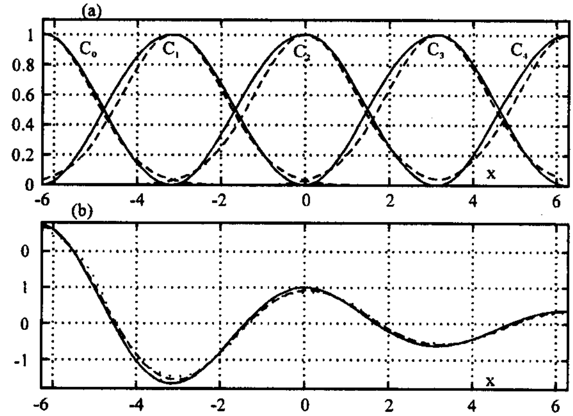

Fig. 4: Exect fuzzy modeling and fuzzy gaussian approximator of the function h (x)= (cos x +1/2π son x) e x/2π. a : continued line fuzzy subset c_i b: continued line

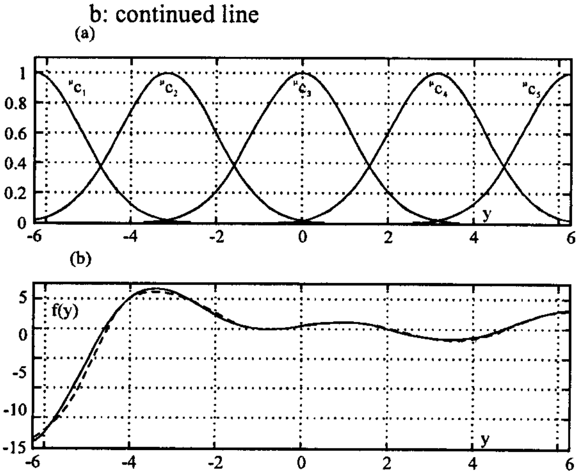

Fig. 5: a: Quantification of variable y. b: approximation of function f(y) by a first order Sugeno fuzzy system. Continued line: f(y). Interrupted line: fuzzy approximator of f(y)

Table 2: fuzzy rule defining function h (x)

Rule	x	h (x)
R_1	C_1:near $c_1=2\pi$	$\gamma_1=e^1$
R_2	C_2:near $c_2=2\pi$	$\gamma_2=-e^{0.5}$
R_3	C_3:near $c_3=0$	$\gamma_2=1$
R_4	C_4:near $c_4=\pi$	$\gamma_4=-c^{-0.5}$
R_5	C_5:near $_52=\pi$.	$\gamma_5=e^{-1}$

$$\dot{x} = f(x) + g(x)u \tag{81}$$

Function f(x) will be modelled by a first order Sugeno fuzzy system whose rules are:

$$\text{if } (x \text{ is } \Omega\, f_i) \text{ Then } f(x) \sim A_i x$$

However, function g(x) will be modelled by azero order Sugeno fuzzy system whose rules are:

$$\text{if } (x \text{ is } \Omega_{gj}) \text{ Then } g(x) \sim B_j$$

Consequently, the nonlinear system (81) can be modelled by the following fuzzy system, whose rules are:

$$\text{if } (x \text{ is } \Omega_r) \text{ Then} \dot{x} \sim A_r x + B_r u$$

where $\Omega_r = \Omega_{fi} \cap \Omega_{gj}$.

It is evident that each rule gives a local linear system for which all systematic method for conrol of linear system can be applied.

For the case of optimal control of nonlinear sys tem, the optimal solution will be expressed by an overlapping of the local optimal control (for each rule). For the case of sliding mode control, equivalent control will be expressed by an overlapping of lo cal equivalent control for each local linear system yielded from each rule[11,31,34,46-50]. This work can be easily extended to any type of control. For example, for the case of the pole placement control, a procedure of pole placement controll will be applied to each local system.

