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K-Out-Of N: G System with Multiple Failure Modes and Single Repair Facility

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Abstract: This study presents models of a K-out-of N: G system subject to multiple failure modes. Times between failures are exponentially distributed and repair times follow general distribution. Solutions for point wise and steady state reliability parameters are derived for two distinct repair policies.

Key words: K-out-of N: G system, multiple failure modes, discrete transform, repair, general distribution

INTRODUCTION

The mathematical models of system reliability on the basis of a Poisson process of single failures cannot reflect integrally the real behaviour of the system (El-Damcese and Temraz, 2010; Hajeeh, 2004a; Moustafa, 1996, 1998). A continuous time Markov chain model meant for analysis of the reliability parameters of an N-unit parallel system with 3 different time varying failure rates and repair rates is presented by El-Damcese and Temraz (2010). The set of differential equations is solved numerically and some estimates of the transient metrics are shown graphically.

There is no formal proof however that under these assumptions the model is suitable and applicable for the steady-state probabilities. Moreover, an attempt to extend the model to cover a large number of failure modes implies enormously large number of system states. K-out-of N: G system with warm and cold standby and multiple failure modes is introduced in (Jain *et al*., 2007). Both failure rates and repair rates are exponentially distributed.

Closed solutions for a system without any repair are derived and methodology for evaluation of a system with repair is also presented. Research by Hajeeh (2004a, b) is a study of the imperfect repair phenomena which often occurs in real world situation. Two models are developed for a system subject to multiple failure modes and having distinct repair rates. Transient behaviour of a repairable K-out-of N: G system with different multiple constant failure rates received comprehensive consideration by Moustafa (1996, 1998). Moustafa (2008) developed the Mean Time to System Failure (MTSF).

Whereas the assumption of exponential failure distribution is adequate for most practical applications especially for electrical and electronic equipment, the distribution of repair time is rather general. We consider in this study K-out-of N: G system with exponential multiple failures and multiple generally distributed repairs. Since, the process (j, t) (j-number of working units, t-time) is not markovian under these assumptions, a supplementary variable x, elapsed time in repair is introduced to obtain suitable equations.

SYSTEM DESCRIPTION AND NOTATION

The system studied is a K-out-of N: G system (at least N-K+1 units must fail for the system to fail). In the case of a failure in a unit it is considered a failed unit i.e., all failures are fatal ones for a single unit. A unit can fail in one of M different modes of failure and times between failures are exponentially distributed with constant rate λ_m (m = 1, 2, ..., M). The failure modes are mutually exclusive i.e., if one of the failure modes occurs it can only be followed by the same failure mode.

We shall consider two repair policies. In the case of the first of them, the failed unit is immediately sent to be repaired and it is put into operation immediately after repair. When N-K+1 units are down the system is stopped and becomes operational again when the repair of unit is completed. In the case of the second policy, the repair is started after N-K+1 units are down and the system becomes operational after all N-K+1 units are repaired. According to these two policies, we shall designate the model related to the first repair policy as Model 1 and the model related to the second policy as Model 2. Throughout this study we use the following notations:

$P_j(t)$ = P [j units are operating at time t] $K \le j \le N$

$P_{j,m}(t, x)$ = P [j units are operating at time t, one unit is under repair due to a failure of type m and the elapsed repair time lies between x and x+dx] $K \le j \le N$, $1 \le m \le M$

$P_{K-1,m}(t, x)$ = P [N-K+1 units are failed due to a failure of type m at time t, system is undergoing repair and the elapsed repair time lies between x and x+dx]

$P_{j,m}(x)$ = P [in equilibrium j units are operating, one unit is under repair due to a failure of type m and the elapsed repair time lies between x and x+dx] $K \le j \le N-1$, $1 \le m \le M$

$$P_N \lim_{t\to\infty} P_N(t)$$

Where:

$P_{j,m}$ = Steady-state probabilities

$P_{j,m}$ = $\int_0^\infty P_{j,m}(x)dx \quad K \le j \le N, 1 \le m \le M$

$\beta_{j,m}$ = $j\lambda_m$ K≤j≤N, 1≤m≤M

β_N = $N\sum_{m=1}^{M}\lambda_m$

F_m (x) = c.d.f. of the service time of type m 1≤m≤M

f_m (x) = p.d.f. of the service time of type m 1≤m≤M

$$\frac{1}{\mu_m} = \int_0^\infty x f_m(x)dx$$

$$h_m(x) = \frac{f_m(x)}{1-F_m(x)}$$

Repair rate for type m 1≤m≤M

Where:

$\underline{fm}(s)$ = Laplace Transform (LT) of f_m (x)

$f_m^*(x)$ = (N-K+1)-th convolution of f_m (x)

$h_m^*(x)$ = (N-K+1)-th convolution of h_m (x)

P_A (t) = Point wise (instantaneous, transient) availability

R (t) = Point wise reliability

P_A = Steady-state availability

MTSF = Mean Time to System Failure

$\delta_{l,n}$ = Kronecker delta

ANALYSIS OF MODEL 1

Viewing the nature of the system following set of integro-differential equation is obtained:

$$\left(\frac{d}{dt}+\beta_N\right)P_N(t)=\sum_{m=1}^{M}\int_0^t P_{N-1,m}(t,x)h_m(x)dx \quad (1)$$

$$\left[\frac{d}{dt}+\frac{\partial}{\partial x}+\beta_{N-1,m}+h_m(x)\right]P_{N-1,m}(t,x)=0 \quad (2)$$
$$1 \le m \le M$$

$$\left[\frac{d}{dt}+\frac{\partial}{\partial x}+\beta_{j,m}+h_m(x)\right]P_{j,m}(t,x)=\beta_{j+1,m}P_{j+1,m}(t,x) \quad (3)$$
$$K \le j \le N-2, 1 \le m \le M$$

$$\left[\frac{d}{dt}+\frac{\partial}{\partial x}+h_m(x)\right]P_{K-1,m}(t,x)=\beta_{K,m}P_{K,m}(t,x) \quad (4)$$
$$1 \le m \le M$$

Having the following boundary and initial conditions:

$$P_N(0) = 1 \quad (5)$$

$$P_{j,m}(t,0) = (1-\delta_{j,K-1})\int_0^\infty P_{j-1,m}(t,x)h_m(x)dx + \delta_{j,N-1}\beta_{N,m}P_N(t) \quad K-1 \le j \le N-1, 1 \le m \le M \quad (6)$$

$$P_{j,m}(0,0) = 0, \quad K\text{-}1 \le j \le N, 1 \le m \le M \quad (7)$$

$$P_{K-1,m}(t, 0) = 0, 1 \le m \le M \quad (8)$$

By using Laplace transform and discrete transform (Itoi *et al*., 1978) the Eq. 1-4 are transformed as follows:

$$(s+\beta_N)P_N(s) = 1+\sum_{m=1}^{M}\int_0^\infty P_{N-1,m}(s,x)h_m(x)dx \quad (9)$$

$$\left[s+\frac{\partial}{\partial x}+\beta_{j,m}+h_m(x)\right]u_{j,m}(s,x)=0 \quad (10)$$
$$K \le j \le N-1, 1 \le m \le M$$

$$\left[s+\frac{\partial}{\partial x}+h_m(x)\right]P_{K-1,m}(s,x)=\beta_{K,m}P_{K,m}(s,x) \quad (11)$$
$$1 \le m \le M$$

Where:

$$u_{j,m}(s,x) = \sum_{n=j}^{N-1,m}\binom{n}{j}P_{n,m}(s,x)$$

And:

$$P_{j,m}(s,x) = \sum_{n=j}^{N-1,m}(-1)^{n-j}\binom{n}{j}u_{n,m}(s,x)$$
$$K \le j \le N-1, 1 \le m \le M$$

$$\text{Let } v_{j,m}(s,x) = \frac{u_{j,m}(s,x)}{1-F_m(x)}$$

And:

$$P'_{K-1,m}(s,x) = \frac{P_{K-1,m}(s,x)}{1-F_m(x)}$$

Then from Eq. 10 and 11, we have after some manipulations:

$$\left[s+\frac{d}{dx}+\beta_{j,m}\right]v_{j,m}(s,x)=0 \quad (12)$$
$$K \le j \le N-1, 1 \le m \le M$$

$$\left[s+\frac{d}{dx}\right]=P'_{K-1,m}(s,x)=P'_{K,m}(s,x) \quad (13)$$
$$1\le m\le M$$

