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Studies on Some Fixed Point Theorems in Terms of Metric and Banach Spaces

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Abstract: In this study we have studied the fixed-point theorem for two, three and Multi-valued Mappings into itself on a metric and Banach spaces.

Key words: Fixed point, metric spaces, banach spaces, complete metric, mapping

INTRODUCTION

In the first section of this paper we have given a few common fixed-point results for two, three and multi-valued self mapping of a complete metric space, which generalize the results of Ray[1] and Taskovic[2].

In section II, we discuss some common fixed-point theorems of three maps f, g and h from a complete metric space (X,d) into itself are proved. The maps f, g and h satisfy:

$$[d(fx,gy)]^2 \le \phi[d(hx,fx)d(hy,gy),d(hx,gy)d(hy,fx),d(hx,fx)d(hx,gy), d(hy,fx)d(hy,gy)]$$

for all x, y∈ X and fh = hf, gh = hg, f (x) ⊂ h (x) and g(x) ⊂ h (x) where ϕ is an upper semi- continuous function from R^{+4} to R^+, satisfying:

$\phi(t, t, a_1t, a_2t) < t$ for any $t > 0$ and $a_i \in \{0, 1, 2\}$ such that $a_1 + a_2 = 2$.

Devi-Prasad[3] and Devi- Prasad[4] have considered such theorems for mappings satisfying a different functional inequality.

In the last section, we modify slightly the proofs of kirk[5] and kirk[6]. We extend result for multi-valued mappings, generalizing also fixed-point theorem 3.1 of Samanta[6] and Samanta[8].

SECTION 1

Taskovic[2] proved a fixed-point theorem on a metric space (X, d) with a mapping T, which is not necessarily continuous and satisfy a condition of the type,

$$ad(Tx, Ty) + bd(x, Tx) + cd(y, Ty) - \min\{d(x, Ty), (y, Tx)\} \le qd(x, y). \text{ for all } x, y \in X.$$

In what follows we give our first result for two mappings f and g generalizing the above result.

Theorem 1.1: Let f and g be two self mappings of a complete metric space (X, d) satisfying:

$$ad(fx, gy) + bd(x, fx) + cd(y, gy) - \min\{d(x, gy), d(y, fx)\} \le pd(x, y).$$

for all x, y ∈ X where a, b, c ≥ 0, q > O with a > q+1 and a+c > 0. Then f and g have unique common fixed point.

Next we have the following generalize theorem of Ray[1] for two commuting self mapping on a metric space.

Theorem 1.2: Let f and g be two mappings of a complete metric space (X, d) into itself with f continuous. Let f and g commute with each other and g (x)⊂ f (x).

Let g satisfy

$$d(g(x), g(y)) \le \alpha(d(f(x), f(y))d(f(x)f(y)) + \beta(d(f(x), f(y))[d(f(x)g(x)) \ d(f(y), g(y))]\ \gamma(d(f(x), f(y))[d(f(x)g(y)) + d(f(y), g(x))] \quad (1)$$

for each f (x) ≠ f (y), where α, β, γ are monotonically decreasing functions from (0, ∞) into [0,1) with $\alpha(t) + 2\beta(t) + 2\gamma(t) < 1$, $t \in (0, \infty)$ then f and g have a unique common fixed point in X.

The proof of this theorem goes in a similar fashion as that of Tanmoy Som[9], Some fixed point theorem on metric and Banach spaces Indian J. pure appl. Math., 16(6); pp: 575 –585, June 1985. So we omit the proof.

Remark 1.1: If f happens to be an identity mapping, we get theorem 1.2 of Ray[1] without the diagram of g being closed. Further if β = γ = 0, we have result of Rokotch[10]

Another generalization of theorem 1.2 of Ray[1] goes as follows:

Theorem 1.3: Let f and g be two self-mappings of a complete metric space (X, d) satisfying

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$$d(fx,gy) \le \alpha(d(x,y))d(x,y) + \beta(d(x,y))[d(x,fx) + d(y,gy)] + \gamma(d(x,y))[d(x,gy) + d(y,fx)]$$

for all $x \neq y \in X$, where α, β, γ are monotonically decreasing functions from $(0, \infty)$ into $[0, 1)$ with $\alpha(t) + 2\beta(t) + 2\gamma(t) < 1$, $t \in (0, \infty)$ Then f and g have a unique common fixed point in X.

SECTION 2

Let ϕ is an upper semi-continuous function from R^{+4} to R^+ which is non-decreasing in each co-ordinate variable and satisfy the condition:

$$\phi(t,t,a_1t.a_2t) \text{ for any } t > 0 \tag{2}$$

and $a_i \in \{0, 1, 2\}$ such that $a_1 + a_2 = 2$.following is the main theorem.