APPLICATION

Let us consider a nonlinear system described by the following nonlinear differential equation:

$$\ddot{y} = y\left(\cos y + \frac{1}{2\pi}\sin y\right)e^{-\frac{y}{2\pi}} + u \tag{82}$$

This equation can be written in the following con densed form: $\ddot{}$ y = f(y) + u, with:

$$f(y) = \left(\cos y + \frac{1}{2\pi}\sin y\right)e^{-\frac{y}{2\pi}} \tag{83}$$

Figure 5a Table 3 shows the quantification of the space y in 5 fuzzy subsets C_i defined by gaussian membership functions μC_i for i = 1 to 5. Fuzzy rules are described in Table 3. Figure 5 b represents the fuzzy approximation of the function f(y). Itis to note that the obtained error between funtion f(y)and its fuzzy estimates is appreciable. It can be improved by the use of any optimisation pro cedure.

In order to test the robustness of the pre sented approaches, we will consider the case where parameters of function f(y) are illdefined (per turbed system):

$$f(y) = y\left(\alpha \cos y + \frac{b}{2\pi}\sin y\right)e^{-\frac{y}{2\pi}} \tag{84}$$

with a variation of +50% on the parameter a (a = 1:5) and a variation of - 50% on the parameter b (b = 0:5).

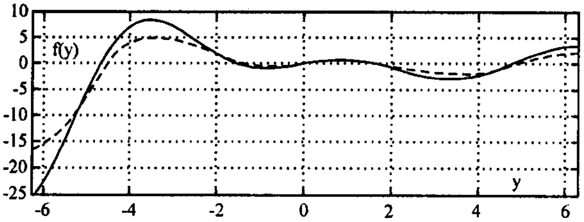

Fig. 6: First order Sugeno fuzzy approximator of function f (y). Continued line: perurbed caes. Inerrupted:line fuzzy approximator without perturbations

Table 4: Fuzzy rules describing the system

Rule	y	state equation
R_1	C_1:near c_1=-2π	x = A_1x+Bu
R_2	C_2:near c_2=-π	x = A_2+xBu
R_3	C_3:near c_3= 0	x = A_3+xBu
R_4	C_4:near c_4=π	x = A_4+xBu
R_5	C_5:near c_5=2π	x = A_5+xBu

The perturbed function f(y) is showes in Figure 6. In this figure, are drawn the variations of f(y) (perturbed case)and its fuzzy estimates. It is well obvious that there are an im portant error between the fuzzy estimationand the perturbed case.

Remarks: f(y) is the function described in Eq .83. Using above description, one can easily write:

$$f(y) \approx \sum_i \lambda_i(y)\gamma_i y, (y \in [-2\pi, 2\pi])$$

Thus, it is clear that system (82) can be ex pressed by a fuzzy dynamic system whose rules are described in Table 4, where x is the state vectorand matrices A_i are given by:

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, A_i \begin{pmatrix} 0 & 1 \\ \gamma_i & 0 \end{pmatrix}$$

with: $\gamma_1 = e^1$, $\gamma_2 = -e^{0.5}$, $\gamma_3 = 1$, $\gamma_4 = -e^{-0.5}$ and$\gamma_5 = e^{-1}$.

The aim is to determine the adequate control to reach the desired state position: xd = 0.

Optimal control: The aim is to determine optimal control strat egy which minimizes the following criterion:

$$J = \int_0^{\infty} \left(qy^2 + q'\dot{y}^2 + u^2 \right) dt \quad (85)$$

with:

$$Q = \begin{bmatrix} q & 0 \\ 0 & q' \end{bmatrix}, R=1 \quad (86)$$

Optimal control will be expressed by a fuzzy approximator. For the local linear system i (x is near ci), the optimal solution can be expressed by: $u = -K_i x$, with:

$$K_i = R^{-1} B^T P_i \quad (87)$$

and Pi is the solution of the nonlinear Riccati equation:

$$P_i A_i + A_i^T P_i - P_i BR^{-1} B^T P_i + Q = 0 \quad (88)$$

This gives:

$$P_i = \begin{bmatrix} \alpha_i & b_i \\ b_i & c_i \end{bmatrix} \quad (89)$$

and:

$$K_i = \begin{bmatrix} b_i & c_i \end{bmatrix} \quad (90)$$

where:

$$\alpha_i = \sqrt{q+\gamma_i^2}\sqrt{q' + 2\gamma_i + 2\sqrt{q+\gamma_i^2}}$$
$$b_i = \gamma_i + \sqrt{q+\gamma_i^2} \quad (91)$$
$$c_i = \sqrt{q' + 2\gamma_i + 2\sqrt{q+\gamma_i^2}}$$

