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Free ΓM-Modules

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Abstract: The characterizations of free ΓM-modules are developed. The cardinality of the basis of the free ΓM-modules is studied. At last we have studied the invariant rank property of free ΓM-modules.

Key words: Group, ring, modules, invariant property, mapping, liner property and rank

INTRODUCTION

Gamma ring: Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M\times\Gamma\times M\to M$ (sending (x, α, y) into $x\alpha y$) such that

- $(x+y)\alpha z = x\alpha z+y\alpha z$
  $x(\alpha+\beta)z = x\alpha z+x\beta z$
  $x\alpha(y+z) = x\alpha y+x\alpha z$
- $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z\in M$ and $\alpha, \beta\in\Gamma$. Then M is called a Γ-ring. This definition is due to Barnes[1].

Ideal of Γ-rings: A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c\in M, \alpha\in\Gamma, a\in A\}$ $(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.

If A and B are both left (respectively right or two sided) ideals of M, then $A+B = \{a+b \mid a\in A, b\in B\}$ is clearly a left (respectively right or two sided) ideal, called the sum of A and B. We can say every finite sum of left (respectively right or two sided) ideal of a Γ-ring is also a left (respectively right or two sided) ideal.

It is clear that the intersection of any number of left (respectively right or two sided) ideal of M is also a left (respectively right or two sided) ideal of M.

If A is a left ideal of M, B is a right ideal of M and S is any non empty subset of M, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i\gamma_i s_i \mid a_i\in A, \gamma_i\in\Gamma, s_i\in S$, n is a positive integer$\}$ is a left ideal of M and SΓB is a right ideal of M. AΓB is a two sided ideal of M.

If $a\in M$, then the principal ideal generated by a denoted by $\langle a\rangle$ is the intersection of all ideals containing a and is the set of all finite sum of elements of the form $na+x\alpha a+a\beta y+u\gamma a\mu v$, where n is an integer, x, y, u, v are elements of M and α, β, γ, μ are elements of Γ. This is the smallest ideal generated by a. Let $a\in M$. The smallest left (right) ideal generated by a is called the principal left (right) ideal $\langle a|$ $(|a\rangle)$.

Division gamma ring: Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non zero ideal is itself.

Zorn's lemma: Let A be a nonempty partially ordered set in which every totally ordered subset has an upper bound in A. Then A contains at least one maximal element.

ΓM-module: Let M be a Γ-ring and let (P, +) be an abelian group. Then P is called a left ΓM-module if there exists a Γ-mapping (Γ-composition) from $M\times\Gamma\times P$ to P sending (m, α, p) to $m\alpha p$ such that

- $(m_1+m_2)\alpha p = m_1\alpha p+m_2\alpha p$
- $m\alpha(p_1+p_2) = m\alpha p_1+m\alpha p_2$
- $(m_1\alpha m_2)\beta p = m_1\alpha(m_2\beta p)$,
  for all $p, p_1, p_2\in P$, $m, m_1, m_2\in M$, $\alpha, \beta\in\Gamma$.

If in addition, M has an identity 1 and $1\gamma p = p$ for all $p\in P$ and some $\gamma\in\Gamma$, then P is called a unital ΓM-module.

Sub ΓM-module: Let M be a Γ-ring. Let P be a left ΓM-module. Let (Q, +) be a subgroup of (P, +). We call Q, a sub left ΓM-module of P if $m\gamma q\in Q$ for all $m\in M$, $q\in Q$ and $\gamma\in\Gamma$.

Quotient ΓM-module: Let M be a Γ-ring and P be left ΓM-module. Let Q be a sub left ΓM- module of P. Then the set $\{p+Q \mid p\in P\}$ is called the quotient ΓM-module of P by Q. It is denoted by P/Q, where $m\gamma(p+Q) = m\gamma p+Q$ for all $m\in M$, $p\in P$ and $\gamma\in\Gamma$ and $(p_1+Q)+(p_2+Q) = (p_1+p_2)+Q$ for all $p_1, p_2\in P$.

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ΓM-homomorphism: Let M be a Γ-ring. Let P and Q be two left ΓM-modules. Let φ be a map of P into Q. Then φ is called a ΓM-homomorphism if and only if $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(m\gamma x) = m\gamma\varphi(x)$ for all $x, y\in P$, $m\in M$ and $\gamma\in\Gamma$. If φ is one-one and onto, then φ is a ΓM-isomorphism and is denoted by $P \cong Q$. If φ is a ΓM-homomorphism of P into Q, then kernel of φ, i.e., $\ker\varphi = \{x\in P\,|\,\varphi(x) = 0\}$, which is a left sub ΓM-module of P and image of φ i.e., $\mathrm{Im}\varphi = \{y\in Q\,|\,y = \varphi(x) \text{ for some } x\in P\}$ is a left sub ΓM-module of Q.

Let M be a Γ-ring and A is an ideal of M. Since every ideal A is a ΓM-module, then the homomorphism between two ideals are the same as that of given above.

Γ-ring homomorphism: Let M and N be two Γ-rings. Let φ be a map from M to N. Then φ is a Γ-ring homomorphism if and only if $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(x\gamma y) = \varphi(x)\gamma\varphi(y)$ for all $x, y\in M$ and some $\gamma\in\Gamma$. If φ is one-one and onto, then φ is Γ-ring isomorphism. If φ is a Γ-ring homomorphism of M into N, then kernel of φ, i. e., $\varphi^{-1}(0) = \{x\in M\,|\,\varphi(x) = 0\}$which is also an ideal of M. More generally, if B is a left (right, two sided) ideal of N, then $\varphi^{-1}(B) = \{x\in M\,|\,\varphi(x)\in B\}$ is also a left (respectively right or two sided of M. Similarly, if φ is a Γ-ring homomorphism of M onto N and A is any left (right, two sided) ideal of M, then $\varphi(A) = \{\varphi(a)\,|\,a\in A\}$ is a left (right, two sided) ideal of N.