Theorem 2.1: Let f, g and h be three self-mappings of a complete metric space (X, d) which satisfy:

$$fh = hf,\ gh = hg,\ f(x) \subset h(x) \text{ and } g(x) \subset h(x) \tag{3}$$

$$[d(fx,gy)]^2 \le \phi[d(hx,fy)d(hy,gy),d(hx,gy)\ d(hy,fx),d(hx,fx)d(hx,gy),d(hy,fx)d(hy,gy)] \tag{4}$$

for all $x, y \in X$. Further, Let h be continuous, then f, g and h have a unique common fixed point in X. Already Devi- prasad[3] proved this theorem in his paper fixed point theorems of three mappings with a new functional Inequality. Indian J. pure appl. math. 16 (10) (October 1985), p. 1073-1077. So we omit the proof.

The following theorem Devi- Prasad[4] follows as a corollary to this theorem by taking
h = I the identity map.

Corollary 2.1: Let f and g be mappings from a complete metric space (X, d) into itself and satisfy:

$$[d(fx,gy)]^2 \le \phi\,[d(x,fx)d(y,gy),d(x,gy)d(x,gy),\ d(y,fx)d(x,gy),d(y,fx)d(y,gy)]$$

for all $x, y \in X$, where ϕ is a defined in (ii), then f and g have a unique common fixed point. The proofs of the following theorems follow from theorem 2.1.

Theorem 2.2: Let f, g and h be three self-mappings of a complete metric space (X, d) such that,

$$fh = hf,\ gh = hg,\ f(x) \subset h^r(x) \text{ and } g(x) \subset h^r(x) \tag{5}$$

There exist positive integers p, q and r satisfying

$$d[(f^px,g^qy)]^2 = \phi\begin{bmatrix} d(h^rx,f^px)d(h^ry,g^qy),d(h^rx,g^qy)d(h^ry,f^px), \\ d(h^rx,f^px)d(h^rx,g^qy),d(h^ry,f^px)d(h^ry,g^qy) \end{bmatrix} \tag{6}$$

for all $x, y \in X$.

and if h^r is continuous, that f, g and h have a common fixed point.

Theorem 2.3: Let $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be sequences of self-mappings of a complete metric space (X, d) such that $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ converge uniformly to self-mappings f, g and h on X with h continuous. Suppose that for each $n \ge 1$, x_n is a common fixed point of f_n and h_n and y_n is a common fixed point of g_n and h_n. Further let f, g and h satisfy conditions(iii) and (iv).
If x_0 is the common fixed point of f, g and h and $\sup d(x_n x_0) < \infty$ and $\sup (y_n, x_0) < \infty$. Then
$x_n \to x_0$ and $y_n \to y_0$.

Theorem 2.4: Let $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be sequences of self-mappings of a complete metric space (X, d) such that h_n is continuous for each n and f_n, g_n and h_n satisfy condition (iii) and (iv) for each $n \ge 1$, if f, g and h are uniform limits of $\{f_n\}$, $\{g_n\}$and $\{h_n\}$respectively, then, f, g and h also satisfy condition (iii) and (iv). Also $\{x_n\}$ be sequence of unique common fixed points of f_n, g_n and h_n converges to the unique common fixed point x_0 of f, g and h, whenever $\sup d(x_n, x_0) < \infty$.

SECTION 3

For the purpose of our theorem we shall take S to be a class of subsets of M which is count ably compact, stable under intersections, normal and contains the closed balls of M.

Theorem 3.1: If $T: M \to F(x)$ be a mapping such that,

$$T(x) \cap M \neq \phi \quad \forall x \in M \tag{7}$$

$$\text{for all } x \in M,\ T(x) \cap S = \{T(x) \cap D: D \in S\} \tag{8}$$

is a compact class of which each non- empty member is a compact subset of X.
For any $G \in S$ satisfying $T(\xi) \cap G \neq \phi$, for all $\xi \in G$,

$$H(T(x) \cap G, T(y) \cap G) \le d(x,y), \forall x,y \in G \tag{9}$$

Then T has a fixed point in M.

Proof: Modifying slightly the proof of Lemma of kirk [6], we show that for each $\varepsilon > 0$, there exists a non-empty set $M(\varepsilon) \in S$ such that $T(x) \cap M(\varepsilon) \neq \phi$ for all $x \in M(\varepsilon)$ and for which $\delta(M(\varepsilon)) \le (h(M)+\varepsilon)\delta(M)$for this take $M(\varepsilon) = M$ if $\delta(M)$ Otherwise, construct $M(\varepsilon)$ as follows:
Let $\rho = (h(M) + \varepsilon)\delta(M)$.
By the definition of h,The set $C = \{z \in M: M \subset B(z,p)\}$ is non-empty.