Hence, solutions of Eq. 9-11 are:

$$P_N(s)=\frac{1+\sum_{m=1}^{M}\underline{f_m}(s+\beta_{N-1,m})u_{N-1,m}(s,0)}{s+\beta_N} \quad (14)$$

$$uj,m\ (s,x)=[1-F_m(x)]u_{j,m}(s,0)e^{-(s+\beta_{j,m})x} \quad (15)$$
$$K\le j\le N-1,\ 1\le m\le M$$

$$P_{K-1,m}(s,x)=[1-F_m(x)]\beta_{K,m}e^{-sx}$$
$$\left[\sum_{n=K}^{N-1}(-1)^{n-K}\binom{n}{K}\frac{1-e^{-\beta_{n,m}x}}{\beta_{n,m}}u_{n,m}(s,0)\right] \quad (16)$$
$$1\le m\le M$$

By integrating Eq. 14 and 15, we obtain the LT of the point wise (instantaneous, transient) probabilities:

$$P_{j,m}\ (s)=\sum_{n=j}^{N-1}(-1)^{n-j}\binom{n}{j}\left[\frac{1-\underline{f_m}(s+\beta_{n,m})}{s+\beta_{n,m}}\right]u_{n,m}\ (s,0) \quad (17)$$
$$K\le j\le N-1,\ 1\le m\le M$$

$$P_{K\text{-}1,m}(s)=\beta_{K,m}=\sum_{n=K}^{N-1}(-1)^{n-K}\binom{n}{K}$$
$$\left[\frac{1-\underline{f_m}(s)}{s}+\frac{1-\underline{f_m}(s+\beta_{n,m})}{s+\beta_{n,m}}\right]\frac{u_{n,m}(s,0)}{\beta_{n,m}},\ 1\le m\le M \quad (18)$$

Taking LT of Eq. 6 and using Eq. 7, 8 and 12-15, we get after some algebra the following system of linear equations:

$$\sum_{n=j}^{N-1}(-1)^{n-j}\binom{n}{j}u_{n,m}(s,0)=\sum_{n=j-1}^{N-1}(-1)^{n-j+1}\binom{n}{j-1}\underline{f_m}(s+ \quad (19)$$
$$\beta_{n,m})u_{n,m}(s,0),\ K+1\le j\le N-1$$

$$\sum_{n=K}^{N-1}(-1)^{n-K}\binom{n}{K}$$
$$u_{n,m}(s,0)=\beta_{K,m}\left[\begin{array}{c}\sum_{n=K}^{N-1}(-1)^{n-K}\binom{n}{K}\\ \frac{\underline{f_m}(s)-\underline{f_m}(s+\beta_{n,m})}{\beta_{n,m}}u_{n,m}(s,0)\end{array}\right] \quad (20)$$
$$1\le m\le M$$

Coefficients $u_{j,m}$ (s, 0) can now be determined from the above equations and Eq. 14. The LT of the point wise availability P_A (s) can be found by summing the probabilities of all non-failure states:

$$P_A(s)=P_N(s)+\sum_{n=K}^{N-1}\sum_{m=1}^{M}P_{n,m}(s) \quad (21)$$

We can apply the final-value theorem to Eq. 14, 17 and 18) to obtain the steady-state probabilities but it will require use of L'Hopital rule and seems difficult and impractical. Instead we set the following differential equations:

$$P_N\beta_N=\sum_{m=1}^{M}\int_0^\infty P_{N-1,m}(x)h_m(x)dx \quad (22)$$

$$\left[\frac{d}{dx}+\beta_{N-1,m}+h_m(x)\right]P_{N-1,m}(x)=0 \quad (23)$$
$$1\le m\le M$$

$$\left[\frac{d}{dx}+\beta_{j,m}+h_m(x)\right]P_{j,m}(x)=\beta_{j+1,m}P_{j+1,m}(x) \quad (24)$$
$$K\le j\le N-2,\ 1\le m\le M$$

$$\left[\frac{d}{dx}+h_m(x)\right]P_{K-1,m}(x)=\beta_{K,m}P_{K,m}(x) \quad (25)$$
$$1\le m\le M$$