Equivalent sets: A set A is called equivalent to a set B, written A~B if There exists a function φ: A→B which is one-one and onto. Clearly two finite sets are equivalent if and only if they contain the same number of elements.

Cardinality of sets: If A is equivalent to B, that is, A ~ B, then we say that A and B have the same cardinality or cardinal number. We write | A | for the cardinality or cardinal number of A. So | A | = | B | if and only if A ~ B.

Theorem (schroeder-bernstein theorem): If | A | = | B | and | B | = | A |, then | A | = | B |. For the above preliminaries we refer to[2-5].

In this study, free ΓM-modules are considered. We have defined free ΓM-modules and some of its properties are developed. We also study invariant rank properties of these modules.

Our results are the generalizations of the results due to[6].

Basic notions of free ΓM-modules

Definition: Let P be a Γ-module over a Γ-ring M. Then for any $\gamma\in\Gamma$ a subset X of P is said to be linearly γ-independent or simply γ-independent over M if there exist distinct elements $x_1, x_2, \ldots, x_n$ in X and elements $m_1, m_2, \ldots, m_n$ in M all of which are zero, such that $m_1\gamma x_1 + m_2\gamma x_2 + \ldots + m_n\gamma x_n = 0$.

If X is linearly γ-independent for every $\gamma\in\Gamma$, then X in called linearly Γ-independent or simply Γ-independent.

Again for any $\gamma\in\Gamma$, a subset X of P is said to linearly γ-dependent or simply γ-dependent if there exist distinct elements $x_1, x_2, \ldots, x_n$ in X and elements $m_1, m_2, \ldots, m_n$ not of all which are zero, such that $m_1\gamma x_1 + m_2\gamma x_2 + \ldots + m_n\gamma x_n = 0$.

If X is linearly γ- dependent for every $\gamma\in\Gamma$, then X is said to be linearly Γ-dependent or simply Γ-dependent.

If $X = \{x_i \mid i\in\Lambda\}$ is a set of distinct elements of a left ΓM-module P, then for every $\gamma\in\Gamma$, an expression $\sum_{i\in\Lambda} m_i\gamma x_i$, where $m_i\in M$ and at most finitely many $m_i \neq 0$, is called a linear Γ-combination of $\{x_i \mid i\in\Lambda\}$. Infact, whenever we write $x = \sum_{i\in\Lambda} m_i\gamma x_i$, we mean that x is a linear Γ-combination of $\{x_i \mid i\in\Lambda\}$.

Definition: Let P be a unital left ΓM-module and let $\{x_i \mid i\in\Lambda\}$ be a subset of P such that each element $p\in P$ can be written in at least one way in the form

$$p = m_1\gamma x_{i_1} + m_2\gamma x_{i_2} + \ldots\ldots\ldots + m_n\gamma x_{i_n}$$

where $m_i\in M$, all $\gamma\in\Gamma$ and $i_j\in\Lambda$;

$\{x_i \mid i\in\Lambda\}$ is called a set of generators of P. If each element $p\in P$ can be written in only one way in this form, then $\{x_i \mid i\in\Lambda\}$ is a basis for P. A unital left ΓM-module P is said to be a free left ΓM-module if it has a basis (finite or infinite) , that is , if each element of P can be written is precisely one way as

$$p = m_1\gamma x_{i_1} + m_2\gamma x_{i_2} + \ldots\ldots\ldots + m_n\gamma x_{i_n}$$

$m_i\in M$, all $\gamma\in\Gamma$ and $x_{i_j} \in \{x_i \mid i\in\Lambda\}$, then P is a free left ΓM-module on the basis.

Definition: Let M be Γ-ring. A left ΓM-module P is called finitely generated if P can be generated by finite set of elements, that is, P is finitely generated if and only if there exist finitely many elements $x_1, x_2, \ldots, x_n\in P$ such that each $p\in P$ can be expressed as a linear Γ-combination $p = \sum_{i=1}^{n} m_i\gamma x_i$ of the x_i with coefficients $m_i\in M$ and all $\gamma\in\Gamma$.

If P is finitely generated, among all generating sets, then there are those with a minimum number of elements. The number of elements in a minimal generating set is called the rank of P. It is denoted by rankP.

Theorem: Let P be a non zero left Γ-module over a Γ-ring M. A non empty subset B of P is a basis of P if and only if every element of P can be uniquely written as a linear Γ-combination of the elements of B.

Proof: Let $B = \{x_i \mid i \in \Lambda\}$ be a basis of P. Let $x \in P$, then x can be written as a linear Γ-combination of the elements of B. Suppose that $x = \sum_{i\in\Lambda} m_i\gamma x_i$ and also $x = \sum_{i\in\Lambda} s_i\gamma x_i$ where $m_i, s_i \in M$ and all $\gamma\in\Gamma$ and are non zero finitely many indices $i\in\Lambda$.

$$\text{Then } \sum_{i\in\Lambda} m_i\gamma x_i = \sum_{i\in\Lambda} s_i\gamma x_i$$
$$\Rightarrow \sum_{i\in\Lambda} m_i\gamma x_i - \sum_{i\in\Lambda} s_i\gamma x_i = 0$$
$$\Rightarrow \sum_{i\in\Lambda} (m_i - s_i)\gamma x_i = 0.$$

Since B is a linearly Γ-independent, then $m_i - s_i = 0$ for all $i\in\Lambda$. Hence $m_i = s_i$ for all $i\in\Lambda$. Therefore every element of P can be expressed uniquely as a linear Γ-combination of the elements of B.