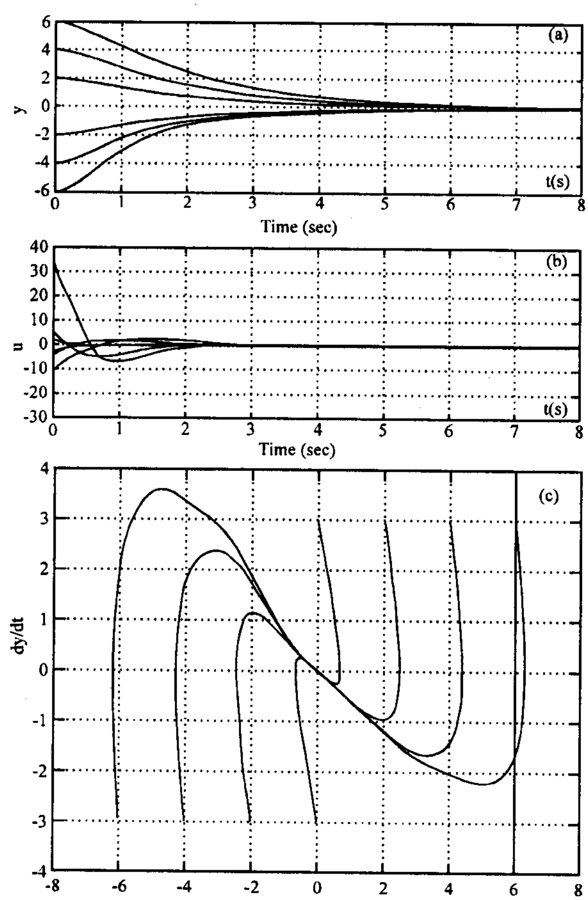

Fig. 7: Curves of y(t), u(t) y and ÿ y(y). Optimal control is expressed by a fuzzy system (nonperturbed case)

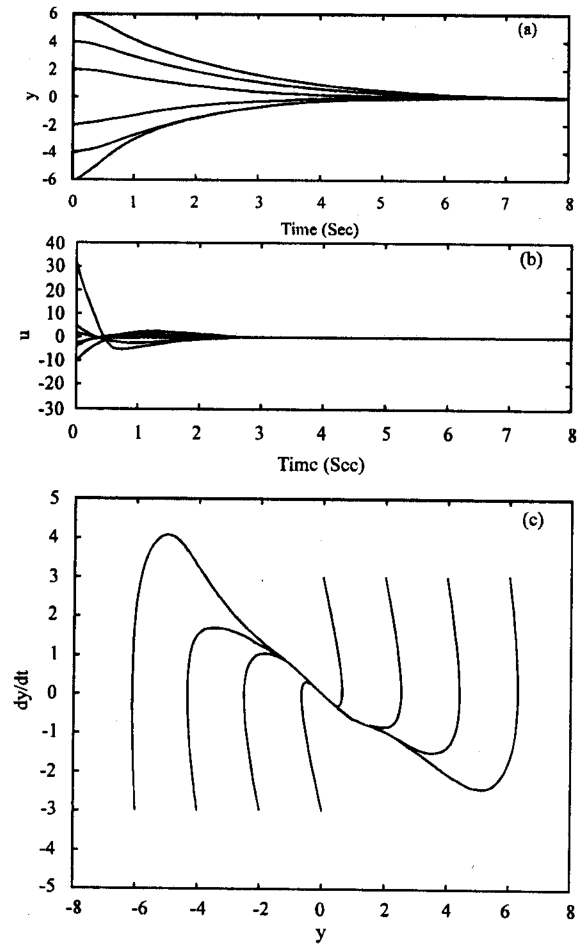

Fig. 8: Curves of y(t), u(t)and ÿ y(y). Optimal control is expressed by a fuzzy system (perturbed case)

Table 5: Fizzy rule describeing optimal control by a local law u= K_ix

Rule	y	K_i
R_1	C_1:near c_1=-2π	[5.78 4.42]
R_2	C_2:near c_2= π	[0.52 3.01]
R_3	C_3:near c_3= 0	[2.73 3.67]
R_4	C_4:near c_4=π	[0.93 3.14]
R_5	C_5:near c_5=2π	[1.83 3.41]