Equations 25-29 are to be solved under the following boundary conditions and normalizing condition:

$$P_{j,m}(0)=(1-\delta_{j,K-1})\int_0^\infty P_{j-1,m}(x)h_m(x)dx$$
$$+\delta d_{j,N\text{-}1}\beta_{N,m}P_N \qquad K\le j\le N-1,\ 1\le m\le M \quad (26)$$

$$P_N+\sum_{j=K-1}^{N-1}\sum_{m=1}^{M}P_{j,m}=1 \quad (27)$$

Solutions of Eq. 30-32 are:

$$P_N=\frac{\sum_{m=1}^{M}\underline{f_m}(\beta_{N-1,m})u_{N-1,m}(0)}{\beta_N} \quad (28)$$

$$P_{j,m}=\sum_{n=j}^{N-1}(-1)^{n-j}\binom{n}{j}\left[\frac{1-\underline{f_m}(\beta_{n,m})}{\beta_{n,m}}\right]u_{n,m}(0) \quad (29)$$
$$K\le j\le N-1,\ 1\le m\le M$$

$$P_{K-1,m} = \sum_{n=K}^{N-1} (-1)^{n-K} \binom{n}{K} \frac{\beta_{K,m}}{\beta_{n,m}} \left[\frac{\frac{1}{\mu_m} - }{\frac{1 - \underline{f_m}(\beta_{n,m})}{\beta_{n,m}}} \right] u_{n,m}(0) \quad (30)$$

For $u_{j,m}(0)$, we have:

$$\begin{aligned} &\sum_{n=j}^{N-1} (-1)^{n-j} \binom{n}{j} u_{n,m}(0) = \sum_{n=j-1}^{N-1} (-1)^{n-j+1} \binom{n}{j-1} \\ &\underline{f_m}(\beta_{n,m}) u_{n,m}(0) + \delta_{j,N-1} \beta_{N,m} P_N \\ &K+1 \le j \le N-1,\ 1 \le m \le M \end{aligned} \quad (31)$$

$$\begin{aligned} &\sum_{n=K}^{N-1} (-1)^{n-K} \binom{n}{K} u_{n,m}(0) = \beta_{K,m} \\ &\sum_{n=K}^{N-1} (-1)^{n-K} \binom{n}{K} \frac{1 - f_m(\beta_{n,m})}{\beta_{n,m}} u_{n,m}(0) \end{aligned} \quad (32)$$

Coefficients $u_{j,m}(0)$ can be determined from Eq. 28, 31 and 32. In order to determine, the point wise reliability we drop Eq. 4 and set $P_{K,m}(t) = 0$ in Eq. 6 then solve Eq. 1-3 and 5-7 simultaneously.

Expressions for the probabilities ($K \le j \le N-1$, $1 \le m \le M$) and coefficients $u_{j,m}(s, 0)$ ($K+1 \le j \le N$, $1 \le m \le M$) are same as in Eq. 14 and 17 and 19, respectively. Instead of Eq. 20 however, we have:

$$\sum_{n=K}^{N-1} (-1)^{n-K} \binom{n}{K} u_{n,m}(0) = 0$$

Thus, using the above solutions the LT of the point wise reliability is given by:

$$R(s) = P_N(s) + \sum_{n=K}^{N-1} \sum_{m=1}^{M} P_{n,m}(s) \quad (33)$$

Moreover, if we set s = 0 in Eq. 33, we obtain the MTSF.