Conversely, suppose that every element of P can be expressed uniquely as a linear Γ-combination of the elements of B. Then clearly B generates P. If B is linearly Γ- dependent, then there exist distinct elements $x_1, x_2, \ldots, x_n$ of B and $m_1, m_2, \ldots, m_n \in M$ not all zero, such that $m_1\gamma x_1 + m_2\gamma x_2 + \ldots + m_n\gamma x_n = 0$ for all $\gamma\in\Gamma$. Also $0 = 0\gamma x_1 + 0\gamma x_2 + \ldots + 0\gamma x_n$. This is a contradiction, as 0 can now be expressed in more than one way as a linear Γ-combination of the elements of B. Hence B is linearly Γ-independent. Therefore B is a basis of P. Thus the theorem is proved.

Theorem: A left ΓM-module P is free if and only if it is isomorphic to a direct sum of copies of the left ΓM-module M^M, where M^M is a left ΓM-module over itself.

Proof: Suppose that P is a free left ΓM-module. Let $B = \{x_i \mid i\in\Lambda\}$ be a basis of P. Then by Theorem 2.4, we have $P = \oplus_{i\in\Lambda} M\gamma x_i$ for all $\gamma\in\Gamma$. Now consider a mapping $\varphi: M^M \to M\gamma x_i$ defined by $\varphi(m) = m\gamma x_i$. Let $m_1, m_2 \in M$, then $\varphi(m_1) = m_1\gamma x_i$ and $\varphi(m_2) = m_2\gamma x_i$. Therefore $\varphi(m_1 + m_2) = (m_1 + m_2)\gamma x_i$

$$= m_1\gamma x_i + m_2\gamma x_i$$
$$= \varphi(m_1) + \varphi(m_2)$$

Let $m\in M$, then $\varphi(m\gamma m_1) = (m\gamma m_1)\gamma x_i$

$$= m\gamma(m_1\gamma x_i)$$
$$= m\gamma\varphi(m_1).$$

Hence φ is a ΓM- homomorphism.

Let $\varphi(m_1) = \varphi(m_2)$

$\Rightarrow \quad m_1\gamma x_i = m_2\gamma x_i$

$\Rightarrow \quad m_1\gamma x_i - m_2\gamma x_i = 0$

$\Rightarrow \quad (m_1 - m_2)\gamma x_i = 0$

$\Rightarrow \quad m_1 - m_2 = 0$, since $x_i \neq 0$.

Thus $m_1 = m_2$. Hence φ is a one-one. Clearly φ is onto. Therefore $M^M \cong M\gamma x_i$. Thus P is isomorphic to a direct sum of copies of the left ΓM-module M^M.

Conversely, let $P \cong \oplus_{i\in\Lambda}$, where $M_i = M^M$ and let $B = \{e_i \mid i\in\Lambda\}$, where $e_i(j) = \delta_{ij}$ for $j\in\Lambda$. Hence δ_{ij} is the Kronicker delta function. Then B is a basis of P. Since if $x\in P$, then $x = \sum_{i\in\Lambda} m_i\gamma e_i$ and if $\sum_{i\in\Lambda} m_i\gamma e_i = 0$, then $(\sum_{i\in\Lambda} m_i\gamma e_i)(j) = 0$, that is, $m_j = 0$ for all $j\in\Lambda$. Hence P is a free left ΓM-module. Thus the theorem is proved.

Our next results show that all left Γ-modules over division Γ-rings are free left Γ-modules.

Theorem: Let Δ be a division Γ-ring and let P be a left ΓΔ-module. Then P is a free left ΓΔ-module.

Proof: We apply Zorn's Lemma to prove this theorem. Let X be a generating set of P and let B_0 be any linearly Γ-independent subset of P(B_0 can be the empty set). Let R be the set of all linearly Γ-independent subset of X containing B_0. Then R is partially ordered by set inclusion. If $\{B_i \mid i\in\Lambda\}$ is a chain in R, $\cup_{i\in\Lambda} B_i$ then is a linearly Γ-independent subset of X containing B_0. Thus every chain in R has an upper bound. By Zorn's Lemma, R has a maximal element. Let B be a maximal element of R. Then B is a maximal linearly Γ-independent subset of X that contains B_0. Now to show that B is a basis of P, all we have to show that $P = \langle B\rangle$, that is; B generates P. For this it is sufficient to show that $X \subseteq \langle B\rangle$. If $x\in X\backslash B$, then by maximality of B, the set $B \cup \{x\}$ is linearly Γ-dependent, so there exist distinct elements $x_1, x_2, \ldots, x_n$ in B and $m_1, m_2, \ldots, m_n$ in Δ, not all zero such that $m\gamma x + \sum_{i=1}^{n} m_i\gamma x_i = 0$ for all $\gamma\in\Gamma$.

Now $m \neq 0$, otherwise $m_i = 0$ for all $i = 1, 2, \ldots, n$ as $\{x_1, x_2, \ldots, x_n\}$ is a linearly Γ-independent set. Therefore

$$m\gamma x = -\sum_{i=1}^{n} m_i\gamma x_i$$

$$m^{-1}\gamma(m\gamma x) = m^{-1}\gamma(-\sum_{i=1}^{n} m_i\gamma x_i)$$

$$(m^{-1}\gamma m)\gamma x = -\sum_{i=1}^{n} m^{-1}\gamma(m_i\gamma x_i)$$

$= -\sum_{i=1}^{n} m^{-1}\gamma m_i\gamma x_i$

Therefore $x = -\sum_{i=1}^{n} m^{-1}\gamma m_i\gamma x_i \in \langle B\rangle$.