Scalars q and qi will be chosen in such awaythat the closed loop system have a non dampedpulsation larger than 2 rad sec and the system isaperiodic. Thus:

$$\sqrt{q+\gamma_i^2} \geq 2$$
$$q'+2\gamma_i - 2\sqrt{q+\gamma_i^2} \geq 0 \quad (92)$$

for i = 1 to 5. This leads to the following choice q = 2 and q = 8. Fuzzy rules are presented in

It is to be noted that the obtained control can be expressed by:

$$u = \sum_{i=1}^{5} \lambda_i K_i x \quad (93)$$

In this case, the dynamic of the system in the closed loop control is described by the following state equation:

$$\ddot{y} = f(x) + \sum_{i=1}^{5} \lambda_i K_i x$$
$$= \sum_{i=1}^{5} \lambda_j \gamma_j y + \sum_{i=1}^{5} \lambda_i K_i x + o(\Delta f)$$
$$= \sum_{j=1}^{5} \lambda_i (\gamma_i y + K_i x) + o(\Delta f) \quad (94)$$

This yields:

$$\dot{x} = \sum_{i=1}^{5} \lambda_i F_i x + o(\Delta f) \quad (95)$$

with:

$$F_i = A_i - BK_i \quad (96)$$

It is clear that the system can be represented by a fuzzy dynamic system whose rules are given in Table 6. In order to test the stability of the system, con sider the definite positive matrix M:

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad (97)$$

It is easy to verify that matrices Qi = (MF_i+; F_i^T are definite positive. Let's consider the following Lyapunov definite positive function V (x) = x T Mx, for which its derivatives with respect to time is expressed by:

$$\frac{d}{dt}\left(x^T M x\right) = \dot{x}^T M x + x^T M \dot{x}$$
$$= \sum \lambda_i \, x^T \left(F_i^T M + MF_i\right) x + o(\Delta f)$$
$$= -\sum \lambda_i \, x^T Q_i \, x + o(\Delta f) < 0 \quad (98)$$

Table 6: Fuzzy rules defining the dynamic of the closed loop system

Rule	y	State Equation
R_1	C_1:near c_1=-2π	x =F_1x
R_2	C_2:near c_2= -2π	x =F_2x
R_3	C_3:near c_3= -0	x =F_3x
R_4	C_4:near c_4=- π	x =F_4x
R_5	C_5:near c_5=-2π	x =F_5x

Note that in this case, all control matrices B are constant. This permits to affirm that if there ex ists a matrix M definite positive in such away that matrices

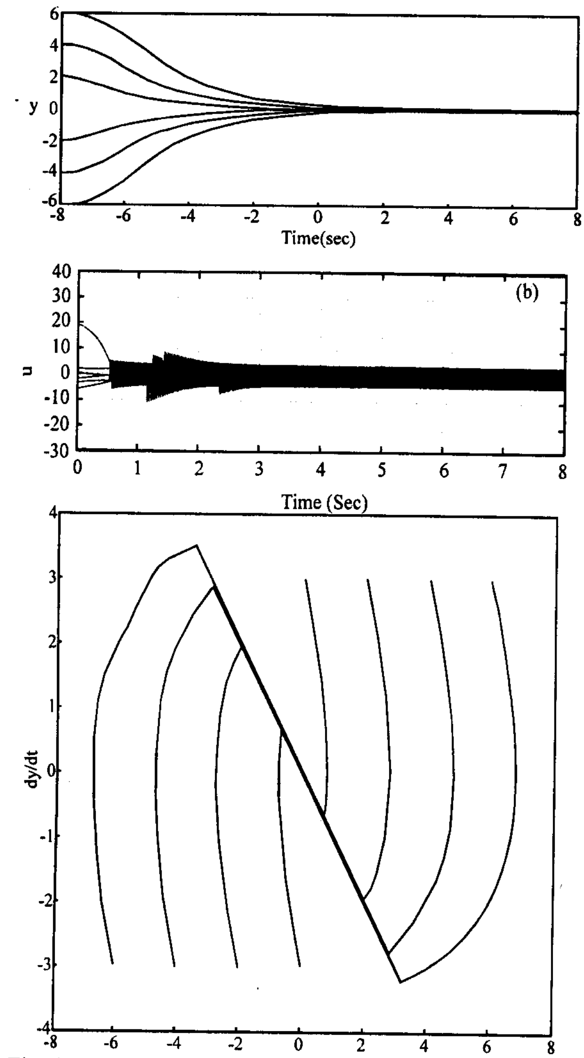

Fig. 9: Stimultion results: Curves of y (t) and u (t) and phase trajectories. Equivalant control is replaced by a fuzzy system (non- perturbed case)

Qi = (MFi + FT M) are also positive definite, then the system is stable.