ANALYSIS OF MODEL 2

The set of integro-differential equations describing the behaviour of the system is:

$$\left(\frac{d}{dt} + \beta_N \right) P_N(t) = \sum_{m=1}^{M} \int_0^t P_{K-1,m}(t,x) h_m^*(x) dx \quad (34)$$

$$\begin{aligned} &\left[\frac{d}{dt} + \beta_{j,m} \right] P_{j,m}(t) = \beta_{j+1,m} P_{j+1,m}(t) \\ &K \le j \le N-1,\ 1 \le m \le M \end{aligned} \quad (35)$$

$$\begin{aligned} &\left[\frac{d}{dt} + \frac{\partial}{\partial x} + h_m^*(x) \right] P_{K-1,m}(t,x) = 0 \\ &1 \le m \le M \end{aligned} \quad (36)$$

With the following boundary and initial conditions:

$$P_{j,m}(0) = \delta_{j,N} \quad (37)$$

$$\begin{aligned} &P_{K-1,m}(t,0) = \beta_{K,m} P_{K,m}(t) \\ &1 \le m \le M \end{aligned} \quad (38)$$

By taking LT of the above equations, we get:

$$(s + \beta_N) P_N = \sum_{m=1}^{M} \int_0^\infty P_{K-1,m}(s,x) h_m^*(x) dx \quad (39)$$

$$\begin{aligned} &(s + \beta_{j,m}) P_{j,m}(t) = \beta_{j+1,m} P_{j+1,m}(s) \\ &K \le j \le N-1,\ 1 \le m \le M \end{aligned} \quad (40)$$

$$\begin{aligned} &\left[s + \frac{d}{dx} + h_m^*(x) \right] P_{K-1,m}(s,x) = 0 \\ &1 \le m \le M \end{aligned} \quad (41)$$

$$P_{j,m}(0) = \delta_{j,N} \quad (42)$$

$$\begin{aligned} &P_{K-1,m}(s,0) = \beta_{K,m} P_{K,m}(s) \\ &1 \le m \le M \end{aligned} \quad (43)$$

After solving the above equations, we get the following expression for the system availability:

$$P_A(s) = \frac{1 + \sum_{m=1}^{M} \sum_{j=K}^{N-1} \prod_{n=j}^{N-1} \frac{\beta_{n+1,m}}{s + \beta_{n,m}}}{s + \beta_N - \sum_{m=1}^{M} \beta_{K,m} \underline{f_m^*}(s) \prod_{n=K}^{N-1} \frac{\beta_{n+1,m}}{s + \beta_{n,m}}} \quad (44)$$

By putting $f_m^*(s) = 0$ in Eq. 44, we obtain the LT of the point wise reliability R(s):

$$R(s) = \frac{1 + \sum_{n=K}^{N-1} \sum_{m=1}^{M} \prod_{n=j}^{N-1} \frac{\beta_{n+1,m}}{s + \beta_{n,m}}}{s + \beta_N} \quad (45)$$

The mean time to system failure MTSF is now:

$$MTSF = \frac{1 + \sum_{n=K}^{N-1} \sum_{m=1}^{M} \frac{\beta_{N,m}}{\beta_{n,m}}}{\beta_N} \quad (46)$$

In order to determine, the steady-state availability we set and solve a set of integro-differential equations similarly to Model 1 and we obtain the following expression:

$$P_A = \frac{1+\sum_{n=K}^{N-1}\sum_{m=1}^{M}\frac{\beta_{N,m}}{\beta_{n,m}}}{1+\sum_{n=K}^{N-1}\sum_{m=1}^{M}\frac{\beta_{N,m}}{\beta_{K,m}}+\sum_{m=1}^{M}\frac{\beta_{N,m}}{\mu_m}} \qquad (47)$$

CONCLUSION

The study presented models for a parallel system with multiple failure modes with different repair policies. The approach gives insight into the transient behaviour of the system. Although, we started with fairly sophisticated set of integro-differential equations, the output of the models is a set of linear equations. Whereas point wise probabilities might need specialized softwarelike Mathematica, Maple or MatLab for the inverse Laplace transform the steady state availabilities and the mean times to system failure can be computed out of very few linear equations or from closed solutions.

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