Hence $X\subseteq\langle B\rangle$. Thus P is a free left $\Gamma\Delta$-module. Hence the theorem is proved.

Corollary: Let P be a left Γ-module over a division Γ- ring Δ. Then a maximal linearly Γ- independent sub set of P is a basis of P .

If P is a free left ΓM-module, then its basis facilates the construction of a ΓM-homomorphism from P to another left ΓM- module N.

Theorem: Let M be a Γ-ring and let P be a free left ΓM-module with basis B. If N is any left ΓM-module and φ: B→ N is any mapping, then there exists a unique ΓM-homomorphism Ψ: P→N such that $\Psi|_B = \varphi$.

Proof: Let $B=\{x_i \mid i \in \Lambda\}$. Then any $x\in P$ can be written uniquely as

$x = \sum_{i\in\Lambda} m_i\gamma x_i$, where $m_i\in M$ and all $\gamma\in\Gamma$ and at most finitely many $m_i \neq 0$. Define

Ψ: P→N by $\Psi(x) = \sum_{i\in\Lambda} m_i\gamma\varphi(x_i)$. Let $x,y \in P$, then

$x = \sum_{i\in\Lambda} m_i\gamma x_i$ and

$y = \sum_{i\in\Lambda} m_i'\gamma x_i$, where m_i, $m_i'\in M$. Then $\Psi(x) = \sum_{i\in\Lambda} m_i\gamma\varphi(x_i)$ and $\Psi(y) = \sum_{i\in\Lambda} m_i'\gamma\varphi(x_i)$. Therefore $\Psi(x)+\Psi(y) = \sum_{i\in\Lambda} m_i\gamma\varphi(x_i) + \sum_{i\in\Lambda} m_i'\gamma\varphi(x_i)$

$= \sum_{i\in\Lambda} (m_i+m_i')\gamma\varphi(x_i) = \Psi(x+y)$.

Let $m\in M$, then $\Psi(m\gamma x) = \sum_{i\in\Lambda} (m\gamma m_i)\gamma\varphi(x_i)$

$= m\gamma \sum_{i\in\Lambda} m_i\gamma\varphi(x_i) = m\gamma\Psi(x)$.

Hence Ψ is a ΓM-homomorphism. Therefore $\Psi|_B = \varphi$. Thus the theorem is proved.

Let P be a left Γ-module over a Γ-ring M. If P is finitely generated, then denoted by $\zeta(P)$, the minimum number of generators of P. If P is not finitely generated, then we define $\zeta(P) = \infty$. Clearly if $P = \{0\}$, then $\zeta(P) = 0$ and $\zeta(P) = 1$ for a cyclic left ΓM-module P.

Let φ: P→ N be a ΓM-homomorphism and let P be a finitely generated left ΓM-module. If $P = \langle x_1,x_2,\ldots,x_n\rangle$, then $\varphi(P) = \langle \varphi(x_1), \varphi(x_2),\ldots, \varphi(x_n)\rangle$. Since if $y\in P$, then $y = \varphi(x)$ for some $x\in P$ and as $x = \sum_{i=1}^{n} m_i\gamma x_i$ for some $m_i\in M$ and all $\gamma\in\Gamma$. So $\varphi(x) = \varphi(\sum_{i=1}^{n} m_i\gamma x_i) = \sum_{i=1}^{n} m_i\gamma\varphi(x_i)$. Therefore $\zeta(\varphi(P))\leq\zeta(P)$. Thus if N is a sub ΓM-module of a finitely generated ΓM-module P, then $\zeta(P/N)\leq\zeta(P)$.

Theorem: Let P be a left Γ-module over a Γ-ring M and let N be a sub ΓM-module of P. If N and P/N are finitely generated ΓM-modules, then P is also finitely generated and $\zeta(P)\leq\zeta(N)+\zeta(P/N)$.

Proof: Let $X = \{x_1, x_2, \ldots, x_n\}$ be a minimal generating set of N and let $Y = \{y_1+N, y_2+N, \ldots, y_t+N\}$ be a minimal generating set of P/N. Now if $x\in P$, then $x+N\in P/N$, so there exist $m_1, m_2, \ldots, m_t \in M$ such that $x+N = \sum_{i=1}^{t} m_i\gamma(y_i+N)$ for all $\gamma\in\Gamma$ and so $x+N = \sum_{i=1}^{t} m_i\gamma y_i+N \Rightarrow x - \sum_{i=1}^{t} m_i\gamma y_i\in N$. Since $N = \langle x_1, x_2, \ldots, x_n\rangle$ then there exist $s_1,s_2,\ldots,s_n \in M$ such that $x - \sum_{i=1}^{t} m_i\gamma y_i = \sum_{j=1}^{t} s_j\gamma x_j$ and so $x = \sum_{i=1}^{t} m_i\gamma y_i + \sum_{j=1}^{t} s_j\gamma x_j$. This proves that $P = \langle x_1, x_2,\ldots, x_n, y_1, y_2,\ldots, y_t\rangle$ Hence $\zeta(P) \leq n+t = \zeta(N)+\zeta(P/N)$. Thus the theorem is proved.

Lemma: Let P be a free left Γ- module over a Γ- ring M. If P has an infinite basis, then no finite subset of P can generate P.