Simulation results are shows in Fig. 7. it presents the evolution of suboptimal trajectories, for different initial conditions. Fig. 8 shows the corresponding results for the perturbed case.

Sliding mode control: The sliding surface is described by:

$$s(x) = K_x \tag{99}$$

$$K = [\lambda \quad 1] \tag{100}$$

yielding the following characteristic equation in the sliding mode, with , = 1:

Table 7: Fuzzy rules expressing the equivalant control $u_{eq} = -L_i x$

Rule	y	L_i
R_1	C_1:near $c_1 = -2\pi$	$[\gamma_1]1$
R_2	C_2:near $c_2 = -\pi$	$[\gamma_2]1$
R_3	C_3:near $c_3 = 0$	$[\gamma_3]1$
R_4	C_4:near $c_4 = -\pi$	$[\gamma_4]1$
R_5	C_5:near $c_5 = 2\pi$	$[\gamma_5]1$

$$p + \lambda = 0 \tag{101}$$

In the following, we propose to express the equivalent control ueq by a fuzzy system. The basic idea is to use the fuzzy approximator of function f(y). The rule Ri suppose that the system is locally linear. This leads to express the local equivalent control using Eq. 37. This results the rules Ri: If y is near ci, Then ueq = ¡ Lix, where Li = [°i ,] (Table 7).

It is obvious that the fuzzy approximation ofthe equivalent control ueq can be expressed by: ueq = P 5 ,iLix. In fact, the exact expressionof the equivalent control is given in equation (36), by replacing g(x) by Band S_x by K:

$$u_{eq} = -(KB)^{-1} Kf(x) \tag{102}$$

Replacing f(x) by its fuzzy approximation, we obtain:

$$u_{eq} \approx -(KB)^{-1} K \sum_{i=1}^{5} \lambda_i A_i x$$

$$\approx -\sum_{i=1}^{5} \lambda_i (KB)^{-1} KA_i x \approx -\sum_{i=1}^{5} \lambda_i L_i x \tag{103}$$

This shows that the fuzzy approximation of ueq is a good approximation of the expression of equiv alent control ueq. It is to be noted that in this case, we have g(x) = B a constant matrix. In the case where g(x) is not constant, we can replace it by a zero order Sugeno fuzzy approximator, yielding to write:

$$f(x) + g(x)u \approx \sum_i \lambda_i A_i x + \sum_j \lambda'_j B_j u \tag{104}$$

Or, we have: $\sum_i \lambda_i = \sum_j \lambda'_j$ Then

$$f(x) + g(x)u \approx \sum_r \eta_r (F_r x + G_r u) \tag{105}$$

with: ηr = ,i,0 , Fr = Ai and Gr = Bj. This result leads to deduce the following fuzzy approximator of the equivalent control ueq, for s(x) = Kx:

$$u_{eq} \approx -\sum_r \eta_r (KB_r)^{-1} KA_r x$$

$$\approx -\sum_{i,j} \lambda_i \lambda'_j (KB_j)^{-1} KA_i x$$

$$\approx -\sum_{i,j} \lambda'_j \left(KB_j\right)^{-1} K \sum_i \lambda_i A_i x$$

$$\approx -\left[Kg(x)\right]^{-1} Kf(x) \tag{106}$$

which is a suitable result.