Let F ={D∈S; C ⊂ D, ✰T (x)∩D ≠ϕ,∀x∈D}. Order the family the F by set inclusion relation.

Let $\tau\{D_i\}_{i\in\Delta}$ be a decreasing chain in F. Let $D_0 = \bigcap_{i\in\Delta} D_i$ then D_0 ∈ S and C⊂ D_0 . Further, since $\tau\{D_i\}_{i\in\Delta}$ decreasing, it follows that for each x ∈ D_0, the family $\{T(x) I D_i\}_{i\in\Delta}$ has finite intersection property. So by hypothesis (viii)

$$T(x)\cap\left(\bigcap_{i\in\Delta} D_i\right)\neq\phi,$$

That is T (x) ∩ D_0 ≠ ϕ, thus every decreasing chain in F has a lower bound. Therefore, by Zorn's lemma, F has a minimal element L (say). Let A = C ∪ T (L), where.

$$T(L)=\bigcup_{x\in\Delta}(T(x)\cap L)$$

Then T (L)⊂ L. So Cov (A) = ∩{D∈S; A⊂ D}⊂ L. Also for x ∈ cov (A), T (x) ∩ L ≠ϕ
and T (x) ∩ L ⊂T (L). So, T (x) ∩A ≠ ϕ. Hence T (x) ∩ cov (A) ≠ ϕ. Thus cov (A) ∈ F.

Since L is a minimal member of F and cov (A)⊂ L, if follows that cov (A)=L. Let M(ε) ={x ∈ L;L ⊂ B(x ,ρ)}Then M(ε) ≠ϕ. Since M(ε)⊂ C. Let x ∈ M (є). Then T (x)∩ L ≠ϕ. Take $x'\in T(x)\cap L$. Let$0 = L\cap B(x',\rho)$. Then C ⊂ 0.Next take η∈ 0, then η∈L and ✰ Now, by (ix).

$$H(T(x)\cap L,T(\eta)\cap L)\leq d(x,\eta)\leq\rho\quad(\because x\in M(\varepsilon),\eta\in L)$$

Since T (x) ∩ L and T (η) ∩L are non-empty compact sets and x′∈ T(x) I L, there exists η′ ∈ T (η) I L, Such that,

$$d(x',\eta')\leq H(T(x)\cap L,T(\eta)\cap L)\leq\rho.$$

So η′∈ 0, Hence T (η)∩0≠ϕ. Thus 0∈F and 0⊂ L, since L is a minimal element in F, it follows that 0 = L. So d(x′, y′) ≤ ρ ∀ y ∈ L. i.e. B (x′ , ρ) ⊂ L . So, x′ ∈ M (ε)hich implies T (x) ∩ M (є) ≠ ϕ. Now let m = {D∈S; D ≠ ϕ, T (x)∩D ≠ ϕ, ∀ x ∈D} and for each D∈ m, let

$$\delta_0(D)=\inf\{\delta(F);F\in m,F\subset D\}.$$

From now on, the proof runs similar as in Kirk[6] and hence it is omitted.

Remark 3.1: The necessity of the condition (ix) of the theorem has been studied by Samanta (Samanta,).

Corollary 3.1(Theorem 1 of kirk[6]): Let (M, d) be a nonempty bounded metric space and suppose M contains a class S of subsets which is count ably compact, Stable under arbitrary intersections and normal. Suppose further that S contains the closed balls of M. Then every non-expansive mapping T of M into itself has a fixed point.

Corollary 3.2(Theorem 3.1 of Samanta[7]): Let X be a reflexive Banach space and K be a bounded closed convex subset of X and if Ψ: K →2^x is a mapping such that,

$$\Psi(x)\cap K\neq\phi,\ \forall x\in K \tag{10}$$

for any closed convex subset G of K satisfying ι(ξ) ∩G ≠ ϕ, ∀ ξ∈G,

$$H(\psi(x)\cap G,\psi(y)\cap G)\leq\|x-y\|\ \text{Whenever } x, y\ (\neq x)\in G \tag{11}$$

Then ι has a fixed point.

CONCLUSIONS

From the foregoing discussion we see that ours is the generalizations of the fixed point theorems of two, three and Multi-valued mappings from complete metric and Banach spaces into itself and while concluding this paper we modify slightly the proof of lemma of kirk (Kirk, 1981). The mapping scheme of the fixed-point theorem is vast no doubt. My study is on the beautiful aspect. We do hope to work again.

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