Proof: Let B be an infinite basis of P and suppose on the contrary that Y is a finite subset of P and $P = \langle Y\rangle$. Since B is a basis of P, for each $y\in Y$, there exist a finite subset $\{x_1, x_2,\ldots, x_k\}$ of distinct elements of B and $m_1, m_2,\ldots, m_k\in M\backslash\{0\}$ so that $y = m_1\gamma x_1+m_2\gamma x_2+\ldots+ m_k\gamma x_k$ for all $\gamma\in\Gamma$.

Thus there is a finite subset X of B such that every element of Y is a linear Γ-combination of the elements of X, that is, $Y\subseteq\langle X\rangle$ and so $P = \langle X\rangle$. Since X is a finite subset of B, so B X is nonempty. But then for $x\in B\backslash X$, the set $X\cup\{x\}$ is a linearly Γ-dependent subset of B; a contradiction. Hence Y is infinite. Thus the lemma is proved.

Theorem: Let M be a Γ-ring and let P be a free left ΓM-module with an infinite basis B. Then every basis of P has the same cardinality as B.

Proof: Let B' be another basis of P. Thus by Lemma 2.10, B' is an infinite set. Now let $F(B')$ be the set of all finite subset of B'. Then for each $x \in B$, there are uniquely determined distinct elements $y_1, y_2, \ldots, y_k$ of B' such that $x = \sum_{i=1}^{k} m_i \gamma y_i$, where $m_1, m_2, \ldots, m_k \in M \setminus \{0\}$ and all $\gamma \in \Gamma$. Thus, we have a well defined mapping $\varphi: B \to F(B')$ given by $\varphi(x) = \{y_1, y_2, \ldots, y_k\}$. Note that $\varphi(B)$ is an infinite set. If on the contrary $\varphi(B)$ is finite, then every element of B is a linear Γ-combination of elements of $\varphi(B)$, that is; $B \subseteq \langle \varphi(B) \rangle$. But then $P = \langle \varphi(B) \rangle$, a contradiction to the Lemma 2.10.

Next, we show that for every $X \in \varphi(B)$, the set $\varphi^{-1}(X)$ is finite. If $x \in \varphi^{-1}(X)$, then by definition of φ, we have $x \in \langle X \rangle$. Since X is a finite subset of B', then there is a finite subset Y of B so that $X \subseteq \langle Y \rangle$. Therefore $x \in \langle X \rangle$ and it implies that either $x \in Y$ or x is a linear Γ- combination of the elements of Y. In the later case, $Y \cup \{x\}$ is a linearly Γ- dependent subset of B, a contradiction. Therefore $x \in Y$ and so $\varphi^{-1}(X) \subseteq Y$. Hence $\varphi^{-1}(X)$ is a finite set.

Now consider the collection of sets $\{\varphi^{-1}(X) \mid X \in F(B)\}$. Clearly $\bigcup_{x \in F(B)} \varphi^{-1}(X) = B$. We claim that $\varphi^{-1}(X) \cap \varphi^{-1}(Y)$ is non empty, whenever $\varphi^{-1}(X) \neq \varphi^{-1}(Y)$. Suppose that $x \in \varphi^{-1}(X) \cap \varphi^{-1}(Y)$. Since $\varphi^{-1}(X) \cap \varphi^{-1}(Y) \subseteq B$ and B is a basis of P, so $x \neq 0$. Let $\varphi^{-1}(X) = \{x_1, x_2, \ldots, x_t\}$ and $\varphi^{-1}(Y) = \{y_1, y_2, \ldots, y_n\}$. Then $x \in \varphi^{-1}(X)$ implies that $x = \sum_{i=1}^{t} m_i \gamma x_i$, where $m_1, m_2, \ldots, m_t \in M$ and all $\gamma \in \Gamma$ and $x \in \varphi^{-1}(Y)$ implies that $x = \sum_{i=1}^{n} s_j \gamma x_j$, where $s_1, s_2, \ldots, s_n \in M$ and all $\gamma \in \Gamma$. But then $\sum_{i=1}^{t} m_i \gamma x_i = \sum_{j=1}^{n} s_j \gamma x_j$.

So $\sum_{i=1}^{t} m_i \gamma x_i - \sum_{j=1}^{n} s_j \gamma x_j = 0$. Thus $m_i = 0$, $i = 1, 2, \ldots, t$ and $s_j = 0$, $j = 1, 2, \ldots, n$. Thus $x = 0$, a contradiction.

Hence, the sets $\varphi^{-1}(X)$, $X \in \varphi(B)$ form a partition of B. Now for each $X \in \varphi(B)$, order the elements of $\varphi^{-1}(X)$, say $x_1, x_2, \ldots, x_n$ and define a mapping $g_X: \varphi^{-1}(X) \to \varphi(B)$ by $g_X(x_k) = (X, k)$. Let $x_k, x_{k'} \in \varphi^{-1}(X)$, then $g_X(x_k) = (X, k)$ and $g_X(x_{k'}) = (X, k')$.

Let $g_X(x_k) = g_X(x_{k'})$

$\Rightarrow$ $(X, k) = (X, k')$

$\Rightarrow$ $k = k'$.

Thus $x_k = x_{k'}$. Hence g_X is one-one. It now follows that the mapping $g: B \to \varphi(B) \times Z^+$ defined by $g(x) = g_X(x)$, where $x \in \varphi^{-1}(X)$.

Let $x = x'$. Then $g_X(x) = g_X(x')$. So $g(x) = g(x')$. Hence g is well defined.