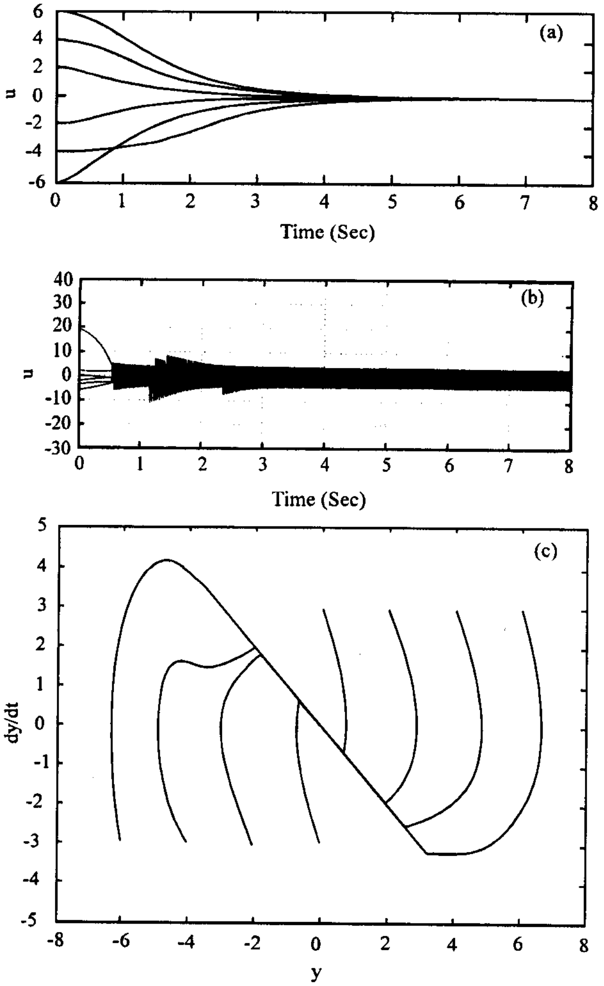

Fig. 10: Simulation results: curves of y (t), u (t) and phase trajectories. Equivelent control is replaced by a fuzzy system (perturbed case)

Simulation results are shows in Fig. 9. It represents the evolution of y(t)and u(t) for different initial conditions on y. Evolution of the obtained trajectories in the state space (y; ÿ y) is also presented. This figure shows that when the system reach the sliding surface, it stays on s(x) = 0 and converge to the desired position.

In order to test the robustness of the system according to parameter variations, function f(y) is perturbed as indicated in equation (84). Simulation results are presented in figure 10, which represents the evolution of y(t)and u(t) for different initial conditionsand the evolution of the phase trajectories. Obtained results show that the system is practically unaffected by the considered perturbations. However, high oscillations on the control are observed (figures 9and 10). In order to overcome this problem, the expression ¢ui = u0i sign si is replaced by a fuzzy system with three rules: [11,30,31,33]:

R_1 : if (s_i is Negetif) Then $u_i = -u_{0i}$

R_2 : if (s_i is Zero) Then $u_i = 0$

R_3 : if (s_i is positive) Then $u_i = u_{0i}$

Simulation results are presented in figures 11 and 12, in the nonperturbedand in the perturbed case,