Thus $|B| \leq |\varphi(B) \times Z^+|$

$= |\varphi(B)||Z^+| = |\varphi(B)|N_0$, where N_0 is the cardinality of Z^+

$\leq |\varphi(B)| \leq |F(B')| = |B'|$. Hence $|B| \leq |B'|$.

Interchanging the role of B and B', we get $|B'| \leq |B|$. Hence by Theorem 1.1, $|B| = |B'|$. Thus every basis of P has the same cardinality. Hence the theorem is proved.

Invariant rank property of free left γm-modules

Definition: Let P be a free left Γ-module over a Γ- ring M such that any two bases of P have same cardinality. Then the cardinality of a basis of P is also called the rank of P over M and we can write rankP = | B |, where B is a basis of P. We say that a Γ- ring M has an invariant rank property if for every free left ΓM-module P, the rank of P over M is defined, that is, any two bases of P have the same cardinality.

We have shown in Theorem 2.6, that a left Γ-module over a division Γ-ring is a free left Γ- module. Now we prove that the rank of such a Γ-module is defined.

Theorem: If P is a Γ-module over a division Γ- ring Δ, then any two bases of P have same cardinality.

Proof: Let B and B' be two bases of P. If either B or B' is infinite, then by Theorem 2.11, $|B| = |B'|$. Therefore, we assume that B and B' are finite. Let $B = \{x_1, x_2, \ldots, x_t\}$ and $B' = \{y_1, y_2, \ldots, y_n\}$. Without any loss we may assume that $n = t$. We write $y_n = m_1\gamma x_1 + m_2\gamma x_2 + \ldots + m_t\gamma x_t$, where $m_i \in M$ and all $\gamma \in \Gamma$. Let k be the first index such that $m_k \neq 0$, so

$m_k\gamma x_k = y_n - m_1\gamma x_1 - m_2\gamma x_2 - \ldots - m_{k-1}\gamma x_{k-1} - m_{k+1}\gamma x_{k+1} - \ldots - m_t\gamma x_t$. Then

$m_k^{-1}(m_k\gamma x_k) = m_k^{-1}\gamma(y_n - m_1\gamma x_1 - m_2\gamma x_2 - \ldots - m_{k-1}\gamma x_{k-1} - m_{k+1}\gamma x_{k+1} - \ldots - m_t\gamma x_t)$

$\Rightarrow (m_k^{-1} m_k)\gamma x_k = m_k^{-1}\gamma y_n - m_k^{-1}\gamma m_1\gamma x_1 - m_k^{-1}\gamma m_2\gamma x_2 - \ldots - m_k^{-1}\gamma m_{k-1}\gamma x_{k-1} - m_k^{-1}\gamma m_{k+1}\gamma x_{k+1} - \ldots - m_k^{-1}\gamma m_t\gamma x_t$

$\Rightarrow 1\gamma x_k = m_k^{-1}\gamma y_n - m_k^{-1}\gamma m_1\gamma x_1 - m_k^{-1}\gamma m_2\gamma x_2 - \ldots - m_k^{-1}\gamma m_{k-1}\gamma x_{k-1} - m_k^{-1}\gamma m_{k+1}\gamma x_{k+1} - \ldots - m_k^{-1}\gamma m_t\gamma x_t$

Therefore $x_k = m_k^{-1}\gamma y_n - m_k^{-1}\gamma m_1\gamma x_1 - m_k^{-1}\gamma m_2\gamma x_2 - \ldots - m_k^{-1}\gamma m_{k-1}\gamma x_{k-1} - m_k^{-1}\gamma m_{k+1}\gamma x_{k+1} - \ldots - m_k^{-1}\gamma m_t\gamma x_t$

Thus the set $\{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m, y_n\}$ generates P. In particular

$y_{n-1} = s\gamma y_n + \sum_{i=1}^{t} s_i\gamma x_i$. Now as $\{y_n, y_{n-1}\}$ is a linearly Γ-independent set, then

$y_{n-1} - s\gamma y_n \neq 0$. Let j be the first index such that $s_j \neq 0$; then x_j is a linear Γ- combination of y_{n-1}, y_n and x_i, $i \neq j, k$ and so $\{y_{n-1}, y_n\} \cup \{x_i \mid i \neq j, k\}$ generates P. In particular y_{n-2} is a linear Γ- combination of y_n, y_{n-1} and x_i, $i \neq j, k$. The above process of adding an element of B' and deleting an element of B may be repeated. If $n < t$, then after n steps, we conclude that $\{y_n, y_{n-1}, \ldots, y_{t-n+1}\}$ generates P. In particular y_{t-n} is a linear

Γ-combination of y_n, y_{n-1}, ..., y_{t-n+1} and this contradicts the linearly Γ-dependence of B′. Therefore t = n. Hence B and B′ have same cardinality. Thus the theorem is proved.

Let P be a left Γ-module over a Γ-ring M and let A be an ideal of M .If

$A\Gamma P = \{ \sum_{i=1}^{k} a_i\gamma x_i \mid a_i \in A, x_i \in P$ and all $\gamma \in \Gamma\}$, then it is easy to verify that $A\Gamma P$ is a sub ΓM- module of P. Also $P/A\Gamma P$ is a Γ-M/A-module with the action of M/A or $P/A\Gamma P$ given by $(m+A)\gamma(x+A\Gamma P) = m\gamma x+A\Gamma P$, where $m\in M$, $x\in P$ and all $\gamma\in\Gamma$. This is a well defined operation; if $m+A = m'+A$ and $x+A\Gamma P = x'+A\Gamma P$, then $m\gamma x - m'\gamma x' = m\gamma x - m\gamma x' + m\gamma x' - m'\gamma x' = m\gamma(x - x') + (m - m')\gamma x' \in A\Gamma P$ and so $m\gamma x+A\Gamma P = m'\gamma x'+A\Gamma P$.

Lemma: Let M be a Γ-ring, let A be a proper ideal of M and let P be a free left ΓM-module with a basis B. Then $P/A\Gamma P$ is a free left Γ-M/A-module with basis $\Pi(B)$ and $| B | = | \Pi(B) |$, where $\Pi: P \to P/A\Gamma P$ is a canonical ΓM-epimerphism of ΓM-modules.

Proof: Let $B = \{x_i \mid i\in\Lambda\}$ be a basis of P. Then $\Pi(B) = \{x_i+A\Gamma P \mid x_i\in B\}$. We now prove this lemma in steps:

Step 1: $\Pi(B)$ generates $P/A\Gamma P$. If $x+A\Gamma P\in P/A\Gamma P$ October 19, 2006, then as B is a basis of P, so, $x = \sum m_i\gamma x_i$, where $m_i\in M$, all $\gamma\in\Gamma$ and $m_i \neq 0$ for finitely many $i\in\Lambda$. Thus

$x+A\Gamma P = \sum m_i\gamma x_i + A\Gamma P$
$=\sum(m_i+A)\gamma(x_i+A\Gamma P)$
$=\sum(x_i+A)\gamma\Pi(x_i)$.
Hence $\Pi(B)$ generates $P/A\Gamma P$.

Step 2: $| B | = |\Pi(B)|$. Let $x+A\Gamma P$ and $x'+A\Gamma P$ be elements of $\Pi(B)$ such that $x \neq x'$ and $x+AGP = x'+AGP$. Then $x - x'\in A\Gamma P$. So $x - x' = \sum_{i=1}^{n} a_j\gamma y_j$,

where $a_j\in A\backslash\{0\}$, $y_j\in P$, for j = 1,2,3, ... , n.

Now writing each y_j as a linear Γ-combination of elements of B over M, we conclude that $x - x'$ is a linear Γ-combination of elements of B with coefficient from A. Since B is a basis of P, on equating the coefficient of x, we get $1\in A$, a contradiction as $A\neq M$. Hence $| B | = |\Pi(B)|$.

Step 3: $\Pi(B)$ is a linearly Γ-independent set over M. Let $x_1+A\Gamma P$, $x_2+A\Gamma P$, ... , $x_k+A\Gamma P$ be k distinct elements of $\Pi(B)$ and let m_1+A, m_2+A, ... , m_k+A be elements of M/A so that

$$\sum_{i=1}^{k} (m_i+A)\gamma(x_i+A\Gamma P) = A\Gamma P$$

$$\Rightarrow \sum_{i=1}^{k} (m_i\gamma x_i+A\Gamma P) = A\Gamma P$$

$$\Rightarrow \sum_{i=1}^{k} m_i\gamma x_i+A\Gamma P = A\Gamma P$$

$$\Rightarrow \sum_{i=1}^{k} m_i\gamma x_i \in A\Gamma P.$$

If $\sum_{i=1}^{k} m_i\gamma x_i = 0$, then Γ- independence of $\{x_1,x_2, ... ,x_k\}$ implies that $m_1 = m_2 = . . . = m_k = 0$. Therefore $m_1+A = m_2+A = ... = m_k+A = A$.

Hence $x_1+A\Gamma P$, $x_2+A\Gamma P$, ... , $x_k+A\Gamma P$ are linearly Γ- independent over M/A. If $\sum_{i=1}^{k} m_i\gamma x_i \neq 0$, then $\sum_{i=1}^{k} m_i\gamma x_i = \sum_{i=1}^{k} a_i\gamma y_i$, where each $a_i\in A\backslash\{0\}$ and $y_i\in P$. Since each y_i is a linear Γ- combination of elements of B over M, we have $\sum_{i=1}^{k} m_i\gamma x_i = \sum_{j=1}^{t} b_j\gamma z_j$, where $z_j\in B$ and $b_j\in A\backslash\{0\}$. Thus, $\sum_{i=1}^{k} m_i\gamma x_i - \sum_{j=1}^{t} b_j\gamma z_j = 0$. If $x_i = z_j$ for some j, then the coefficients of x_i is $m_i - b_j$ and then Γ-independence of B implies that $m_i - b_j = 0$, so $m_i = b_j\in A$. If $x_i \neq z_j$ for any j = 1, 2, 3, ... , t, then $m_i = 0$. Thus, in any case $m_i+A = A$ for all i = 1, 2, 3, ..., k. Hence $\{ x_1+A\Gamma P, x_2+A\Gamma P, ... , x_k+A\Gamma P\}$ is a linearly Γ-independent set over M. This shows that every finite subset of $\Pi(B)$ is linearly Γ-independent over M. Therefore $\Pi(B)$ is linearly Γ-independent over M/A. Hence $\Pi(B)$ is a basis of left Γ-M/A-module $P/A\Gamma P$. Therefore $P/A\Gamma P$ is a free left Γ-M/S-module. Thus the lemma is proved.

Theorem: Let $\varphi: M\to S$ be a nonzero Γ- ring epimorphism. If S has the invariant rank property, then M has the invariant rank property.

Proof: Let P be a free left ΓM-module with basis B and B′ and $K = \ker\varphi$. Note that $K\neq M$ as φ is a nonzero Γ- ring homomorphism. If K = {0}, then φ is a Γ- ring isomorphism. Now P can be viewed as a ΓS-module with scalar multiplication given by $s\gamma x = \varphi^{-1}(s)\gamma x$ for all $s\in S$, $x\in P$ and $\gamma\in\Gamma$. Clearly B and B′ are also basis of ΓS-module P. Hence $| B | = | B' |$.