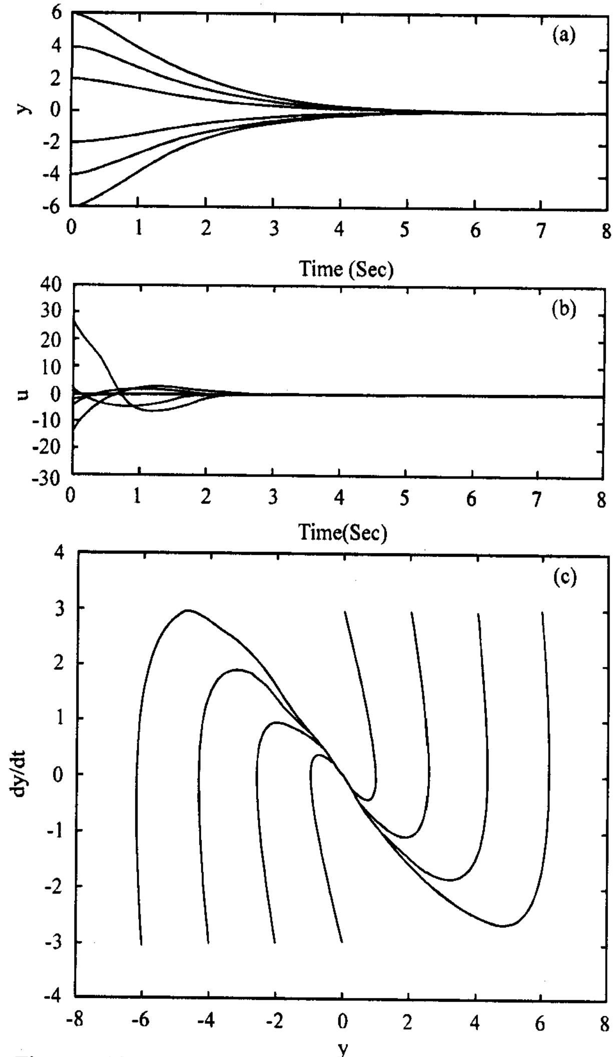

Fig. 11: Simulation results: curves of y(t)and u(t)and phase trajectories. Equivalent control is replaced by a fuzzy systemand the function sign is replaced by a fuzzy system. (nonperturbed case)

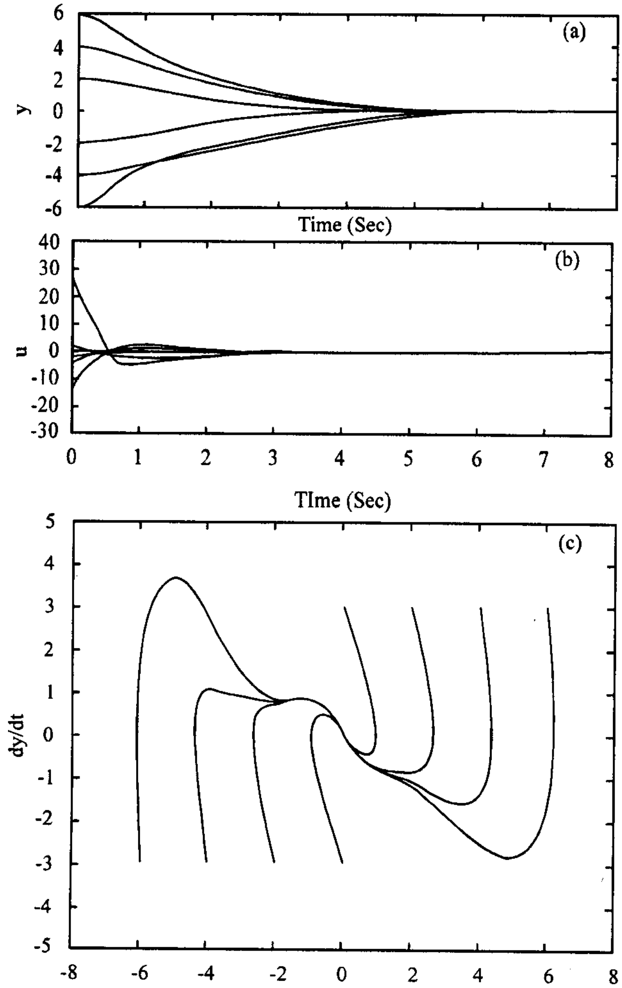

Fig. 12: Simulation results: curves of y(t)and u(t)and phase trajectories. Equiv alent control is replaced by a fuzzy systemand the function sign is replaced by a fuzzy system. (nonperturbed case).

respectively. These figures show that the chattering phenomenon was suppressed.

CONCLUSIONS

In this work, we detailed two type of control: the optimal controland the sliding mode con trol. For the case of nonlinear systems, these con trol law becomes difficult if not impossible to ex press analytically. For these reasons, these control strategies are modelled by fuzzy systems. Only Sugeno fuzzy systems are considered in this paper. In fact, we belive that Sugeno fuzzy systems are the well adapted ones for the the control systems. Then, fundamentalsand generalities on fuzzy sys temsand fuzzy approximators are presented. Af ter that, a modeling by fuzzy approximators of nonlinear boundedand piecewise continuousand monoton functions is then presented. As an illustration, a nonlinear application is considered, for which optimal control law is ex pressedand the sliding mode control is designed. These two control types are modelled by fuzzy approximators. In order to study the robust ness of the implemented control strategies, pertur bations yielded by parameter variations are con siderd. Simulation results show that the perfor mances of the obtained solutions are practically unaffected by these perturbations.

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