Now let $K \neq \{0\}$. By 1st isomorphism Theorem of Γ-rings, $M/K \cong S$.

Therefore M/K has invariant rank property. Also by Lemma 3.3, $P/K\Gamma P$ is a free left Γ-M/K-module with basis $\Pi(B)$ and $\Pi(B')$ such that $| \Pi(B)| = | B |$ and $| \Pi(B')| = |B'|$, where $\Pi: P\to P/K\Gamma P$ is the canonical Γ- epimorphism. Hence $| B | = | B' |$. Thus the theorem proved.

Theorem: Let M be a Γ-ring with the invariant rank property and let P and Q be free left ΓM-modules. Then P and Q are isomorphic if and only if rankP = rankQ.

Proof: Let φ: P$\rightarrow$Q be a ΓM-isomorphism. If B is a basis of P, then we verify that φ(B) is a basis of Q. Since φ is one-one and onto, then $|B| = |\varphi(B)|$. Hence rankP = rankQ.

Conversely, let rankP = rankQ. Let B be a basis of P and B′ be a basis of Q.

Since rankP = rankQ, so $|B| = |B'|$. Thus, there is a one-one and onto mapping Ψ: B$\rightarrow$B′. By Theorem 2.8, there is a ΓM-homomorphism φ: P$\rightarrow$Q such that $\varphi|_B = \Psi$. Clearly φ is also one-one and onto. Hence P$\cong$Q. Thus the theorem is proved.

Theorem: Let P be a left free Γ-module over a Γ-ring M and let Q be a sub ΓM-module of P. Then rankP = rankQ+rankP/Q.

Proof: Let B′ is a basis of Q. Then it can be extended to form a basis B of P. Let $B = \{x_i \mid i\in\Lambda\}$ be such that $B' = \{x_j \mid j\in\Lambda'\}$ and $\Lambda'\subseteq\Lambda$. Therefore $|B| = |\Lambda| = |\Lambda'|+|\Lambda\backslash\Lambda'|$ = rankQ+$|\Lambda\backslash\Lambda'|$. We now show that rank P/Q= $|\Lambda\backslash\Lambda'|$ and this will prove the theorem. Let $B'' = \{x_i \mid i\in\Lambda\backslash\Lambda'\}$ and $\overline{B} = \{x_i+Q \mid i\in\Lambda\backslash\Lambda'\}$. Then for each $i\in\Lambda\backslash\Lambda'$, $x_i+Q\neq Q$, since otherwise $x_i\in Q$ implies that x_i is a linear Γ- combination of elements of Q, a contradiction. Similarly for $i,j\in\Lambda\backslash\Lambda'$, $i\neq j$, we have $x_i+Q\neq x_j+Q$. Hence $|B| = |B''| = |\Lambda\backslash\Lambda'|$. Now we show that $\overline{B}$ is a basis of P/Q. First we show that $\overline{B}$ is a linearly Γ-independent set. Let $x_{i1}+Q, x_{i2}+Q, \ldots, x_{it}+Q$ be distinct elements of $\overline{B}$ such that $\sum_{r=1}^{t} m_i\gamma(x_{ir}+Q) = Q$, where $m_i\in M$, so $\sum_{r=1}^{t} m_i x_{i_r}+Q = Q$. then $\sum_{r=1}^{t} m_i\gamma x_{i_r}Q$. If $\sum_{r=1}^{t} m_i\gamma x_{i_r}\neq 0$, then it is a linear Γ-combination of element of B′ and this contradicts the linear Γ- independence of B. Therefore $\sum_{r=1}^{t} m_i\gamma x_{i_r}= 0$ and it implies that $m_r = 0$ for all r = 1,2, ..., t as $\{x_{i_1}, x_{i_2}, \ldots, x_{i_T}\}\subseteq B$. Hence every finite subset of $\overline{B}$ is linearly Γ- independent and so $\overline{B}$ is linearly Γ- independent.

Now, if $x+Q \in P/Q$, then as $x\in P$ so $x = \sum_{i\in\Lambda} m_i\gamma x_i$, where $m_i \in M$ and $m_i\neq 0$ for finitely many indices $i\in\Lambda$. Thus $x+Q = \sum_{i\in\Lambda} m_i\gamma x_i+Q = \sum_{i\in\Lambda} m_i\gamma(x_i+Q) = \sum_{i\in\Lambda'\backslash\Lambda} m_i\gamma(x_i+Q)$, as $x_i\in Q$ for all $i\in\Lambda'$. Hence $\overline{B}$ generates P/Q. Therefore $\overline{B}$ is a basis of P/Q. Hence rank P/Q=$|\Lambda\backslash\Lambda'|$. Thus rank P = rankQ+rank P/Q. Hence the theorem is proved.

REFERENCES

1. Barnes, W.E., 1966. On the gamma rings of Nobusawa, Pacific J. Math, 18: 411- 422.
2. Coppage, W.E. and J. Luh, 1971. Radicals of gamma rings, J. Math. Soc. Japan, 23: 40-52.
3. Luh, J., 1968. On primitive gamma rings with minimal one sided ideals, Osaka J. Math, 5: 165-173.
4. Luh, J., 1969. On the theory of simple gamma rings, Michigan Math J. 16: 65-75.
5. Nobusawa, N., 1964. On a generalization of the ring theory, Osaka J. Math, 1: 81-89.
6. Sahai, V. and V. Bist, 1999. Algebra, Narosa Publishing House, New Delhi, pp: 190-201.